COMBINATORIAL PROPERTIES OF GROUP DIVISIBLE INCOMPLETE BLOCK DESIGNS

By R. C. Bose and W. S. Connor[1]

University of North Carolina

1. Summary and introduction. Group divisible incomplete block designs are an important subclass of partially balanced designs [1], [2] with two associate classes, and they may also be regarded as a special case of intra- and inter-group balanced incomplete block designs [3], [4]. They may be defined as follows.

An incomplete block design with v treatments each replicated r times in b blocks of size k is said to be group divisible (GD) if the treatments can be divided into m groups, each with n treatments, so that the treatments belonging to the same group occur together in λ_1 blocks and treatments belonging to different groups occur together in λ_2 blocks. If $\lambda_1 = \lambda_2 = \lambda$ (say), then every pair of treatments occurs in λ blocks, and the design becomes a balanced incomplete block design, which has been extensively studied [5], [6], [7], [8]. We shall therefore confine ourselves to the case $\lambda_1 \neq \lambda_2$.

The object of this paper is to study the combinatorial properties of these designs. It is shown that the GD designs can be divided into three exhaustive and mutually exclusive classes:

(a) Singular GD designs characterized by $r - \lambda_1 = 0$;

(b) Semi-regular GD designs characterized by $r - \lambda_1 > 0$, $rk - v\lambda_2 = 0$;

(c) Regular GD designs characterized by $r - \lambda_1 > 0$, $rk - v\lambda_2 > 0$.

Certain inequality relations between the parameters necessary for the existence of the design have been derived in each case. Some other interesting theorems about the structure of these designs have also been obtained. Methods of constructing GD designs will be given in a separate paper.

2. Group divisible designs regarded as special cases of partially balanced and intra- and inter-group incomplete block designs. A partially balanced design with two associate classes is defined as follows.

(i) There are v treatments, each replicated r times in b blocks of size k.

(ii) There can be established a relation of association between any two treatments satisfying the following conditions. (a) Two treatments are either first associates or second associates. (b) Each treatment has n_1 first associates and n_2 second associates. (c) Given any two treatments which are ith associates, the number of treatments which is common to the jth associates of the first and kth associates of the second is p^i_{jk} and is independent of the pair of treatments with which we start $(i, j, k = 1, 2)$. Also $p^i_{jk} = p^i_{kj}$.

(iii) Two treatments which are ith associates occur together in exactly λ_i blocks $(i = 1, 2)$.

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[1] Now with the National Bureau of Standards, Washington, D. C.

It is known [1] that the following relations hold among the parameters:

$$bk = vr,\ v = n_1 + n_2 + 1,\ n_1\lambda_1 + n_2\lambda_2 = r(k-1), \tag{2.0}$$

$$1 + p_{11}^1 + p_{12}^1 = p_{11}^2 + p_{12}^2 = n_1, \tag{2.1}$$

$$p_{21}^1 + p_{22}^1 = 1 + p_{21}^2 + p_{22}^2 = n_2, \tag{2.2}$$

$$n_1 p_{12}^1 = n_2 p_{11}^2,\ n_1 p_{22}^1 = n_2 p_{12}^2. \tag{2.3}$$

If in a GD design with parameters $v, b, r, k, \lambda_1, \lambda_2, m, n$, treatments belonging to the same group are considered as first associates, and treatments belonging to different groups are considered as second associates, then it is easy to see that it is a partially balanced design with two associate classes for which

$$n_1 = n - 1, \qquad n_2 = n(m-1), \tag{2.4}$$

$$p_{11}^1 = n - 2, \qquad p_{12}^1 = 0, \qquad p_{22}^1 = n(m-1), \tag{2.5}$$

$$p_{11}^2 = 0, \qquad p_{12}^2 = n - 1, \qquad p_{22}^2 = n(m-2). \tag{2.6}$$

Conversely suppose for a partially balanced design $p_{12}^1 = 0$. Then from (2.1), $p_{11}^1 = n_1 - 1$. The relation of first association between two treatments is by definition commutative. We shall show that in the present case it is also transitive. Let the treatments θ_0 and θ_1 be first associates. Let the other first associates of θ_0 be $\theta_2, \theta_3, \cdots, \theta_{n_1}$. Now since θ_0 and θ_1 have $n_1 - 1$ common first associates, they can be no other than $\theta_2, \theta_2, \cdots, \theta_{n_1}$. Also since θ_1 has exactly n_1 first associates, so all its first associates are $\theta_0, \theta_2, \theta_3, \cdots, \theta_{n_1}$. This shows that any first associate of θ_0 (other than θ_1) is also a first associate of θ_1. These conditions are sufficient to insure that the v treatments can be divided into groups of $n_1 + 1$ such that two treatments in the same group are first associates, and two treatments in different groups are second associates. Hence the design is a GD design where the treatments of the same group are first associates. Similarly if $p_{12}^2 = 0$, we can show that the partially balanced design is a GD design, the treatments of the same group being second associates. We can therefore state

THEOREM 1. *The necessary and sufficient condition for a partially balanced design to be group divisible is the vanishing of p_{12}^1 or p_{12}^2. If $p_{12}^i = 0$ then the treatments in the same group are ith associates* ($i = 1, 2$).

A GD design is also a special case of intra- and inter-group balanced incomplete block designs which have been defined by Nair and Rao [3] in the following manner.

(i) there are v treatments arranged in b blocks of size k.

(ii) the v treatments fall into m groups consisting of $v_1, v_2, \cdots, v_m$ treatments, treatments of the ith group being replicated r_i times ($i = 1, 2, \cdots, m$).

(iii) every pair of treatments of the ith group occurs in λ_{ii} blocks ($i = 1, 2, \cdots, m$), and every pair of treatments belonging to the ith and jth groups, $i \neq j$, occurs in λ_{ij} blocks ($i, j = 1, 2, \cdots, m$).

If in particular

$$r_i = r, \qquad \lambda_{ii} = \lambda_1 \qquad (i = 1, 2, \cdots, m),$$
$$\lambda_{ij} = \lambda_2 \qquad (i \neq j;\, i, j = 1, 2, \cdots, m),$$

then the design reduces to a GD design.

GD designs may therefore be characterized as the class of designs which are partially balanced and intra- and inter-group balanced at the same time.

3. Relations between the parameters, and the classification of GD designs. Clearly

$$v = mn, \qquad bk = vr. \tag{3.0}$$

Any given treatment θ occurs in r blocks. Since each of these blocks contains $k - 1$ other treatments, there are $r(k - 1)$ pairs of which one member is θ. But θ must form λ_1 pairs with each of the $n - 1$ treatments belonging to the same group as θ, and λ_2 pairs with each of the $n(m - 1)$ treatments not in the same group as θ. Hence

$$(n - 1)\lambda_1 + n(m - 1)\lambda_2 = r(k - 1). \tag{3.1}$$

Also

$$r \geqq \lambda_1, \qquad r \geqq \lambda_2. \tag{3.2}$$

The eight parameters v, b, r, k, λ_1, λ_2, m, n are therefore connected by the three relations (3.0) and (3.1), so that only five parameters are free.

Let $n_{ij} = 1$ or 0 according as the ith treatment does or does not occur in the jth block. Then the matrix

$$N = (n_{ij}) \tag{3.3}$$

is defined to be the incidence matrix of the design. From the conditions satisfied by the design, it is easy to see that

$$\sum_{j=1}^{v} n_{ij}^2 = r, \tag{3.4a}$$

$$\sum_{j=1}^{r} n_{ij} n_{uj} = \lambda_1 \text{ or } \lambda_2, \tag{3.4b}$$

according as the ith and uth treatments $(i \neq u)$ do belong or do not belong to the same group.

If in numbering the treatments we follow the convention that the lth group consists of the treatments number $n(l - 1) + 1$, $n(l - 1) + 2$, $\cdots$, nl, then from (3.4) and (3.5) we can write

$$NN' = \begin{bmatrix} A & B & \cdots & B \\ B & A & \cdots & B \\ \vdots & \vdots & & \vdots \\ B & B & \cdots & A \end{bmatrix} \tag{3.5}$$

where N' is the transpose of the matrix N, and A and B are $n \times n$ matrices defined by

$$(3.6) \qquad A = \begin{bmatrix} r & \lambda_1 & \cdots & \lambda_1 \\ \lambda_1 & r & \cdots & \lambda_1 \\ \vdots & \vdots & & \vdots \\ \lambda_1 & \lambda_1 & \cdots & r \end{bmatrix}, \qquad B = \begin{bmatrix} \lambda_2 & \lambda_2 & \cdots & \lambda_2 \\ \lambda_2 & \lambda_2 & \cdots & \lambda_2 \\ \vdots & \vdots & & \vdots \\ \lambda_2 & \lambda_2 & \cdots & \lambda_2 \end{bmatrix}.$$

Each row or column in the matrix on the right-hand side in (3.5) contains A in the diagonal position, and contains B in the other $m - 1$ positions.

To evaluate $| NN' |$ proceed as follows.

(i) Add the 2nd, $\cdots$, mnth rows in $| NN' |$ to the first row, every element of which now becomes $r + (n - 1)\lambda_1 + n(m - 1)\lambda_2 = rk$ (cf. (3.1)).

(ii) Take rk outside the determinant, and subtract the first row multiplied by λ_2 from all other rows. Then

$$| NN' | = rk \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 - \lambda_2 & r - \lambda_2 & \cdots & \lambda_1 - \lambda_2 \\ \vdots & \vdots & & \vdots \\ \lambda_1 - \lambda_2 & \lambda_1 - \lambda_2 & \cdots & r - \lambda_2 \end{vmatrix} \cdot \begin{vmatrix} r - \lambda_2 & \lambda_1 - \lambda_2 & \cdots & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & r - \lambda_2 & \cdots & \lambda_1 - \lambda_2 \\ \vdots & \vdots & & \vdots \\ \lambda_1 - \lambda_2 & \lambda_1 - \lambda_2 & \cdots & r - \lambda_2 \end{vmatrix}^{m-1}$$

$$= rk \{(r - \lambda_2) + (n - 1)(\lambda_1 - \lambda_2)\}^{m-1} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 - \lambda_2 & r - \lambda_2 & \cdots & \lambda_1 - \lambda_2 \\ \vdots & \vdots & & \vdots \\ \lambda_1 - \lambda_2 & \lambda_1 - \lambda_2 & \cdots & r_1 - \lambda_2 \end{vmatrix}^{m}.$$

Using (3.0) and (3.1) we can finally write

$$(3.7) \qquad | NN' | = rk(rk - v\lambda_2)^{m-1}(r - \lambda_1)^{m(n-1)}.$$

The quantity $rk - v\lambda_2$ occurring above is nonnegative. Here we shall prove this statement for the case $r - \lambda_1 > 0$, and complete the proof later in Section 4.

Let N_1 be the submatrix formed from the matrix N given by (3.3), by taking the first $2n$ rows (which correspond to the treatments of the first two groups).

Then

$$N_1 N_1' = \begin{bmatrix} A & B \\ B & A \end{bmatrix}. \tag{3.8}$$

$$| N_1 N_1' | = \{ r + (n - 1)\lambda_1 + n\lambda_2 \} (r - \lambda_1)^{2(n-1)} (rk - v\lambda_2). \tag{3.9}$$

Since $| N_1 N_1' | \geqq 0$, it follows that $rk - v\lambda_2 \geqq 0$ if $r - \lambda_1 > 0$.

Hence we can divide all GD designs into three exhaustive and mutually exclusive classes:

(a) Singular GD designs characterized by $r = \lambda_1$;

(b) Semi-regular GD designs characterized by $r > \lambda_1$, $rk - v\lambda_2 = 0$;

(c) Regular GD designs characterized by $r > \lambda_1$, $rk - v\lambda_2 > 0$.

The main combinatorial properties of each of these three classes will be given in the succeeding sections.

4. Singular GD designs. Consider a balanced incomplete block design with v^* treatments, each replicated r^* times in b^* blocks of size k^*, such that any two treatments occur together in λ^* blocks. Replace each treatment with a group of n treatments. Now there are $v = nv^*$ treatments divided into v^* groups (each group corresponding to one of the original treatments). Two treatments belonging to the same group now occur together r^* times and two treatments belonging to different groups occur together λ^* times. We thus get a GD design with parameters

$$\begin{aligned} v &= nv^*, & b &= b^*, & r &= r^*, & k &= nk^*, \\ \lambda_1 &= r^*, & \lambda_2 &= \lambda^*, & m &= v^*, & n &= n, \end{aligned} \tag{4.0}$$

which belongs to the singular class since $r - \lambda_1 = 0$.

Conversely consider a singular GD design with parameters v, b, r, k, λ_1, λ_2, m, n, where $r = \lambda_1$. Let θ and ϕ be any two treatments belonging to the same group. θ occurs in r blocks, and since $r = \lambda_1$, ϕ must occur in each of these r blocks and nowhere else. Hence if a treatment occurs in a certain block, every treatment belonging to the group occurs in that block. Let each group of treatments be replaced by a single treatment in the design, then there are $v^* = m$ treatments in the new design, and because any two treatments belonging to different groups occur together λ_2 times in the original design, the new design is a balanced incomplete block design with parameters

$$v^* = m, \quad b^* = b, \quad r^* = r, \quad k^* = k/n, \quad \lambda^* = \lambda_2. \tag{4.1}$$

We may therefore state

THEOREM 2. *If in a balanced incomplete block design with parameters v^*, b^*, r^*, k^*, λ^* each treatment is replaced by a group of n treatments, we get a singular GD design with parameters given by* (4.0). *Conversely, every singular GD design is obtainable in this way from a corresponding balanced incomplete block design.*

COROLLARY. *For a singular GD design* $b \geqq m$.

For example, consider the balanced incomplete block design with parameters $v^* = b^* = 7$, $r^* = k^* = 3$, $\lambda^* = 1$. The plan for this is given below, the columns representing the blocks.

a	b	c	d	e	f	g
b	c	d	e	f	g	a
d	e	f	g	a	b	c

If $n = 2$, then we may replace the treatment a by a_1, a_2, and do the same for the other treatments. We then get the singular GD design with the parameters

$$v = 14, \quad b = 7, \quad r = 3, \quad k = 6,$$
$$\lambda_1 = 3, \quad \lambda_2 = 1, \quad m = 7, \quad n = 2,$$

the plan for which is shown below.

a_1	b_1	c_1	d_1	e_1	f_1	g_1
a_2	b_2	c_2	d_2	e_2	f_2	g_2
b_1	c_1	d_1	e_1	f_1	g_1	a_1
b_2	c_2	d_2	e_2	f_2	g_2	a_2
d_1	e_1	f_1	g_1	a_1	b_1	c_1
d_2	e_2	f_2	g_2	a_2	b_2	c_2

As before the columns represent the blocks.

The relation $rk - v\lambda_2 \geqq 0$ is true by definition for semi-regular and regular GD designs. We shall show that it holds for singular GD designs also.

Let the parameters of a singular GD design be given by (4.0). Then remembering the relation $\lambda^*(v^* - 1) = r^*(k^* - 1)$ which holds for a balanced incomplete block design

$$\begin{aligned} rk - v\lambda_2 &= n(r^*k^* - v^*\lambda^*) \\ &= n(r^* - \lambda^*) \\ &\geqq 0. \end{aligned}$$

Hence we may state

THEOREM 3. *For any GD design* $rk - v\lambda_2 \geqq 0$.

5. Semi-regular GD designs. For a semi-regular GD design we have by definition

$$r - \lambda_1 > 0, \qquad rk - v\lambda_2 = 0. \tag{5.0}$$

Hence

$$r + (n - 1)\lambda_1 = n\lambda_2 . \tag{5.1}$$

We shall now prove the following

THEOREM 4. *For a semi-regular GD design k is divisible by m. If $k = cm$ then every block must contain c treatments from every group.*

Let e_j treatments from the first group occur in the jth block ($j = 1, 2, \cdots, b$). Then

$$\sum_{j=1}^{b} e_j = nr, \tag{5.2}$$

$$\sum_{j=1}^{b} e_j(e_j - 1) = n(n-1)\lambda_1, \tag{5.3}$$

since each treatment from the first group occurs in r blocks, and every pair of treatments from the first group occurs in λ_1 blocks. Using (5.1), (5.2) and (5.3),

$$\sum_{j=1}^{b} e_j^2 = n^2\lambda_2.$$

Let

$$\bar{e} = \frac{1}{b}\sum_{j=1}^{b} e_j = \frac{nr}{b} = \frac{k}{m}$$

from (3.0). Hence

$$\sum_{j=1}^{b} (e_j - \bar{e})^2 = n^2\lambda_2 - \frac{bk^2}{m^2} = 0 \tag{5.4}$$

from (3.0) and (5.0). Therefore

$$e_1 = e_2 = \cdots = e_b = \bar{e} = \frac{k}{m}. \tag{5.5}$$

Since e_j must be integral, k must be divisible by m. If $k = cm$ then $e_j = c$ ($j = 1, 2, \cdots, b$). The same argument applies to treatments of any other group. This proves our theorem.

The relation (3.7) shows that the $v \times v$ matrix NN' given by (3.5) is singular, for the case of semi-regular GD designs. We shall now show that its rank is not less than $v - m + 1$. Without changing the rank of NN' we can transform it into M_1 and then into M_2, where

$$M_1 = \begin{bmatrix} D & D & \cdots & D \\ B & A & \cdots & B \\ \vdots & \vdots & & \vdots \\ B & B & \cdots & A \end{bmatrix},$$

where $D = A + (m - 1)B$, and

$$M_2 = \begin{bmatrix} D & 0 & \cdots & 0 \\ B & A - B & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ B & 0 & \cdots & A - B \end{bmatrix},$$

where 0 is the null $n \times n$ matrix.

If we strike out the last row and column from $A - B$ we get the $(n - 1) \times (n - 1)$ matrix C given by

$$C = \begin{bmatrix} r - \lambda_2 & \lambda_1 - \lambda_2 & \cdots & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & r - \lambda_2 & \cdots & \lambda_1 - \lambda_2 \\ \vdots & \vdots & & \vdots \\ \lambda_1 - \lambda_2 & \lambda_1 - \lambda_2 & \cdots & r - \lambda_2 \end{bmatrix}.$$

From (5.0) and (5.1)

$$| D | = rk(r - \lambda_1)^{n-1}, \qquad | C | = (\lambda_2 - \lambda_1)(r - \lambda_1)^{n-2}.$$

If from M_2 we strike out the $2n$th, $\cdots$, mnth rows and columns, then the determinant of the resulting matrix is

$$| D | \, | C |^{m-1} = rk(\lambda_2 - \lambda_1)^{m-1}(r - \lambda_1)^{mn-2m+1} \neq 0,$$

since $\lambda_1 \neq \lambda_2$ and $r - \lambda_1 \neq 0$. Hence

(5.6) $$\text{Rank } M_2 \geqq n + (m - 1)(n - 1) = v - m + 1.$$

But

(5.7) $$\text{Rank } M_2 = \text{Rank } NN' \leqq b.$$

Hence the following theorem.

THEOREM 5. *For a semi-regular GD design* $b \geqq v - m + 1$.

If the design is resolvable, that is, the blocks can be divided into r groups, of b/r blocks each, such that each group of blocks gives a complete replication, then

(5.8) $$\text{Rank } NN' = \text{Rank } M_2 \leqq b - r + 1,$$

since in N the sum of the columns corresponding to a complete replication must give a column consisting of unities. Thus not more than $b - r + 1$ column vectors are independent. Hence the following corollary.

COROLLARY. *For a resolvable semi-regular GD design* $b \geqq v - m + r$.

6. Regular GD designs. For a regular GD design we have by definition

$$r > \lambda_1, \qquad rk - v\lambda_2 > 0. \tag{6.0}$$

From (3.7), $| NN' | > 0$. Hence

$$v = \text{Rank } NN' = \text{Rank } N \leqq b. \tag{6.1}$$

This gives

THEOREM 6. *For a regular GD design* $b \geqq v$.

If the design is resolvable then as before Rank $N \leqq b - r + 1$. Hence we can state the

COROLLARY. *For a resolvable regular GD design* $b \geqq v + r - 1$.

A GD design is said to be symmetrical if $b = v$, and in consequence $r = k$. Shrikhande [9] and Chowla and Ryser [10] have obtained conditions necessary for the existence of symmetrical balanced incomplete block designs. We shall extend their results to symmetrical regular GD designs. With this in view we give a brief resume of the important properties of the Hilbert norm residue symbol $(a, b)_p$, where a and b are rational numbers and p is a prime, and of the Minkowski-Hasse invariant [11], [12] $C_p(A)$ of a symmetric matrix with rational elements. We shall use these symbols in the sense defined by Pall [13]. Detailed proofs and a more general theory are available in a book by Jones [14].

7. The Hilbert norm residue symbol $(a, b)_p$. If a and b are any nonzero rational numbers, we define the Hilbert symbol $(a, b)_p$ to have the value $+1$ or -1 according as the congruence

$$ax^2 + by^2 \equiv 1 (\text{mod } p^r) \tag{7.0}$$

has or has not for each value r, rational solutions x_r and y_r. Here p is any prime including the conventional prime $p_\infty = \infty$.

It is clear that the value of $(a, b)_p$ is unchanged if we replace a and b by their square-free parts, that is, if s and t are rational numbers then

$$(as^2, bt^2)_p = (a, b)_p. \tag{7.1}$$

Pall [13] has given the following formulae for calculating $(a, b)_p$ in general, where $(a \mid p)$ is the well known Legendre symbol, and m and m' are prime to p.

$$(a, b)_\infty = -1, \qquad \text{if and only if } a \text{ and } b \text{ are negative;} \tag{7.2}$$

$$(p^\alpha m, p^{\alpha'} m')_p = (-1 \mid p)^{\alpha\alpha'} (m \mid p)^{\alpha'} (m' \mid p)^\alpha \text{ if } p > 2; \tag{7.3}$$

$$(2^\alpha m, 2^{\alpha'} m')_2 = (2 \mid m)^{\alpha'} (2 \mid m')^\alpha (-1)^{(m-1)(m'-1)/4}. \tag{7.4}$$

The following properties of $(a, b)_p$ stated by Pall can be easily verified.

$$(a, b)_p = (b, a)_p; \tag{7.5}$$

$$(a, -a)_p = 1, \qquad (a, a)_p = (a, -1)_p; \tag{7.6}$$

$$(a, b_1 b_2)_p = (a, b_1)_p (a, b_2)_p, \qquad (a_1 a_2, b)_p = (a_1, b)_p (a_2, b)_p. \tag{7.7}$$

For any odd prime p, $(a, b)_p$ is evidently $+1$ unless p actually appears in a or b. For a given a, b, $(a, b)_p$ is -1 for only a finite even number (possibly zero) of primes p counting in $p_\infty = \infty$. This is expressed by writing

$$\prod_p (a, b)_p = +1. \tag{7.8}$$

8. **The Hasse-Minkowski invariant** $C_p(A)$. Instead of considering invariants of quadratic forms $f = \sum a_{ij}x_ix_j$ we might consider the invariants to refer to the corresponding symmetric matrices (a_{ij}).

Let $A = (a_{ij})$ be any $n \times n$ matrix with rational elements. The matrix B is said to be rationally congruent to A, written $A \sim B$, provided there exists a nonsingular matrix C with rational elements such that

$$A = C'BC, \tag{8.0}$$

where C' is the transpose of C. If D_i $(i = 1, 2, \cdots, n)$ denotes the leading principal minor determinant of order i in the matrix A, then if none of the D_i vanishes, the quantity

$$C_p = C_p(A) = (-1, -D_n)_p \prod_{i=1}^{n-1} (D_i, -D_{i+1})_p \tag{8.1}$$

is invariant for all matrices rationally congruent to A [11].

The invariant C_p may be expressed in a more symmetrical form as follows:

$$\begin{aligned} C_p(A) &= (-1, -1)_p (D_n, -1)_p \prod_{i=1}^{n-1} (D_i, D_{i+1})_p (D_i, -1)_p \\ &= (-1, -1)_p (D_1, -1)_p \prod_{i=1}^{n-1} (D_{i+1}, D_i)_p (D_{i+1}, -1)_p \\ &= (-1, -1)_p \prod_{i=0}^{n-1} (D_{i+1}, -D_i)_p, \end{aligned} \tag{8.2}$$

where $D_0 = 1$.

If A_{n-1} denotes the matrix obtained by leaving out the last row and column of A, we at once get from (8.2)

$$c_p(A) = C_p(A_{n-1})(D_n, -D_{n-1})_p. \tag{8.3}$$

From this and the properties of the Hilbert symbol we can at once deduce the following Lemma, stated by Pall [13] in equivalent form.

LEMMA I. *If A and B are symmetric matrices with rational elements, and*

$$U = \begin{bmatrix} A & \\ & B \end{bmatrix} \tag{8.35}$$

is the direct sum of A and B, then

$$C_p(U) = (-1, -1)_p C_p(A) C_p(B)(|A|, |B|)_p. \tag{8.4}$$

LEMMA II. *If Δ_m is the direct sum of A taken m times, where A is a symmetric matrix with rational elements,* that is,

$$\Delta_m = \begin{bmatrix} A & & & & \\ & A & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & A \end{bmatrix}, \tag{8.45}$$

then

$$C_p(\Delta_m) = \{ C_p(A) \}^{m} (| A | , -1)_p^{m(m-1)/2}(-1, -1)_p^{m-1}. \tag{8.5}$$

PROOF. From Lemma I

$$\begin{aligned} C_p(\Delta_2) &= (-1, -1)_p \{ C_p(A) \}^2(| A | , | A |)_p \\ &= (-1, -1)_p \{ C_p(A) \}^2(| A | , -1)_p . \end{aligned}$$

Hence the lemma holds for $m = 2$. Supposing it to hold for $m - 1$, we have

$$\begin{aligned} C_p(\Delta_m) &= C_p(\Delta_{m-1})C_p(A)(| A | ^{m-1}, | A |)_p(-1, -1)_p \\ &= C_p(\Delta_{m-1})C_p(A)(| A | , -1)_p^{m-1}(-1, -1)_p \\ &= \{ C_p(A) \}^m(| A | , -1)_p^{m(m-1)/2}(-1, -1)_p^{m-1}. \end{aligned} \tag{8.6}$$

COROLLARY. *If d is a rational number and Δ_m is the diagonal matrix of order m given by*

$$\Delta_m = \begin{bmatrix} d & & & & \\ & d & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & d \end{bmatrix}, \tag{8.65}$$

then

$$C_p(\Delta_m) = (-1, -1)_p(d, -1)_p^{m(m+1)/2}. \tag{8.7}$$

This follows at once from the main lemma by noting that

$$C_p(d) = (-1, -1)_p(d, -1)_p$$

when d is regarded as a matrix of order one (cf. (8.2)).

We shall prove another lemma on a property of the Hilbert symbol which we shall use later.

LEMMA III. *For any two rational numbers $a \neq 0$, $b \neq 0$, and any prime p,*

$$(a, b)_p = (-ab, a + b)_p. \tag{8.8}$$

Let

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \sim \begin{bmatrix} a + b & b \\ b & b \end{bmatrix}.$$

Hence from (8.2)

$$(-1, -1)_p(a, -1)_p(ab, -a)_p = (-1, -1)_p(a + b, -1)_p(ab, -a-b)_p,$$

or

$$(a, -1)_p(b, a)_p(b, -1)_p = (a + b, -1)_p(ab, -1)_p(ab, a + b)_p,$$

or

$$(a, -1)_p(a, b)_p(b, -1)_p = (a + b, -ab)_p(a, -1)_p(b, -1)_p,$$

or

$$(a, b)_p = (-ab, a + b)_p.$$

Putting $a = n + 1$, $b = -1$ we obtain the following result given as a lemma by Bruck and Ryser [15].

COROLLARY.

$$(n + 1, -1)_p = (n + 1, n)_p. \tag{8.9}$$

9. Necessary conditions for the existence of symmetrical regular GD designs. Consider a symmetrical regular GD design with parameters $v, b, r, k, m, n, \lambda_1, \lambda_2$, where

$$v = b = mn, \qquad r = k, \tag{9.0}$$

$$(n - 1)\lambda_1 + n(m - 1)\lambda_2 = r(r - 1), \tag{9.1}$$

$$Q = r - \lambda_1 > 0, \qquad P = r^2 - v\lambda_2 > 0. \tag{9.2}$$

Since the incidence matrix N given by (3.3) is now a square matrix it follows from (3.7) that

$$| NN' | = | N |^2 = r^2P^{m-1}Q^{m(n-1)}. \tag{9.3}$$

It follows that $P^{m-1}Q^{m(n-1)}$ must be a perfect square. Hence the theorem

THEOREM 7. *A necessary condition for the existence of a symmetrical regular GD design with parameters $v, b, r, k, m, n, \lambda_1, \lambda_2$ satisfying* (9.0), (9.1) *and* (9.2) *is that $P^{m-1}Q^{m(n-1)}$ is a perfect square.*

Let $X = NN'$ be the matrix given by (3.5), then X is rationally congruent to the unit matrix I. Hence a necessary condition for the design to exist is that

$C_p(X) = C_p(NN') = C_p(I)$ for every prime p. In order to calculate $C_p(X)$ we shall start by obtaining $C_p(Y)$, where Y is the $n \times n$ matrix given by

$$(9.4) \qquad Y = \begin{bmatrix} r - \lambda_2 & \lambda_1 - \lambda_2 & \cdots & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & r - \lambda_2 & \cdots & \lambda_1 - \lambda_2 \\ \vdots & \vdots & & \vdots \\ \lambda_1 - \lambda_2 & \lambda_1 - \lambda_2 & \cdots & r - \lambda_2 \end{bmatrix}.$$

Referring to the derivation of (3.7) it is readily seen that $| Y | = PQ^{n-1}$. Let us set

$$Y_1 = \begin{bmatrix} Y & \\ & -\lambda \end{bmatrix},$$

where $\lambda = \lambda_1 - \lambda_2$.

Hence from (8.3),

$$(9.45) \qquad \begin{aligned} C_p(Y_1) &= C_p(Y)(-\lambda PQ^{n-1}, -PQ^{n-1})_p \\ &= C_p(Y)(-\lambda, -1)_p(-\lambda, P)_p(-\lambda, Q)_p^{n-1}. \end{aligned}$$

If we add the last row in Y_1 to each of the other rows, and then the last column to each of the other columns, we get

$$Y_1 \sim Y_2 = \begin{bmatrix} r - \lambda_1 & 0 & \cdots & 0 & -\lambda \\ 0 & r - \lambda_1 & \cdots & 0 & -\lambda \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & r - \lambda_1 & -\lambda \\ -\lambda & -\lambda & \cdots & -\lambda & -\lambda \end{bmatrix}.$$

Since $| Y_2 | = | Y_1 | = -\lambda PQ^{n-1}$ it follows from (8.3) and (8.7) that

$$(9.5) \qquad \begin{aligned} C_p(Y_1) = C_p(Y_2) &= (-1, -1)_p(Q, -1)_p^{n(n+1)/2}(-\lambda PQ^{n-1}, -Q^n)_p \\ &= (-1, -1)_p(Q, -1)_p^{n(n+1)/2}(-\lambda, -1)_p(-\lambda, Q)_p^n \\ &\qquad \cdot (P, -1)_p(P, Q)_p^n(Q, -1)_p^{n-1}. \end{aligned}$$

Comparing (9.45) and (9.5) we finally get

$$(9.6) \qquad C_p(Y) = (-1, -1)_p(Q, -1)_p^{n(n-1)/2}(P, \lambda)_p(Q, \lambda)_p(P, Q)_p^n,$$

where $\lambda = \lambda_1 - \lambda_2$.

To calculate $C_p(X) = C_p(NN')$ we now proceed as follows. Set

$$X_1 = \begin{bmatrix} X & \\ & -\lambda_2 \end{bmatrix}.$$

From (8.3) and (9.3),

$$C_p(X_1) = C_p(X)(-\lambda_2 r^2 P^{m-1}Q^{m(n-1)}, -r^2P^{m-1}Q^{m(n-1)})_p \tag{9.65}$$
$$= C_p(X)(-\lambda_2, -1)_p,$$

since from Theorem 7 $P^{m-1}Q^{m(n-1)}$ is a perfect square.

Again we may transform X_1 to the rationally congruent matrix X_2, by adding the last row of X_1 to each of the other rows, and then the last column of X_1 to each of the other columns. Thus

$$X_1 \sim X_2 = \begin{bmatrix} Y & & & & L \\ & Y & & & L \\ & & \ddots & & \vdots \\ & & & Y & L \\ L' & L' & \cdots & L' & -\lambda_2 \end{bmatrix},$$

where Y is given by (9.4), L is an $n \times 1$ column matrix with each element as $-\lambda_2$ and L' is the transpose of L. We note that

$$|X_2| = |X_1| = -\lambda_2 r^2 P^{m-1}Q^{m(n-1)}.$$

Hence from (8.3) and (8.5),

$$C_p(X_1) = C_p(X_2) = \{C_p(Y)\}^m(PQ^{n-1}, -1)_p^{m(m-1)/2}(-1, -1)_p^{m-1}$$
$$\cdot(-\lambda_2 r^2 P^{m-1}Q^{m(n-1)}, -P^mQ^{m(n-1)})_p \tag{9.7}$$
$$= \{C_p(Y)\}^m(PQ^{n-1}, -1)_p^{m(m-1)/2}(-1, -1)_p^{m-1}(-\lambda_2, -1)_p(-\lambda_2, P)_p,$$

since $P^{m-1}Q^{m(n-1)}$ is a perfect square.

Comparing (9.65) and (9.7)

$$C_p(X) = C_p(NN') = \{C_p(Y)\}^m(PQ^{n-1}, -1)_p^{m(m-1)/2}(-1, -1)_p^{m-1}(-\lambda_2, P)_p.$$

Substituting from (9.6),

$$C_p(NN') = (-1, -1)_p(Q, -1)_p^{m(n-1)(m+n-1)/2}(P, -1)_p^{m(m-1)/2} \tag{9.8}$$
$$\cdot(PQ, \lambda_1 - \lambda_2)_p^m(P, Q)_p^{mn}(P, -\lambda_2)_p.$$

Now

$$(P, -1)_p^{m-1}(P, Q)_p^{m(n-1)} = (P, P^{m-1})_p(P, Q^{m(n-1)})_p \tag{9.85}$$
$$= (P, P^{m-1}Q^{m(n-1)})_p$$
$$= +1.$$

Also from (9.1)

$$n(\lambda_1 - \lambda_2) = P - Q.$$

Therefore,

$$(PQ, \lambda_1 - \lambda_2)_p = (PQ, P - Q)_p(PQ, n)_p$$
$$= (P, -Q)_p(P, n)_p(Q, n)_p$$

from (8.8), whence

$$(PQ, \lambda_1 - \lambda_2)_p = (P, -1)_p(P, Q)_p(P, n)_p(Q, n)_p .$$

Hence finally

$$C_p(NN') = (-1, -1)_p(Q, -1)_p^{m(n-1)(m+n-1)/2}(P, -1)_p^{m(m-1)/2} \cdot (P, n)_p^m(Q, n)_p^m(P, \lambda_2)_p . \tag{9.9}$$

We thus get

THEOREM 8. *A necessary condition for the existence of a symmetrical regular GD design with parameters $v, b, r, k, m, n, \lambda_1, \lambda_2$ satisfying* (9.0), (9.1) *and* (9.2) *is that $C_p(NN')$, given by* (9.9), *is equal to $C_p(I)$ for all primes p, where I is the unit matrix.*

Note that from (7.2), (7.4), (7.6) and (8.2),

$$C_p(I) = (-1, -1)_p = -1 \quad \text{if } p = 2 \text{ or } p_\infty, \qquad = +1 \quad \text{if } p \text{ is an odd prime.} \tag{9.35}$$

Confining ourselves only to the case when p is an odd prime, we can combine Theorems 7 and 8 into the following single theorem.

THEOREM 9. *If a symmetrical regular GD design with parameters $v, b, r, k, m, n, \lambda_1, \lambda_2$ satisfying* (9.0), (9.1) *and* (9.2) *exists then*

(a) *if m is even P must be a perfect square and if further $m = 4t + 2$ and n is even $(Q, -1)_p = +1$ for all odd primes p;*

(b) *if m is odd and n is even Q is a perfect square, and $((-1)^\alpha n\lambda_2, P)_p = +1$ for all odd primes p, where $\alpha = m(m - 1)/2$;*

(c) *if m and n are both odd $((-1)^\alpha n\lambda_2, P)_p((-1)^\beta n, Q)_p = +1$ for all odd primes p, where $\alpha = m(m - 1)/2$ and $\beta = n(n - 1)/2$.*

PROOF.

(a) If m is even then in $P^{m-1}Q^{m(n-1)}$ the index of Q is even and the index of P is odd. Hence from Theorem 7, P must be a perfect square. Noting that $(-1, -1)_p = +1$ for all odd primes p, $C_p(NN')$ reduces to

$$(Q, -1)_p^{m(n-1)(m+n-1)/2}.$$

If now m is of the form $4t + 2$ and n is even, then the index of $(Q, -1)_p$ is odd. Theorem 8 shows that $(Q, -1)_p = +1$.

(b) If m is odd and n is even then in $P^{m-1}Q^{m(n-1)}$ the index of P is even and the index of Q is odd. Hence from Theorem 7, Q is a perfect square. Also $C_p(NN')$ reduces to $((-1)^\alpha n\lambda_2, P)_p$ for all odd primes p, if $\alpha = m(m - 1)/2$. The result follows from Theorem 8.

(c) The result follows from Theorem 8 by noting that if m and n are both odd then for all odd primes p

$$\begin{aligned} C_p(NN') &= (n\lambda_2, P)_p(n, Q)_p && \text{if } m = 4t+1, && n = 4t+1; \\ &= (-n\lambda_2, P)_p(n, Q)_p && \text{if } m = 4t+3, && n = 4t+1; \\ &= (n\lambda_2, P)_p(-n, Q)_p && \text{if } m = 4t+1, && n = 4t+3; \\ &= (-n\lambda_2, P)_p(-n, Q)_p && \text{if } m = 4t+3, && n = 4t+3. \end{aligned}$$

We give below a table of some symmetrical regular GD designs whose impossibility can be proved by using Theorem 9. The last column gives the reason for impossibility. Thus for the design "Ref. no. 1," the statement $P = 5$, Th. 9(a) in the last column means that $P = 5$ contradicts Theorem 9(a). Hence the design is impossible.

Some impossible symmetrical regular GD designs

Ref. no.	$v=b$	$r=k$	m	n	λ_1	λ_2	Reason for impossibility
1	44	7	22	2	0	1	$P = 5$, Th. 9(a)
2	92	10	46	2	0	1	$P = 8$, Th. 9(a)
3	56	8	28	2	2	1	$P = 8$, Th. 9(a)
4	88	10	22	4	2	1	$P = 12$, Th. 9(a)
5	20	5	10	2	2	1	$P = 5$, Th. 9(a)
6	18	7	2	9	3	2	$P = 13$, Th. 9(a)
7	30	9	2	15	3	2	$P = 21$, Th. 9(a)
8	22	5	11	2	0	1	$Q = 5$, Th. 9(b)
9	42	7	21	2	2	1	$Q = 5$, Th. 9(b)
10	90	10	45	2	2	1	$Q = 8$, Th. 9(b)
11	30	8	15	2	0	2	$Q = 8$, Th. 9(b)
12	33	6	11	3	0	1	$(-n\lambda_2, P)_3(-n, Q)_3 = -1$, Th. 9(c)
13	93	10	31	3	0	1	$(-n\lambda_2, P)_5(-n, Q)_5 = -1$, Th. 9(c)
14	95	10	19	5	0	1	$(-n\lambda_2, P)_5(n, Q)_5 = -1$, Th. 9(c)
15	39	7	13	3	3	1	$(n\lambda_2, P)_5(-n, Q)_5 = -1$, Th. 9(c)
16	27	7	3	9	3	1	$(-n\lambda_2, P)_{11}(n, Q)_{11} = -1$, Th. 9(c)
17	69	9	23	3	3	1	$(-n\lambda_2, P)_3(-n, Q)_3 = -1$, Th. 9(c)
18	65	9	13	5	3	1	$(n\lambda_2, P)_5(n, Q)_5 = -1$, Th. 9(c)
19	15	6	5	3	3	2	$(n\lambda_2, P)_3(-n, Q)_3 = -1$, Th. 9(c)
20	35	9	7	5	3	2	$(-n\lambda_2, P)_3(n, Q)_3 = -1$, Th. 9(c)
21	33	10	11	3	0	3	$(-n\lambda_2, P)_5(-n, Q)_5 = -1$, Th. 9(c)

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