Moreover, we cannot have $\mu_i = -1$ because that would mean by (3) that

$$0 = \bar{z}_i' A_1 z_i + \bar{z}_i' A_2 z_i = \bar{z}_i' A z_i .$$

Relation (12) thus implies

$$1 - |\mu_i|^2 > 0 \tag{14}$$

i.e. $|\mu_i| < 1$ as was to be proved.

The part of the theorem giving the sufficient condition was already obtained by L. Seidel [1] and G. Temple in a somewhat more indirect fashion.

REFERENCES

[1] L. Seidel, "Über ein Verfahren die Gleichungen, auf welche die Methode der kleinsten Quadrate führt, sowie lineare Gleichungen überhaupt, durch successive Annä herung aufzulösen," *Abhandlungen der Mathematisch-Physikalischen Classe der Königlich Bayerischen Akademie der Wissenschaften*, Vol. 11 (1874), pp. 81–108.

[2] C. E. Berry, "A criterion of convergence for the classical iterative method of solving linear simultaneous equations," *Annals of Math. Stat.*, Vol. 16 (1945), pp. 398–400.

[3] L. Cesari, "Sulla risoluzione dei sistemi di equazioni lineari per approssimazioni successive," *Rassegna delle Poste, dei Telegrafi e dei Telefoni*, Anno 9 (1931).

[4] L. Cesari, "Sulla risoluzione dei sistemi di equazioni lineari per approssimazioni successive," *Reale Accademia Nazionale dei Lincei, Serie 6, Classe di Szienze fisiche, matematiche e naturali, Rendiconti*, Vol. 25 (1937), pp. 422–428.

[5] J. Morris, *The Escalator Method in Engineering Vibration Problems*, Chapman and Hall Ltd., London 1947, pp. 63–70.

[6] R. J. Schmidt, "On the numerical solution of linear simultaneous equations by an iterative method," *Phil. Mag.*, Ser. 7, Vol. 32 (1941), pp. 369–383.

SOME RECURRENCE FORMULAE IN THE INCOMPLETE BETA FUNCTION RATIO

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1. Introduction. It is well known that the incomplete beta function ratio, defined by

$$I_x(p, q) = \frac{B_x(p, q)}{B(p, q)}, \tag{1}$$

where

$$B_x(p, q) = \int_0^x x^{p-1}(1 - x)^{q-1}\, dx, \tag{2}$$

and

$$B(p, q) = B_1(p, q), \tag{3}$$

is of importance in probability distribution theory, and, hence, also in obtaining exact probability values in making tests of statistical hypotheses. In constructing certain extensions [1] of Karl Pearson's "Tables of the Incomplete Beta-Function" [2], the recurrence formulae contained in the following sections were derived.

2. Derivation of formulae. The incomplete beta function, $B_x(p, q)$ may be considered as a special case of the hypergeometric series, $F(a, b, c, x)$, thus

$$B_x(p, q) = \frac{x^p}{p} F(p, 1 - q, p + 1, x). \tag{4}$$

The series converges for $|x| \leq 1$, if and only if $a + b < c$. By setting $a = p$, $b = 1 - q$, and $c = p + 1$, as in (4), all conditions are satisfied, if we also take $q > 0$.

Recurrence formulae for $F(a, b, c, x)$, e. g., in the work of Magnus and Oberhettinger [3], may now be directly converted for use with $B_x(p, q)$ or $I_x(p, q)$. In particular, using the three identities on page 9 of [3], with x replacing z, we have

$$cF(a, b, c, x) + (b - c)F(a + 1, b, c + 1, x) - b(1 - x)F(a + 1, b + 1, c + 1, x) = 0, \tag{5}$$

$$c(c - ax - b)F(a, b, c, x) - c(c - b)F(a, b - 1, c, x) + abx(1 - x)F(a + 1, b + 1, c + 1, x) = 0, \tag{6}$$

$$cF(a, b, c, x) - cF(a, b + 1, c, x) + axF(a + 1, b + 1, c + 1, x) = 0, \tag{7}$$

with $a = p$, $b = 1 - q$, and $c = p + 1$, we obtain in turn

$$xI_x(p, q) - I_x(p + 1, q) + (1 - x)I_x(p + 1, q - 1) = 0 \tag{8}$$

$$(p + q - px)I_x(p, q) - qI_x(p, q + 1) - p(1 - x)I_x(p + 1, q - 1) = 0 \tag{9}$$

$$qI_x(p, q + 1) + pI_x(p + 1, q) - (p + q)I_x(p, q) = 0. \tag{10}$$

Formula (8) is the basic recurrence formula used in the construction of Karl Pearson's [2] tables. Formula (10) was obtained, incidentally, by the author [4] in a different connection and manner.

Formulae (8), (9), and (10) may now be combined to give other useful formulae, e. g.,

$$qI_x(p + 1, q + 1) + (\overline{p + q}x - q)I_x(p + 1, q) - (p + q)xI_x(p, q) = 0, \tag{11}$$

$$pI_x(p + 1, q + 1) + (q - \overline{p + q}x)I_x(p, q + 1) - (p + q)(1 - x)I_x(p, q) = 0, \tag{12}$$

$$(13)\quad (p+q-1)xI_x(p-1,q) - (\overline{p+q-1}x+p)I_x(p,q) + pI_x(p+1,q) = 0,$$

$$(14)\quad (p+q)(1-x)I_x(p+1,q-1) - \{(p+q)(1-x)+q\}I_x(p+1,q) + pI_x(p+1,q+1) = 0.$$

Notice that the sum of the coefficients is always zero.

By a repeated use of (10) it is possible to obtain the formulae

$$(15)\quad I_x(p+n,q) = \frac{1}{(p+n-1)^{(n)}}\sum_{r=0}^{n}(-1)^r \cdot \binom{n}{r}(p+q+n-1)^{(n-r)}(q+r-1)^{(r)}I_x(p,q+r),$$

$$(16)\quad I_x(p,q+n) = \frac{1}{(q+n-1)^{(n)}}\sum_{r=0}^{n}(-1)^r \cdot \binom{n}{r}(p+q+n-1)^{(n-r)}(p+r-1)^{(r)}I_x(p+r,q),$$

where $(p+q+n-1)^{(n-r)}$, etc., refer to the factorial notation, e. g.,

$$[p+q+(n-1)]^{(n-r)} = (p+q+n-1)(p+q+n-2)\cdots(p+q+r).$$

3. An application. Formulae (15) and (16) may be used to write general formulae for obtaining values of $I_x(p,q)$ where p or q may be greater than 50, i. e., for such values outside the range of Karl Pearson's tables. In particular,

$$(17)\quad I_x(50+n,q) = \frac{1}{49+n)^{(n)}}\Big[n+q+49)^{(n)}I_x(50,q) - \binom{n}{1}q(n+q+49)^{(n-1)}I_x(50,q+1)\cdots(-1)^n(q+n-1)^{(n)}I_x(50,q+n)\Big]$$

and

$$(18)\quad I_x(p,50+n = \frac{1}{(49+n)^{(n)}}\Big[(n+p+49)^{(n)}I_x(p,50) - \binom{n}{1}p(n+p+49)^{(n-1)}I_x(p+1,50)\cdots(-1)^n(p+n-1)^{(n)}I_x(p+n,50)\Big].$$

It should be noted for (17) that as n increases the range of values that can be obtained outside Karl Pearson's tables are reduced since the last term of (17) contains $I_x(50, q+n)$. A similar observation is noted for (18). From a practical standpoint the computational labor restricts n to fairly small values. Using (17) we may easily compute for example,

$$I_{.60}(52,48) = I_{.60}(50+2,48)$$

$$= \frac{1}{(51)(50)}[(99)(98)I_{.60}(50,48) - 2(99)(48)I_{.60}(50,49) + (49)(48)I_{60}(50,50)].$$

Substituting the necessary values from Karl Pearson's tables we calculate

$$I_{.60}(52, 48) = .9465248.$$

Similarly using (18) we may calculate

$$I_{.40}(48, 52) = .0534752.$$

As a check on the computations, we use the well-known identity

$$I_x(p, q) = 1 - I_{1-x}(p', q'),$$

where $p' = q$ and $q' = p$. Then

$$\begin{aligned} I_{.40}(48, 52) &= 1 - I_{.60}(52, 48) \\ &= 1 - .9465248 \\ &= .0534752. \end{aligned}$$

In like manner formulae (15) and (16) may be used to write general formulae for obtaining half values for p or q greater than 10.5, i. e., for values not included in Karl Pearson's tables. In particular,

$$\begin{aligned} I_x(10.5 + n, q) = \frac{1}{(9.5 + n)^{(n)}} \Big[& (9.5 + q + n)^{(n)} I_x(10.5, q) - \binom{n}{1} \\ & \cdot q(9.5 + q + n)^{(n-1)} I_x(10.5, q + 1) \cdots (-1)^n (q + n - 1)^{(n)} I_x(10.5, q + n) \Big], \end{aligned} \tag{19}$$

and

$$\begin{aligned} I_x(p, 10.5 + n) = \frac{1}{(9.5 + n)^{(n)}} \Big[& (9.5 + p + n)^{(n)} I_x(p, 10.5) - \binom{n}{1} \\ & \cdot p(9.5 + p + n)^{(n-1)} I_x(p + 1, 10.5) \cdots (-1)^n (p + n - 1)^{(n)} I_x(p + n, 10.5) \Big]. \end{aligned} \tag{20}$$

Using (19) we may compute

$$\begin{aligned} I_{.60}(12.5, 8) = \frac{1}{(11.5)^{(2)}} [& (19.5)^{(2)} I_{.60}(10.5, 8) - 2(8)(19.5) I_{.60}(10.5, 9) \\ & + (9)(8) I_{.60}(10.5, 10)], = .4512367. \end{aligned}$$

Similarly using (20) we obtain

$$I_{.40}(8, 12.5) = .5487633.$$

Employing the check formula,

$$\begin{aligned} I_{.40}(8, 12.5) &= 1 - I_{.60}(12.5, 8) \\ &= 1 - .4512367 \\ &= .5487633. \end{aligned}$$

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REFERENCES

[1] T. A. Bancroft, "Some extensions of the incomplete beta function tables." (in preparation)

[2] Karl Pearson, *Tables of the Incomplete Beta-Function*, Cambridge University Press, 1934.

[3] Wilhelm Magnus und Fritz Oberhettinger, *Formeln and Sätze für die Speziellen Funktionen der Mathematischen Physik*, Julius Springer, Berlin, 1943.

[4] T. A. Bancroft, "On biases in estimation due to the use of preliminary tests of significance, *Annals of Math. Stat.*," Vol. 15 (1944).

ON A THEOREM BY WALD AND WOLFOWITZ

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Let $\mathfrak{H}_n = (h_1, \cdots, h_n)$, $(n = 1, 2, \cdots)$, be sequences of real numbers and for all n denote by $H_{e_1 \cdots e_m}$ the symmetrical function generated by $h_1^{e_1} \cdots h_m^{e_m}$, i.e., $H_{e_1 \cdots e_m} = \Sigma\, h_{i_1}^{e_1} \cdots h_{i_m}^{e_m}$ where the summation is extended over the $n(n-1) \cdots (n-m+1)$ possible arrangements of the m integers $i_1, \cdots, i_m$, such that $1 \leq i_j \leq n$ and $i_j \neq i_k$, $(j, k = 1, \cdots, m)$. According to Wald and Wolfowitz [1] the sequences $\mathfrak{H}_n$ are said to satisfy condition W, if for all integral $r > 2$

$$\frac{\frac{1}{n}\sum_{i=1}^{n}(h_i - \bar{h})^r}{\left[\frac{1}{n}\sum_{i=1}^{n}(h_i - \bar{h})^2\right]^{r/2}} = O(1),^1$$

where $\bar{h} = 1/n\, \Sigma_{i=1}^{n} h_i$.

Given sequences $\mathfrak{A}_n = (a_1, \cdots, a_n)$ and $\mathfrak{D}_n = (d_1, \cdots, d_n)$, consider the chance variable

$$L_n = d_1x_1 + \cdots + d_nx_n,$$

where the domain of $(x_1, \cdots, x_n)$ consists of the $n!$ equally likely permutations of the elements of $\mathfrak{A}_n$. Then it is shown in [1] that if the sequences $\mathfrak{A}_n$ and $\mathfrak{D}_n$ satisfy condition W, the distribution of $L_n^0 = (L_n - EL_n)/\sigma(L_n)$ approaches the normal distribution with mean 0 and variance 1 as $n \to \infty$. These conditions

[1] The symbol O, as well as the symbols o and $\sim$ to be used later, have their usual meaning. See e. g. Cramér [2, p. 122].