ON THE STOCHASTIC MATRICES ASSOCIATED WITH CERTAIN QUEUING PROCESSES

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1. Summary and introduction. We shall be concerned with an irreducible Markov chain, which we shall call "the system." For simplicity we shall assume that the system is aperiodic, but this is not essential. The reader is referred to [1] for explanations of the terminology used. We first state some general theorems which provide criteria for determining whether the system is *transient*, *recurrent-null* or *ergodic* (*recurrent-nonnull*). These are then applied to the Markov chains associated with certain queuing processes recently studied by D. G. Kendall [4], [5]; most of the results have already been obtained by Kendall using direct methods, and the main purpose of the present paper is to illustrate the application of general theorems to this type of problem.

2. Let $[p_{ij}]$ $(i, j = 0, 1, 2, \cdots)$ be the infinite stochastic matrix of the system, and denote by $[p_{ij}^{(n)}]$ its nth power.

Theorem 1. *The system is ergodic if there exists a nonnull solution of the equations*

$$\sum_{i=0}^{\infty} x_i p_{ij} = x_j \qquad (j = 0, 1, 2, \cdots) \tag{1}$$

such that $\sum | x_i | < \infty$; *and only if this property is possessed by any nonnegative solution of the inequalities*

$$\sum_{i=0}^{\infty} x_i p_{ij} \leqq x_j \qquad (j = 0, 1, 2, \cdots). \tag{2}$$

Proof. It is known (cf. [1]) that $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$ always exists and is independent of i; and further that *either* $\pi_j > 0$ for all j *or* $\pi_j \equiv 0$. The system is ergodic if and only if $\pi_j > 0$. For any nonnull absolutely convergent solution $\{x_i\}$ of (1)

$$\sum_{i=0}^{\infty} x_i p_{ij}^{(n)} = x_j \qquad (j = 0, 1, 2, \cdots) \tag{3}$$

for all n, and so

$$\sum_{i=0}^{\infty} x_i \pi_j = x_j \qquad (j = 0, 1, 2, \cdots). \tag{4}$$

Therefore $\pi_j > 0$ (for otherwise the solution would be null), and so the system is ergodic.

Conversely, suppose the system to be ergodic, so that $\pi_j > 0$. Let $\{x_i\}$ be any nonnegative solution of (2). Then we have also (3) and (4) with inequalities. Therefore $\sum x_i < \infty$.

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THEOREM 2. *The system is ergodic if there exists a nonnegative solution of the inequalities*

$$(5) \qquad \sum_{j=0}^{\infty} p_{ij} y_j \leqq y_i - 1, \qquad i \neq 0,$$

such that $\sum_{j=0}^{\infty} p_{0j} y_j < \infty$.

PROOF. Let $\sum_{j=0}^{\infty} p_{0j} y_j = \lambda$. Define

$$y_i^{(n+1)} \equiv \sum_{j=0}^{\infty} p_{ij}^{(n)} y_j, \qquad y_i^{(1)} \equiv y_i.$$

Then

$$y_i^{(n+2)} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} p_{ij}^{(n)} p_{jk} y_k = \sum_{j=0}^{\infty} p_{ij}^{(n)} y_j^{(2)} \leqq \lambda p_{i0}^{(n)} + \sum_{j=1}^{\infty} (y_j - 1) p_{ij}^{(n)}$$
$$\leqq (1 + \lambda) p_{i0}^{(n)} + y_i^{(n+1)} - 1.$$

It follows that if $y_i^{(n)}$ is finite, then $y_i^{(n+1)}$ is finite. But $y_i^{(1)}$ is finite for all i. Therefore $y_i^{(n)}$ is finite for all i and n. We have now a recurrence relation from which we obtain

$$y_i^{(n+2)} \leqq (1 + \lambda) \sum_{r=1}^{n} p_{i0}^{(r)} + y_i^{(2)} - n.$$

Therefore,

$$n^{-1} y_i^{(n+2)} \leqq (1 + \lambda) n^{-1} \sum_{r=1}^{n} p_{i0}^{(r)} + n^{-1} y_i^{(2)} - 1.$$

Letting $n \to \infty$, we have $0 \leqq (1 + \lambda)\pi_0 - 1$. Therefore, $\pi_0 \geqq (1 + \lambda)^{-1} > 0$, and so the system is ergodic.

The corresponding necessary condition can be given the following sharper form.

THEOREM 3. *If the system is ergodic, then the (finite) mean first-passage times,* d_j, *from the jth to the zero state satisfy the equations*

$$(6) \qquad \sum_{j=1}^{\infty} p_{ij} d_j = d_i - 1, \qquad i \neq 0,$$

and

$$\sum_{j=1}^{\infty} p_{0j} d_j < \infty.$$

(For a proof see [1], p. 335.)

In the following theorems we do not distinguish between recurrent-nonnull and recurrent-null systems.

THEOREM 4. *The system is transient if and only if there exists a bounded non-constant solution of the equations*

$$(7) \qquad \sum_{j=0}^{\infty} p_{ij} y_j = y_i, \qquad i \neq 0.$$

PROOF. The system will be recurrent if and only if there is probability unity that the zero state is eventually attained from any state $i \neq 0$. Consider therefore the modified system in which the zero state is made completely absorbing. Denote the modified stochastic matrix by $[p'_{ij}]$. [Thus $p'_{00} = 1$, $p'_{ij} \equiv p_{ij}$, $i \neq 0$.] Then we have $\lim_{n\to\infty} p'^{(n)}_{ij} = \pi'_{ij}$, where $\pi'_{ij} = 0, j \neq 0$, and π'_{i0} is the probability that the zero state is eventually attained from the ith state. Now $\sum_{j=0}^{\infty} p'_{ij}\pi'_{j0} = \pi'_{i0}$ for all i. If the original system is transient, $\pi'_{i0} < 1$ for some i, and in all cases $\pi'_{00} = 1$. Therefore, defining $y_j \equiv \pi'_{j0}$, we have a bounded nonconstant solution for (7).

Conversely, suppose a bounded nonconstant solution of (7) to exist. Since (for any constants α, β) $\{\alpha + \beta y_j\}$ is also a solution, we may suppose without loss of generality that $y_0 = 1$, $0 \leqq y_i \leqq 2$ for all i. We have

$$\sum_{j=0}^{\infty} p'_{ij} y_j = y_i \qquad \text{for all } i, \tag{8}$$

so that $\sum_{j=0}^{\infty} p'^{(n)}_{ij} y_j = y_i$ for all i, n. Therefore, letting $n \to \infty$, we obtain $\sum_{j=0}^{\infty} \pi'_{ij} y_j \leqq y_i$ for all i, and hence $\pi'_{i0} \leqq y_i$ for all i. But we must have either $y_i < 1$ or $y_i > 1$ for some i. In the former case we have $\pi'_{i0} < 1$. In the latter case by considering the solution $\{2 - y_i\}$ we reach the same conclusion.

(A different proof of essentially this theorem has been given by Feller ([1], p. 334). The above proof is included since the technique is required for the following two theorems.)

THEOREM 5. *The system is recurrent if there exists a solution* $\{y_i\}$ *of the inequalities*

$$\sum_{j=0}^{\infty} p_{ij} y_j \leqq y_i, \qquad i \neq 0, \tag{9}$$

such that $y_i \to \infty$ *as* $i \to \infty$.

PROOF. Using the same "matrix modification" technique as in Theorem 4, we have (8) with inequalities, and we may assume without loss of generality that $y_i \geqq 0$ for all i. A proof that π'_{i0} is now identically equal to unity has been given by Kendall [3].

The condition given in Theorem 5 would appear to be necessary for recurrence only under certain additional assumptions. (Cf. Foster [2].)

The following variant of Theorem 4 is sometimes useful.

THEOREM 6. *The system is transient if and only if there exists a bounded solution* $\{y_i\}$ *of the inequalities* (9) *such that* $y_i < y_0$ *for some* i.

PROOF. The proof of the necessity is as in Theorem 4. To prove the sufficiency, we have as before, assuming $y_0 = 1$, $\pi'_{i0} \leqq y_i$, so that $\pi'_{i0} < 1$ for some i.

It may be noted that since the numbering of the states is conventional, there is no particular virtue in selecting the zero state for its special rôle in the above theorems.

In the sequel we shall require the following familiar lemma from branching-process theory. (For a proof of it see e.g., [1], p. 226.)

THEOREM 7. *Given that* $\{p_n\}$ $(n = 0, 1, 2, \cdots)$ *is a probability distribution with* $p_0 > 0$, *the equation*

$$\sum_{n=0}^{\infty} z^n p_n = z$$

possesses a root ξ *in the range* $0 < \xi < 1$ *if and only if* $\sum_1^\infty np_n > 1$.

3. The queuing system *M/G/1*. (For an explanation of this labelling see [5].) The associated stochastic matrix has the form

$$[p_{ij}] \equiv \begin{bmatrix} k_0 & k_1 & k_2 & \cdots \\ k_0 & k_1 & k_2 & \cdots \\ 0 & k_0 & k_1 & \cdots \\ 0 & 0 & k_0 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{bmatrix},$$

in which $k_i > 0$ for all i. Define $\rho \equiv \sum_1^\infty nk_n$. We shall prove that the system is ergodic if and only if $\rho < 1$; and that it is recurrent if and only if $\rho \leqq 1$.

Suppose first that $\rho < 1$. Define $y_j \equiv j(1 - \rho)^{-1}$. We find that $\{y_j\}$ satisfies the conditions of Theorem 2, and so the system is ergodic.

Conversely, suppose the system to be ergodic. From the structure of the matrix it will be clear that if μ_{ij} is the mean first-passage time from the ith to the jth state then

$$\mu_{i,i-1} = \mu_{10} \qquad (i \neq 0).$$

Moreover, from the ith state the zero state can be attained only via the $(i - 1)$st state. Therefore

$$\mu_{i0} = \mu_{i,i-1} + \mu_{i-1,0}\,.$$

It follows by an induction that

$$\mu_{i0} = i\mu_{10} \qquad (i \neq 0).$$

Therefore, using Theorem 3, we have

$$\mu_{10}\sum_1^\infty p_{1j}\, j = \mu_{10} - 1.$$

Therefore

$$\mu_{10}\,\rho = \mu_{10} - 1,$$

so that

$$\rho = 1 - \mu_{10}^{-1} < 1.$$

We turn now to the problem of recurrence. If $\rho \leqq 1$, defining $y_j \equiv j$, we find that the inequalities (9) are satisfied, and $y_j \to \infty$ as $j \to \infty$. Therefore by Theorem 5 the system is recurrent. (We have here merely reproduced the method employed by Kendall [4].)

Conversely, suppose $\rho > 1$. By Theorem 7 the equation $\sum_0^\infty z^n k_n = z$ has a root ξ in the range $0 < \xi < 1$. Define $y_j \equiv \xi^j$. We find that the equations (7) are satisfied and $y_j \to 0$ as $j \to \infty$, with $y_0 = 1$. Therefore by Theorem 4, the system is transient.

4. The queuing system *GI/M/1*. The associated stochastic matrix has the form

$$[p_{ij}] \equiv \begin{bmatrix} \alpha_0 & a_0 & 0 & 0 & \cdots \\ \alpha_1 & a_1 & a_0 & 0 & \cdots \\ \alpha_2 & a_2 & a_1 & a_0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{bmatrix},$$

where the elements a_i, α_i $[\equiv \sum_{i+1}^\infty a_j]$ are all positive and $\sum a_n = 1$.

(The more general system *GI/M/s* studied in [5] would necessitate only trivial alterations to the treatment given below.) Define ρ by $\rho^{-1} \equiv \sum_1^\infty n a_n$. We shall prove that the system is ergodic if and only if $\rho < 1$; and that it is recurrent if and only if $\rho \leqq 1$.

Suppose first that $\rho < 1$. By Theorem 7, the equation

$$\sum_0^\infty z^n a_n = z \tag{10}$$

has a root ξ in the range $0 < \xi < 1$. Define $x_i \equiv \xi^i$. We find that equations (1) are satisfied and we have $\sum x_i < \infty$. Therefore the system is ergodic. On the other hand, if $\rho \geqq 1$, $x_i \equiv 1$ is a solution for (2). But in this case $\sum x_i$ is infinite, and so by Theorem 1 the system cannot be ergodic. (The sufficiency only of the condition $\rho < 1$ was previously proved by Kendall [5] using this method.)

We turn now to the problem of recurrence and consider the possible solutions of the equations (7). We may without loss of generality suppose $y_0 = 0$. Define

$$Y(z) \equiv \sum_1^\infty y_n z^n,$$

$$A(z) \equiv \sum_0^\infty a_n z^n.$$

whenever these power-series converge. We find that

$$Y(z) = a_0 y_1 z \{A(z) - z\}^{-1},$$

y_1 being arbitrary. By Theorem 7, if $\rho < 1$ the right-hand side has a singularity at some point $z = \xi$, $0 < \xi < 1$. Therefore the sequence $\{y_n\}$ cannot be bounded and nonconstant, and so, by Theorem 4, the system is recurrent.

We now consider $\rho \geqq 1$. Write

$$A(z) - z = (1 - z)\left\{1 - \frac{1 - A(z)}{1 - z}\right\}.$$

Following Kendall ([4], p. 159), we have

$$\frac{1 - A(z)}{1 - z} = \sum_{n=0}^{\infty} z^n \sum_{i=1}^{\infty} a_{n+1}.$$

Therefore for $|z| < 1$,

$$\left|\frac{1 - A(z)}{1 - z}\right| < 1/\rho \leq 1,$$

and so $|A(z) - z| \neq 0$. It follows that the power-series expansion,

$$(1 - z)\left\{A(z) - z\right\}^{-1} = \sum_0^{\infty} b_n z^n$$

is valid for $|z| < 1$. It will further be observed that the coefficients b_n are nonnegative, and so, by Abel's theorem,

$$\sum_0^{\infty} b_n = \rho/(\rho - 1).$$

Now

$$Y(z) = a_0 y_1 z \left\{\sum_0^{\infty} z^n\right\}\left\{\sum_0^{\infty} b_n z^n\right\}$$

$$= a_0 y_1 \sum_{n=0}^{\infty} z^{n+1} \sum_{i=0}^{n} b_i.$$

Therefore $y_{n+1} = a_0 y_1 \sum_0^n b_i$. Therefore the sequence $\{y_n\}$ is bounded and nonconstant if and only if $\rho > 1$. Therefore, by Theorem 4, the system is recurrent if and only if $\rho \leqq 1$.

I am indebted to Mr. Kendall for showing me a copy of his paper [5] prior to publication, and for valuable criticism in preparing the present paper.

REFERENCES

[1] William Feller, *An Introduction to Probability Theory and its Applications*, John Wiley and Sons, 1950.

[2] F. G. Foster, "On Markov chains with an enumerable infinity of states," *Proc. Cambridge Philos. Soc.*, Vol. 48 (1952), pp. 587-591.

[3] D. G. Kendall, "On non-dissipative Markoff chains with an enumerable infinity of states," *Proc. Cambridge Philos. Soc.*, Vol. 47 (1951), pp. 633-634.

[4] D. G. Kendall, "Some problems in the theory of queues," *J. Roy. Stat. Soc.*, (B), Vol. 13 (1951), pp. 151-185.

[5] D. G. Kendall, "Stochastic processes occurring in the theory of queues and their analysis by the method of the imbedded Markov chain," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 338–354.