NOTE ON THE DISTRIBUTION OF A DEFINITE QUADRATIC FORM[1, 2, 3]

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1. Summary. An expression is derived for the distribution of a definite quadratic form in independent $N(0, 1)$ variates which depends only on the value of the determinant of the form and on the moments of a quadratic form whose matrix is the inverse of the original quadratic form. This expression is an alternating series which converges absolutely and is such that if we stop with any even power of the series we have an upper bound and if we stop with any odd power of the series a lower bound to the cumulative distribution function. The result given in this note seems to be in several ways an improvement over the method given in Robbins [2].

2. Notation. All vectors are column vectors and primes indicate their transposes. Thus $X' = [X_1, \cdots, X_n]$. In addition $d\underline{x}_n$ stands for $dx_1 \cdots dx_n$, while $|A|$ stands for the product $a_1 \cdots a_n$ where $a_1, \cdots, a_n$ are the latent roots of the matrix A, and $E(Q_n)^k$ stands for the kth moment of Q_n.

3. The problem. Suppose we have a quadratic form $Q_n = \frac{1}{2}Y'AY$ in $Y_1, \cdots, Y_n$ where the Y_i are independent $N(0, 1)$ variates, and where A denotes a definite $n \times n$ matrix. It is well known that we can make an orthogonal transformation reducing Q_n to its canonical form, that is $Q_n = \frac{1}{2}\sum a_i X_i^2$ where the elements $a_1, \cdots, a_n$ are the latent roots of the matrix A. Under such a transformation, $X_1, \cdots, X_n$ remain independent $N(0, 1)$. Let $F_n(t) = \Pr(Q_n \leqq t)$, then the problem is to find $F_n(t)$.

4. The solution.

Theorem. *Let* $Q_n = \frac{1}{2}\sum_1^n a_i X_i^2$, *where the* X_i *are independent* $N(0, 1)$ *variates and where* $a_i > 0$ *for* $i = 1, 2, \cdots, n$. *Let* $Q_n^* = \frac{1}{2}\sum_1^n a_i^{-1} X_i^2$. *Then*

(a) $$F_n(t) = \frac{t^{n/2}}{|A|^{1/2}} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \frac{E(Q_n^*)^k}{\Gamma(\frac{1}{2}n + k + 1)};$$

(b) *the series in* (a) *is absolutely convergent*;

(c) *for any two nonnegative integers* r *and* s *and every* $t > 0$,

$$S_{2s} = \sum_{k=0}^{2s} d_k > F_n(t) > \sum_{k=0}^{2r+1} d_k = S_{2r+1},$$

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[2] Part of a Ph.D. thesis presented to the Department of Mathematical Statistics, University of North Carolina.

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where

$$d_k = \frac{t^{n/2}}{|A|^{1/2}} \frac{(-t)^k}{k!} \frac{E(Q_n^*)^k}{\Gamma(\frac{1}{2}n + k + 1)}.$$

PROOF. Let $[R]$ represent the region where $\frac{1}{2}\sum_1^n a_i x_i^2 \leqq t$. Then

$$F_n(t) = (2\pi)^{-n/2} \int_{[R]} \cdots \int \exp\left[-\tfrac{1}{2}\sum_1^n x_i^2\right] d\underline{x}_n. \tag{1}$$

Expanding the exponential in the integrand, we get

$$F_n(t) = (2\pi)^{-n/2} \int_{[R]} \cdots \int \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\tfrac{1}{2}\sum_1^n x_i^2\right)^k d\underline{x}_n.$$

To evaluate integrals like $\int_{[R]} \cdots \int (\sum_1^n x_i^2)^k \, d\underline{x}_n$, we expand the integrand according to the multinomial theorem and get

$$\sum_{i_1+\cdots+i_n=k} \frac{k!}{i_1! \cdots i_n!} \int_{[R]} \cdots \int \prod_{j=1}^n x_j^{2i_j} \, dx_j.$$

We shall make use of the Dirichlet integral

$$\int \cdots \int \prod_{j=1}^n x_j^{l_j-1} \, d\underline{x}_j = \prod_{j=1}^n \Gamma\left(\frac{l_j}{p_j}\right) c_j^{l_j} \Big/ \Gamma\left[\sum_1^n (l_j/p_j) + 1\right], \quad -\infty < x_i < \infty,$$

where l_j, c_j, and p_j are all positive and the integration is over the region where $\sum_1^n (x_i/c_i)^{p_i} \leqq 1$. (See Edwards [1].) Putting

$$c_j = (2t/a_j)^{1/2}, \qquad p_j = 2, \qquad l_j = 2i_j + 1,$$

we find that

$$\int_{[R]} \cdots \int \left(\sum_1^n x_i^2\right)^k \underline{dx}_n = \frac{k!(2t)^{k+n/2}}{|A|^{1/2}\Gamma(\frac{1}{2}n + k + 1)} \sum_{i_1+\cdots+i_n=k} \frac{\Gamma(i_1 + \frac{1}{2}) \cdots \Gamma(i_n + \frac{1}{2})}{i_1! \cdots i_n! \, a_1^{i_1} \cdots a_n^{i_n}}.$$

The problem now is to evaluate this last expression. Recalling that if X_i is $N(0, 1)$, then $E(\frac{1}{2}X_i^2)^k = \Gamma(k + \frac{1}{2}) / \Gamma(\frac{1}{2})$, we find that

$$\begin{aligned} E(Q_n^*)^k &= E\left(\tfrac{1}{2}\sum_1^n a_i^{-1} X_i^2\right)^k \\ &= 2^{-k} E \sum_{i_1+\cdots+i_n=k} \frac{k!}{i_1! \cdots i_n!} \frac{X_1^{2i_1} \cdots X_n^{2i_n}}{a_1^{i_1} \cdots a_n^{i_n}} \\ &= \pi^{-n/2} k! \sum_{i_1+\cdots+i_n=k} \frac{\Gamma(i_1 + \frac{1}{2}) \cdots \Gamma(i_n + \frac{1}{2})}{i_1! \cdots i_n! \, a_1^{i_1} \cdots a_n^{i_n}}. \end{aligned}$$

Consequently

$$\int_{[R]}\cdots\int\left(\sum_1^n x_i^2\right)^k d\underline{x}_n = \frac{\pi^{n/2}(2t)^{k+n/2}}{|A|^{1/2}\Gamma(\frac{1}{2}n+k+1)}E(Q_n^*)^k, \tag{2}$$

$$F_n(t) = \frac{t^{n/2}}{|A|^{1/2}}\sum_{k=0}^{\infty}\frac{(-t)^k}{k!}\,\frac{E(Q_n^*)^k}{\Gamma(\frac{1}{2}n+k+1)}.$$

This proves part (a) of the theorem.

To show absolute convergence for part (b) we note that if $a = \min a_i$, then

$$Q_n^* \leqq \frac{1}{2a}\sum_1^n X_i^2, \qquad E(Q_n^*)^k \leqq \frac{\Gamma(\frac{1}{2}n+k)}{a^k\Gamma(\frac{1}{2}n)}.$$

Thus if $F_n^+(t)$ is the sum of the absolute values of the terms of the series for $F_n(t)$, then

$$F_n^+(t) \leqq \frac{t^{n/2}}{|A|^{1/2}}\sum_{k=0}^{\infty}\frac{t^k}{k!}\,\frac{\Gamma(\frac{1}{2}n+k)}{a^k\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}n+k+1)}$$

$$\leqq \frac{t^{n/2}}{|A|^{1/2}\Gamma(\frac{1}{2}n+1)}e^{t/a} < \infty, \qquad |t| < \infty.$$

This proves part (b) of the theorem.

The bounds of part (c) are based on the fact that if r and s are any two non-negative integers, then, for $t > 0$ the Taylor expansion gives

$$\sum_{k=0}^{2s}\frac{(-t)^k}{k!} > e^{-t} > \sum_{k=0}^{2r+1}\frac{(-t)^k}{k!}.$$

Replacing t by $\sum x_i^2$ and using (1) and (2) then proves part (c) of the theorem.

Remarks. (i) In case some of the latent roots are zero, that is the form is positive semidefinite of rank r, say, we need only replace n by r in the theorem and in the proof. (ii) The moments of Q_n^* are easy to obtain from the cumulants of Q_n^*. The rth cumulant of Q_n^* is $\frac{1}{2}(r-1)!\sum_1^n a_i^{-r}$ for $r = 1, 2, \cdots$. (iii) Clearly $S_0, S_2, S_4, \cdots$ is a sequence of upper bounds and $S_1, S_3, S_5, \cdots$ a sequence of lower bounds for $F_n(t)$. In practice we would compute a finite number of terms and then state that $\min S_{2s} > F_n(t) > \max S_{2r+1}$. The absolute value of the error thus committed is not greater than $\min S_{2s} - \max S_{2r+1}$, where r and s are non-negative integers.

5. Application to the distribution of a sum of squares in dependent variates. Let $X_1, \cdots, X_n$ have a joint multivariate normal distribution with zero means and inverse covariance matrix A. We wish to find the distribution of $R = \frac{1}{2}X'X$. Now

$$\Pr(R \leqq t) = \frac{|A|^{1/2}}{(2\pi)^{n/2}}\int_{[\frac{1}{2}x'x\leqq t]}\cdots\int \exp\left[-\tfrac{1}{2}x'Ax\right]d\underline{x}_n.$$

We make an orthogonal transformation such that

$$\Pr(R \leqq t) = \frac{|A|^{1/2}}{(2\pi)^{n/2}}\int_{[\frac{1}{2}y'y\leqq t]}\cdots\int \exp\left[-\tfrac{1}{2}\sum_1^n a_i y_i^2\right]d\underline{y}_n,$$

where $a_1, \cdots, a_n$ are the latent roots of the matrix A. We now let $z_i^2 = a_i y_i^2$, getting

$$\Pr(R \leqq t) = (2\pi)^{-n/2} \int \cdots \int_{\mathfrak{D}} \exp[-\tfrac{1}{2}z'z]\, dz_n, \qquad \mathfrak{D} = \left[\tfrac{1}{2}\sum_1^n a_i^{-1} z_i^2 \leqq t\right]$$

$$= \Pr(Q_n^* \leqq t).$$

Thus we can make use of the theorem.

Remark. Combining the results of the theorem and the above application, it is easy to show that we could find the distribution of a definite quadratic form $X'AX$ where $X_1, \cdots, X_n$ have a multivariate normal distribution with covariance matrix B^{-1}, which distribution involves as parameters the latent roots of AB^{-1}.

We shall now state, without proof, an obvious corollary to the theorem, obtained by letting the first m_1 latent roots be a_1, the next m_2 latent roots be a_2, etc.

Corollary. *Let $S_r = \frac{1}{2}(a_1\chi_{m_1}^2 + \cdots + a_r\chi_{m_r}^2)$, where the $\chi_{m_i}^2$, for $i = 1, \cdots, r$, are independent random variables having a central chi square distribution with m_i degrees of freedom. Let*

$$S_r^* = \tfrac{1}{2}(a_1^{-1}\chi_{m_1}^2 + \cdots + a_r^{-1}\chi_{m_r}^2),$$

$$M = \sum_1^r m_i, \qquad a_i > 0, \qquad G_r(t) = \Pr(S_r \leqq t).$$

Then

(a) $$G_r(t) = \frac{t^{M/2}}{(a_1^{m_1} \cdots a_r^{m_r})^{1/2}} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \frac{E(S_r^*)^k}{\Gamma(\frac{1}{2}M + k + 1)};$$

(b) *The series in* (a) *is absolutely convergent*;

(c) *For any two nonnegative integers s and j and every $t > 0$,*

$$\sum_{k=0}^{2s} d_k > G_r(t) > \sum_{k=0}^{2j+1} d_k, \qquad d_k = \frac{t^{M/2}}{(a_1^{m_1} \cdots a_r^{m_r})^{1/2}} \frac{(-t)^k}{k!} \frac{E(S_r^*)^k}{\Gamma(\frac{1}{2}M + k + 1)}.$$

Remark. The moments of S_r^* are easy to compute from the cumulants. The jth cumulant of S_r^* is $\frac{1}{2}(j-1)!\sum_{i=1}^r m_i a_i^{-j}$.

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