Since $\sigma^2 > u'^2$, this is the characteristic function for two variables which are normally distributed. Thus, the simultaneous distribution of ξ and M is asymptotically normal. It is of interest to note that, if the pdf $f(x)$ is symmetric, the correlation coefficient is zero, and M and ξ are asymptotically independent. We might also note that $\phi(t_1, 0)$ is the characteristic function for the mean deviation from the sample median. Thus, the random variable M is asymptotically normal with asymptotic mean and variance u' and $((m - \theta)^2 + \sigma^2 - u'^2)/2k$ respectively.

The author wishes to express his appreciation to Professor A. T. Craig for valuable suggestions in the study of this problem.

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NOTE ON THE EXTENSION OF CRAIG'S THEOREM TO NON-CENTRAL VARIATES

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A theorem due to A. T. Craig [1] and H. Hotelling [3] concerning the distribution of real quadratic forms in normal variates is extended to the case of non-central normal variates with equal variance.

The following notation is used: A, A_1, A_2 are real symmetric matrices, L is an orthogonal matrix, Γ is a diagonal matrix of latent roots, and X, Y, M and U are column vectors.

Theorem. *Let $X' = (x_1, \cdots, x_n)$ be a set of normally and independently distributed variates with equal variance σ^2 and means $M' = (m_1, \cdots, m_n)$. Then, a necessary and sufficient condition that a real symmetric quadratic form $Q(X) = X'AX$ of rank r be distributed as $\sigma^2\chi^2$, where*

$$p(\chi^2, r, \lambda^2) = \tfrac{1}{2}e^{-\lambda^2}(\chi^2/2)^{(r-2)/2}\, e^{-\chi^2/2}$$

$$\sum_{j=0}^{\infty} (\lambda^2\chi^2/2)^j/j!\,\Gamma[(r - 2j)/2], \tag{1}$$

is that $A^2 = A$. If $Q(X)/\sigma^2$ is distributed by $p(\chi^2, r, \lambda^2)$, then $\lambda^2 = Q(M)/2\sigma^2$.

Further, let $Q_1(X) = X'A_1X$ and $Q_2(X) = X'A_2X$ be real symmetric quadratic forms of ranks r_1 and r_2. Then a necessary and sufficient condition that $Q_1(X)$ and $Q_2(X)$ be statistically independent is that $A_1A_2 = 0$.

Proof. The theorem is proved by establishing the equivalence and factorization of moment generating functions [4]. The moment generating function of

$p(\chi^2, r, \lambda^2)$ is

$$G(t) = Ee^{t\chi^2/2} = e^{\lambda^2 t/(1-t)}(1-t)^{-r/2}. \tag{2}$$

Let $x_1, \cdots, x_n$ be normally and independently distributed with means $E(x_i) = m_i$ and common variance σ^2. Without loss of generality, we may take $\sigma^2 = 1$, changing to the general case when necessary with the transformation $x_i = z_i/\sigma$.

Let $Q(X) = X'AX$ be a real symmetric quadratic form of rank r. Then the moment generating function of $Q(X)$ is

$$G_Q(t) = Ee^{tQ(X)/2} = (2\pi)^{-n/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(X-M)'(X-M)-X'tAX]} \prod_1^n dx_i. \tag{3}$$

If t is restricted to values such that $|t| < |1/\gamma_0|$, where γ_0 is the dominant latent root of A, then $I - tA$ is positive definite and

$$\begin{aligned} G_Q(t) &= (2\pi)^{-n/2} e^{\frac{1}{2}M'tA(I-tA)^{-1}M} \\ &\quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}[X-(I-tA)^{-1}M]'(I-tA)[X-(I-tA)^{-1}M]} \prod_1^n dx_i \\ &= e^{\frac{1}{2}M'tA(I-tA)^{-1}M} \, | I - tA |^{-\frac{1}{2}}. \end{aligned} \tag{4}$$

If L is an orthogonal matrix such that

$$L'AL = \Gamma = \begin{pmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \gamma_n \end{pmatrix},$$

where the γ_i are the latent roots of A, then the transformation $M = LU$ gives

$$G_Q(t) = e^{\frac{1}{2}U't\Gamma(I-t\Gamma)^{-1}U} \cdot | I - t\Gamma |^{-\frac{1}{2}}. \tag{5}$$

A necessary and sufficient condition that $G_Q(t) = G(t)$ is that $A^2 = A$. If $A^2 = A$, then all of the latent roots of A are $+1$ or 0, and sufficiency can be established by substituting the appropriate value of each γ_i into equation (5), giving

$$G_Q(t) = e^{\lambda^2 t/(1-t)}(1-t)^{-r/2} = G(t). \tag{6}$$

Also $\lambda^2 = \sum_1^n \gamma_i u_i^2/2 = \frac{1}{2}(U'\Gamma U) = \frac{1}{2}(M'AM) = Q(M)/2$.

It is apparent from the form of $G_Q(t)$ that a necessary condition for $G_Q(t) = G(t)$ is that $| I - tA |^{-\frac{1}{2}} = (1-t)^{-r/2}$. But it has been proved by Craig [1] that the condition $A^2 = A$ is necessary, as well as sufficient, for this equality.

Next, let $Q_1(X) = X'A_1X$ and $Q_2(X) = X'A_2X$ be real symmetric quadratic forms of ranks r_1 and r_2. Then from (4)

$$\begin{aligned} G(t_1, t_2) &= Ee^{t_1Q_1/2 + t_2Q_2/2} \\ &= e^{\frac{1}{2}M'(t_1A_1+t_2A_2)(I-t_1A_1-t_2A_2)^{-1}M} \, | I - t_1A_1 - t_2A_2 |^{-\frac{1}{2}}, \end{aligned} \tag{7}$$

t_1, t_2 being restricted to values for which $(I - t_1A_1 - t_2A_2)$ is positive definite. A necessary and sufficient condition that $G(t_1, t_2) = G_Q(t_1) \cdot G_Q(t_2)$ is $A_1A_2 = 0$. The required equation in the moment generating functions is

$$G(t_1, t_2) = e^{\frac{1}{2}M' t_1 A_1 (I - t_1A_1)^{-1} M} \mid I - t_1A_1 \mid^{-\frac{1}{2}} \cdot e^{\frac{1}{2}M' t_2 A_2 (I - t_2A_2)^{-1} M} \mid I - t_2A_2 \mid^{-\frac{1}{2}}. \tag{8}$$

Assume $A_1A_2 = 0$. Then $(I - t_1A_1 - t_2A_2) = (I - t_1A_1)(I - t_2A_2)$ and $\mid I - t_1A_1 - t_2A_2 \mid = \mid I - t_1A_1 \mid \cdot \mid I - t_2A_2 \mid$. Also

$$(t_1A_1 + t_2A_2)(I - t_1A_1 - t_2A_2)^{-1} = t_1A_1(I - t_1A_1)^{-1} + t_2A_2(I - t_2A_2)^{-1},$$

for using the identity $tA(I - tA)^{-1} = (I - tA)^{-1} - I$, this becomes

$$(I - t_2A_2)^{-1}(I - t_1A_1)^{-1} = (I - t_1A_1)^{-1} + (I - t_2A_2)^{-1} - I.$$

Multiplying both sides on the left by $(I - t_2A_2)$ and on the right by $(I - t_1A_1)$, the identity follows. Thus the condition is sufficient.

It is apparent from the form of the moment generating functions that a necessary condition for $G(t_1, t_2) = G_Q(t_1)G_Q(t_2)$ is that $\mid I - t_1A_1 - t_2A_2 \mid = \mid I - t_1A_1 \mid \cdot \mid I - t_2A_2 \mid$. However, it has been proved by Hotelling [3] and Craig [2] that the condition $A_1A_2 = 0$ is necessary for this equality.

An extension can be made to correlated variates. Let $X' = (x_1, \cdots, x_n)$ be normally distributed with non-singular correlation matrix B and means $M' = (m_1, \cdots, m_n)$. Then there exists a non-singular transformation $X = TZ$, such that the variates Z are independent and have unit variance. Thus $T^{-1}BT'^{-1} = I, B = TT'$ and $Q(X) = X'AX = Z'T'ATZ$. Applying the theorem proved above, a necessary and sufficient condition that $Q(X)$ be distributed as χ^2 is that $(T'AT)^2 = T'ABAT = T'AT$, or that $ABA = A$. As before, $\lambda^2 = Q(M)/2$. In the same manner, a necessary and sufficient condition for independence of $Q_1(X)$ and $Q_2(X)$ is that $(T'A_1T)(T'A_2T) = T'A_1BA_2T = 0$, or that $A_1BA_2 = 0$.

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