TOLERANCE REGIONS

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1. Summary. In this paper definitions are given for three types of tolerance regions. For distribution-free tolerance regions, an analytic condition is derived for the characteristic function of the region. Examples of the application of the condition are considered. For β-expectation tolerance regions, a criterion for a good tolerance region is introduced, and it is shown that the problem of finding such a tolerance region can be reduced to that of finding a good test for an equivalent hypothesis-testing problem. Best tolerance regions are obtained for a number of single variate and multivariate problems involving normal distributions.

2. Introduction. Let $\mathfrak{X}$ ($\mathfrak{A}$) be a measurable space and $\{P_X^\theta \mid \theta \varepsilon \Omega\}$ be a class of probability measures defined over $\mathfrak{X}(\mathfrak{A})$. For the theory in this paper we assume that an experiment corresponds to a sample of n from a component experiment. Hence our sample space is $\mathfrak{W} = \mathfrak{X}^n$, and the probability measures are the nth power product of the measures $\{P_X^\theta \mid \theta \varepsilon \Omega\}$. We designate these measures by $\{P_W^\theta \mid \theta \varepsilon \Omega\}$.

A statistical tolerance region is a mapping from the sample space $\mathfrak{W}$ to the space of subsets $\mathfrak{A}$ of the component space.

Definition 2.1. A statistical tolerance region, $S(x_1, \cdots, x_n)$, is a statistic defined over $\mathfrak{W} = \mathfrak{X}^n$ and taking values in the σ-algebra $\mathfrak{A}$.

In application the statistician calculates from the outcome $(x_1, \cdots, x_n)$ a region $S(x_1, \cdots, x_n)$ in the space $\mathfrak{X}$ which is being sampled, and then makes some probability or expectation statement about the probability measure of this set.

We first consider distribution-free tolerance regions. Heretofore the term "nonparametric" has generally been applied to these regions, but in accordance with the use of the term "distribution free" in other branches of statistics, and because these regions can also be considered for parametric problems, we prefer the term distribution free.

Definition 2.2. $S(x_1, \cdots, x_n)$ is a distribution-free tolerance region for $\{P_X^\theta \mid \theta \varepsilon \Omega\}$ if the induced probability distribution of

$$P_X^\theta(S(x_1, \cdots, x_n))$$

corresponding to the measure P_W^θ over $\mathfrak{X}^n$ is independent of the parameter $\theta \varepsilon \Omega$.

Because the probability measure or coverage of a distribution-free tolerance region has a "known" distribution independent of the "unknown" parameter,

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the statistician is able to make a probability statement about the coverage of the region.

The next definition is proposed more with a view toward the immediate requirements of a statistician.

DEFINITION 2.3. $S(x_1, \cdots, x_n)$ is a β-content tolerance region at confidence level C if

$$\Pr_\theta\{P_X^\theta(S(X_1, \cdots, X_n)) \geqq \beta\} \geqq C$$

for all $\theta \varepsilon \Omega$.

For such a region, the statistician has confidence C that the probability content of the region $S(x_1, \cdots, x_n)$ is at least β, regardless of the measure being sampled. Of course in some situations he may prefer that $S(x_1, \cdots, x_n)$ satisfy the relation

$$\Pr_\theta\{\beta_1 \leqq P_X^\theta(S(X_1, \cdots, X_n)) \leqq \beta_2\} \geqq C$$

for all $\theta \varepsilon \Omega$.

The next type of tolerance region has had perhaps less attention from the applied statistician than it deserves.

DEFINITION 2.4. $S(x_1, \cdots, x_n)$ is a β-expectation tolerance region if

$$E_W^\theta\{P_X^\theta(S(X_1, \cdots, X_n))\} \leqq \beta \qquad \text{for all } \theta \varepsilon \Omega.$$

For such a region the average probability content of the region is at most β.

In hypothesis testing the reduction to similarity is sometimes helpful for finding a whole class of tests in convenient form. For tolerance regions, we therefore propose the following definition:

DEFINITION 2.5. $S(x_1, \cdots, x_n)$ is a similar β-expectation tolerance region if

$$E_W^\theta\{P_X^\theta(S(X_1, \cdots, X_n))\} = \beta$$

for all $\theta \varepsilon \Omega$.

A similar β-expectation region can also be viewed as a β-confidence region for a future observation from the distribution being sampled. For by noting that $P_X^\theta\ (S(x_1, \cdots, x_n))$ is the probability that another observation falls in S given $x_1, \cdots, x_n$, we see that the left-hand side of the expression in Definition 2.5 is the marginal probability of such an event. This probability is equal to β; hence there is β confidence that the future observation falls in S.

3. Distribution-free tolerance regions. For distribution-free tolerance regions we are able to give a necessary and sufficient analytic condition. To do this we need the definition of a characteristic function, $\varphi_y(x_1, \cdots, x_n)$, of a region $S(x_1, \cdots, x_n)$:

$$\begin{aligned} \varphi_y(x_1, \cdots, x_n) &= 1 \qquad \text{if } y \varepsilon S(x_1, \cdots, x_n) \\ &= 0 \qquad \text{if } y \not\varepsilon S(x_1, \cdots, x_n) \end{aligned} \tag{3.1}$$

where $y \varepsilon \mathfrak{X}$. Then it is easily seen that

$$P_X^\theta(S(x_1, \cdots, x_n)) = E_Y^\theta\{\varphi_Y(x_1, \cdots, x_n)\}$$

where the expectation applies to the random variable Y with probability measure P_X^θ.

THEOREM 3.1. *A necessary and sufficient condition that $S(x_1, \cdots, x_n)$ be a distribution-free tolerance region is that there exist a sequence of real numbers $\alpha_1, \alpha_2, \cdots$ such that*

$$\varphi_{y_1}(x_1, \cdots, x_n) - \alpha_1, \qquad \varphi_{y_1}(x_1, \cdots, x_n)\varphi_{y_2}(x_1, \cdots, x_n) - \alpha_2, \cdots$$

are respectively unbiased estimates of zero over $\mathfrak{X}^{n+1}, \mathfrak{X}^{n+2}, \cdots$ for the power product measures of $\{P_X^\theta \mid \theta \varepsilon \Omega\}$. The sequence $\alpha_1, \alpha_2, \cdots$ is the moment sequence for the distribution of $V = P_X^\theta(S(X_1, \cdots, X_n))$, where the X_i have measure P_X^θ.

PROOF. A distribution-free tolerance region has the distribution function, say $F_\theta(v)$, independent of θ. Now, since a distribution function on a bounded interval is uniquely determined by the corresponding moment sequence and conversely (see [1]), it is equivalent to state that the moment sequence for $F_\theta(v)$ is independent of θ.

Letting α_r be the rth moment of $F_\theta(v)$, then

$$\begin{aligned}
\alpha_r &= \int_0^1 v^r \, dF_\theta(v) \\
&= \int_{\mathfrak{X}^n} [P_X^\theta(S(x_1, \cdots, x_n))]^r \prod_{i=1}^n dP_X^\theta(x_i) \\
&= \int_{\mathfrak{X}^n} [E_Y^\theta\{\varphi_Y(x_1, \cdots, x_n)\}]^r \prod_{i=1}^n dP_X^\theta(x_i) \\
&= \int_{\mathfrak{X}^n} \left[\int_{\mathfrak{X}} \varphi_y(x_1, \cdots, x_n) \, dP_X^\theta(y)\right]^r \prod_{i=1}^n dP_X^\theta(x_i) \\
&= \int_{\mathfrak{X}^{n+r}} \prod_{i=1}^i \varphi_{y_i}(x_1, \cdots, x_n) \prod_{j=1}^r dP_X^\theta(y_j) \prod_{i=1}^n dP_X^\theta(x_i).
\end{aligned}$$

Therefore, $\prod_{j=1}^r \varphi_{y_j}(x_1, \cdots, x_n) - \alpha_r$ is an unbiased estimate of zero over $\mathfrak{X}^{n+r}$. Thus, the statement that $F_\theta(v)$ is independent of θ is equivalent to the existence of the sequence $\alpha_1, \alpha_2, \cdots$ such that the above expression is an unbiased estimate of zero for all r.

For some theoretical developments it is convenient to have a definition of a randomized tolerance region. Let Z be a random variable whose probability measure is a measurable function of $x_1, \cdots, x_n$.

DEFINITION 3.1. $S(x_1, \cdots, x_n; z)$ is a randomized distribution-free tolerance region for $\{P_X^\theta \mid \theta \varepsilon \Omega\}$ if the induced probability distribution of $P_X^\theta(S(x_1, \cdots, x_n; z))$, corresponding to the measure P_W^θ over $\mathfrak{X}^n$ and the random variable Z for z, is independent of the parameter θ. It is assumed that $S(x_1, \cdots, x_n; z)$ is a measurable function of $(x_1, \cdots, x_n, z)$.

As for the nonrandomized case, we define a function

$$\Phi_y(x_1, \cdots, x_n; z) = 1 \quad \text{if } y \,\varepsilon\, S(x_1, \cdots, x_n; z)$$
$$= 0 \quad \text{if } y \not\varepsilon\, S(x_1, \cdots, x_n; z). \tag{3.2}$$

Taking the expectation with respect to Z, we define a related function

$$\varphi_{y_1 \cdots y_r}(x_1, \cdots, x_n) = E_Z\left\{\prod_{j=1}^{r} \Phi_{y_j}(x_1, \cdots, x_n; Z)\right\} \tag{3.3}$$

a function which is characteristic of the tolerance region. Then we have the extension of Theorem 3.1:

THEOREM 3.2. *A necessary and sufficient condition that $S(x_1, \cdots, x_n; z)$ be a distribution-free randomized tolerance region is that there exist a sequence of real numbers $\alpha_1, \alpha_2, \cdots$ such that $\varphi_{y_1}(x_1, \cdots x_n) - \alpha_1, \varphi_{y_1 y_2}(x_1, \cdots, x_n) - \alpha_2, \cdots$ are respectively unbiased estimates of zero over $\mathfrak{X}^{n+1}, \mathfrak{X}^{n+2}, \cdots$ for the power product measures of $\{P_X^\theta \mid \theta \,\varepsilon\, \Omega\}$. The sequence $\alpha_1, \alpha_2, \cdots$ is the moment sequence for the distribution of $V = P_X^\theta(S(X_1, \cdots, X_n; Z))$ where the X_i have the measure P_X^θ.*

PROOF. This follows the method of proof given for Theorem 3.1.

We give now some examples of the application of the above theorem for the nonrandomized case.

EXAMPLE 3.1. Consider sampling from an arbitrary discrete distribution on the real line. We have $W = R^n$, and the class of probability measures is $\{P_X^\theta \mid \theta \,\varepsilon\, \Omega\}$, where θ here indexes the discrete distributions on R^1. We establish that there do not exist distribution-free tolerance regions $S(x_1, \cdots, x_n)$ symmetric in the x's, other than the trivial tolerance region $S = \mathscr{I}$ or $\mathfrak{X}$.

Let $S(x_1, \cdots, x_n)$ be a distribution-free tolerance region which is symmetric in the x's. We show that either $S(x_1, \cdots, x_n) = \mathscr{I}$ or $S(x_1, \cdots, x_n) = \mathfrak{X}^n$.

If $\varphi_y(x_1, \cdots, x_n)$ is the characteristic function of $S(x_1, \cdots, x_n)$, then by Theorem 3.1 we have the existence of $\alpha_1, \alpha_2, \cdots$ such that

$$\prod_{j=1}^{r} \varphi_{y_j}(x_1, \cdots, x_n) - \alpha_r \tag{3.4}$$

is an unbiased estimate of zero over $\mathfrak{X}^{n+r}$.

For samples from $\mathfrak{X}^n$ we define a statistic called the order statistic

$$t(x_1, \cdots, x_n) = \{x_1, \cdots, x_n\}.$$

This statistic gives the values of the x's in the outcome $(x_1, \cdots, x_n)$, but not the order in which they occur in this outcome. Now it is easily shown that for the class of power product measures over $\mathfrak{X}^n$ considered here, this statistic is sufficient. Halmos [2] has shown that $t(x_1, \cdots, x_n)$ is complete for the measures above.

We have that (3.4) is an unbiased estimate of zero:

$$E_\theta\left\{\prod_{j=1}^{r} \varphi_{X_{n+j}}(X_1, \cdots, X_n) - \alpha_r\right\} = 0. \tag{3.5}$$

Since $t(x_1, \cdots, x_{n+r})$ is a sufficient statistic, the expression

$$E\left\{\prod_{j=1}^{r} \varphi_{X_{n+j}}(X_1, \cdots, X_n) - \alpha_r \,\middle|\, t(\underline{X}) = t\right\}$$

is independent of θ, that is, it is a statistic. But (3.5) can be written as

$$E_\theta\left\{E\left\{\prod_{j=1}^{i} \varphi_{X_{n+j}}(X_1, \cdots, X_n) - \alpha_r \,\middle|\, t(\underline{X}) = T\right\}\right\} = 0,$$

where the first expectation operator applies to the induced distribution of $t(x_1, \cdots, x_{n+r})$. From completeness over $\mathfrak{X}^{n+r}$, we have

$$E\left\{\prod_{j=1}^{r} \varphi_{X_{n+j}}(X_1, \cdots, X_n) - \alpha_r \,\middle|\, t(\underline{X}) = t\right\} = 0 \tag{3.6}$$

almost everywhere with respect to the induced measures of $t(x_1, \cdots, x_{n+r})$. Since the class $\{P_X^\theta \mid \theta \in \Omega\}$ is the class of all discrete distributions, almost everywhere means everywhere.

We consider (3.6) with $r = 1$. The conditional distribution given the statistic $t(x_1, \cdots, x_{n+1})$ gives equal probability to all permutations of $(x_1, \cdots, x_{n+1})$. Hence (3.6) with $r = 1$ becomes

$$\frac{1}{(n+1)!}\sum_P \varphi_{x_{i_{n+1}}}(x_{i_1}, \cdots, x_{i_n}) - \alpha_1 = 0 \tag{3.7}$$

everywhere; P designates summation with respect to all permutations $i_1, \cdots, i_{n+1}$ of $(1, \cdots, n+1)$. Since $S(x_1, \cdots, x_n)$ is symmetric in the x's, so also is $\varphi_y(x_1, \cdots, x_n)$ symmetric in $x_1, \cdots, x_n$. Therefore (3.7) becomes

$$\begin{aligned}\varphi_{x_{n+1}}(x_1, \cdots, x_n) + \varphi_{x_n}(x_1, \cdots, x_{n-1}, x_{n+1}) + \cdots + \varphi_{x_1}(x_2, \cdots, x_{n+1}) \\ = (n+1)\alpha_1,\end{aligned} \tag{3.8}$$

and (3.8) holds for all $x_1, \cdots, x_{n+1}$. Taking $x_1 = x_2 = \cdots = x_{n+1} = x$, we have

$$(n+1)\varphi_x(x, \cdots, x) = (n+1)\alpha_1.$$

The quantity $\varphi_x(x, \cdots, x)$ can be either zero or one. Hence $(n+1)\alpha_1 = 0$ or $(n+1)$; that is, $\alpha_1 = 0$ or $= 1$. Thus the first moment of a random variable restricted to the interval 0, 1 is either zero or one. Obviously, the random variable (the coverage of the tolerance region) takes the value zero or one with probability one. Because the class of measures is the class of discrete distributions, this means either that

$$S(x_1, \cdots, x_n) = \mathscr{I}$$

or that

$$S(x_1, \cdots, x_n) = \mathfrak{X}.$$

EXAMPLE (3.2) Consider sampling from an arbitrary absolutely continuous distribution on the real line. We have $\mathfrak{X} = R$ and we let θ in $\{P_X^\theta \mid \theta \in \Omega\}$ index the absolutely continuous distributions. We find the form of distribution-free upper tolerance limits in special cases. Suppose a distribution-free tolerance region $S(x_1, \cdots, x_n)$ has the form

$$S(x_1, \cdots, x_n) =]-\infty, u(x_1, \cdots, x_n)]. \tag{3.9}$$

(Intervals are open at the end where a reversed bracket appears, and closed otherwise.) Then $u(x_1, \cdots, x_n)$ is called a distribution-free upper tolerance limit. We assume that $u(x_1, \cdots, x_n)$ is symmetric in $x_1, \cdots, x_n$.

For convenience we define L_r to be the Lebesgue measure over R^r. As in Example 3.1, it can easily be shown that the order statistic is sufficient for the class of absolutely continuous distributions. It has been proved complete by Lehmann [3] and Fraser [4]. Following the argument in Example 3.1, we obtain from Theorem 3.1 with $r = 1$ that

$$\frac{1}{(n+1)} \sum_P \varphi_{x_{i_{n+1}}}(x_{i_1}, \cdots, x_{i_n}) = \alpha_1$$

almost everywhere (Lebesgue) over R^{n+1}. Since $\varphi_y(x_1, \cdots, x_n)$ is symmetric in the x's, we have

$$\varphi_{x_{n+1}}(x_1, \cdots, x_n) + \cdots + \varphi_{x_1}(x_2, \cdots, x_{n+1}) = (n+1)\alpha_1 \tag{3.10}$$

almost everywhere. Because φ is a characteristic function, $(n+1)\alpha_1$ is one of the integers $0, 1, \cdots, n, n+1$. We find the form of $u(x_1, \cdots, x_n)$ when $(n+1)\alpha_1$ is 0, 1, n, or $n+1$.

Consider the case $(n+1)\alpha_1 = 1$. We shall prove that

$$\begin{aligned} \varphi_y(x_1, \cdots, x_n) &= 1 \quad \text{if } y \leqq x_{(1)} \\ &= 0 \quad \text{if } y > x_{(1)} \end{aligned}$$

almost everywhere in R^{n+1}. In terms of $u(x_1, \cdots, x_n)$, this means that

$$u(x_1, \cdots, x_n) = x_{(1)}$$

almost everywhere in R^n. Let $x_{(1)}, x_{(2)}, \cdots, x_{(n)}$ designate the numbers $x_1, \cdots, x_n$ arranged in order of increasing magnitude: $x_{(1)} \leqq \cdots \leqq x_{(n)}$.

Suppose $\varphi_y(x_1, \cdots, x_n) = 1$ on a set of positive Lebesgue measure in the region of R^{n+1} for which $y > x_{(1)}$. From the properties of measure, it follows that there exists a positive δ such that $\varphi_y(x_1, \cdots, x_n) = 1$ on a set of positive measure in the region $y > x_{(1)} + \delta$; call this set A. Divide the space R^n of $(x_1, \cdots, x_n)$ into "cubes" having sides of length ϵ, a typical set being

$$\{(x_1, \cdots, x_n) \mid m_i \varepsilon < x_i \leqq (m_i + 1)\varepsilon; \quad (i = 1, \cdots, n)\}. \tag{3.11}$$

We let, of course, m_i for each i range over all real integers. There is a countable number of such sets. Consider the following set B:

$$B = \{(x_1, \cdots, x_n) \mid L_1\{y \mid (x_1, \cdots, x_n, y) \varepsilon A\} > 0\}.$$

From the properties of measure, there exists at least one of the above-defined cubes which intersects B on a set of positive measure. For a later purpose we require $\epsilon < \delta$.

Now by choosing ϵ sufficiently small we can ensure the existence of at least one cube which intersects B on a set of positive measure and at the same time is disjoint from each of the diagonal sets,

$$\{(x_1, \cdots, x_n) \mid x_i = x_j\}, \tag{3.12}$$

which have measure zero. Designate one of these cubes by C. We summarize the results so far. For $(x_1, \cdots, x_n) \varepsilon B \cap C$, we have $\varphi_y(x_1, \cdots, x_n) = 1$ at least for y belonging to a set of positive measure in $y > x_{(1)} + \delta$.

From the definition of $\varphi_y(x_1, \cdots, x_n)$ we know it is monotone nonincreasing in y and takes the values 0, 1. That is, if $\varphi_{y^*}(x_1, \cdots, x_n) = 1$, then $\varphi_y(x_1, \cdots, x_n) = 1$ for $y < y^*$. Now from the last statement in the above paragraph, it follows that if $(x_1, \cdots, x_n) \varepsilon B \cap C$, then $\varphi_y(x_1, \cdots, x_n) = 1$ for $y \leqq x_{(1)} + \delta$.

We now derive a contradiction to (3.10). Without loss of generality, let x_1 be the smallest of the co-ordinates for points in C (it will always be the same co-ordinate because C does not intersect the diagonal sets (3.12.)). Consider the set D in R^{n+1}:

$$D = \{(x_1, \cdots, x_{n+1}) \mid (x_1, \cdots, x_n) \varepsilon C \cap B \quad ; \quad (x_{n+1}, x_2, \cdots, x_n) \epsilon C \cap B\}.$$

From the first condition defining D, we have $\varphi_{x_{n+1}}(x_1, \cdots, x_n) = 1$ for $x_{n+1} \leqq x_1 + \delta$. From the second condition defining D, we have $\varphi_{x_1}(x_{n+1}, x_2, \cdots, x_n) = 1$ for $x_1 \leqq x_{n+1} + \delta$. But for $(x_1, \cdots, x_{n+1}) \varepsilon D$ we have x_1 and x_{n+1} both as possible first co-ordinates for a point in C; hence $|x_1 - x_{n+1}| < \varepsilon$. Therefore if $(x_1, \cdots, x_{n+1}) \varepsilon D$, $\varphi_{x_{n+1}}(x_1, \cdots, x_n) = 1 = \varphi_{x_1}(x_{n+1}, x_2, \cdots, x_n)$, since the two conditions above are fulfilled by reason of our choice of $\epsilon < \delta$. For $(x_1, \cdots, x_{n+1}) \varepsilon D$ we have the left-hand side of (3.10) equal to at least 2, while the right-hand side of (3.10) by assumption was 1. This is a contradiction if we show D has positive measure.

Let L_n designate Lebesgue measure in R^n and let $\psi(x_1, \cdots, x_n)$ be the characteristic function of $B \cap C$ in R^n.

$$\begin{aligned} L_{n+1}(D) &= \int_{R^{n+1}} \psi(x_1, \cdots, x_n)\psi(x_{n+1}, x_2, \cdots, x_n) \prod_{i=1}^{n+1} dL_1(x_i) \\ &= \int_{R^{n-1}} \int_R \psi(x_1, \cdots, x_n)\, dL_1(x_1) \int_R \psi(x_{x+1}, \cdots, x_n)\, dL_1 \\ &\quad \times (x_{n+1}) \prod_{j=2}^{n} dL_1(x_j) \\ &= \int_{R^{n-1}} M^2(x_2, \cdots, x_n) \prod_{j=2}^{n} dL_1(x_j). \end{aligned} \tag{3.13}$$

But

$$(3.14)\qquad \begin{aligned} L_n(B \cap C) &= \int_{R^n} \psi(x_1, \cdots, x_n) \prod_{i=1}^{n} dL_1(x_i) \\ &= \int_{R^{n-1}} M(x_2, \cdots, x_n) \prod_{i=2}^{n} dL_1(x_i). \end{aligned}$$

Since $L_n\ (B \cap C) > 0$ by construction of $B \cap C$, then by comparing (3.13) and (3.14) we obtain $L_{n+1}(D) > 0$. This is the contradiction we worked toward. Therefore our assumption that $\varphi_y(x_1, \cdots, x_n) = 1$ on a set of positive measure in the region of R^{n+1} for which $y > x_{(1)}$ was false.

We have that $\varphi_y(x_1, \cdots, x_n) = 1$ on a set of positive measure only if that set is in $y \leqq x_{(1)}$. Thus for $x_{i_1} < \cdots < x_{i_{n+1}}$ there is only one term of (3.10) which can be 1 on a set of positive measure. However the right-hand side of (3.10) by assumption was 1 almost everywhere over $x_{i_1} < \cdots < x_{i_{n+1}}$. Therefore $\varphi_{x_{i_1}}(x_{i_2}, \cdots, x_{i_{n+1}}) = 1$ almost everywhere when $x_{i_1} < \cdots < x_{i_{n+1}}$. That is,

$$\begin{aligned} \varphi_y(x_1, \cdots, x_n) &= 1 \qquad \text{if } y \leqq x_{(1)} \\ &= 0 \qquad \text{if } y > x_{(1)}, \end{aligned}$$

and the distribution-free upper tolerance bound $u(x_1, \cdots, x_n)$ equals $x_{(1)}$.

Similarly if $(n + 1)\alpha_1 = n$, then $u(x_1, \cdots, x_n) = x_{(n)}$, and by an almost trivial argument $u(x_1, \cdots, x_n) = -\infty, +\infty$ according as $(n + 1)\alpha_1 = 0$, $n + 1$.

4. β-Content tolerance regions. Any tolerance region satisfying Definition 2.1 will produce a β-content tolerance region for suitably chosen C; for example,

$$C = \inf_{\theta \varepsilon \Omega} \Pr_\theta\{P_X^\theta(S(X_1, \cdots, X_n)) \geqq \beta\}.$$

Also, a distribution-free tolerance region will produce a β-content tolerance region with a property of similarity. For if $S(x_1, \cdots, x_n)$ satisfies Definition 2.2, then, letting C equal the expression

$$\Pr_\theta\{P_X^\theta(S(X_1, \cdots, X_n)) \geqq \beta\},$$

which does not depend on θ, we have a similar β-content tolerance region given by

$$\Pr_\theta\{P_X^\theta(S(X_1, \cdots, X_n)) \geqq \beta\} = C.$$

5. β-expectation tolerance regions. First we prove some general properties of β-expectation tolerance regions. In Section 3 we defined by Formula 3.1 the characteristic function $\varphi_y(x_1, \cdots, x_n)$ of a nonrandomized tolerance region, and by Formula 3.3 with $r = 1$ we defined a characteristic function $\varphi_y(x_1, \cdots, x_n)$ of a randomized tolerance region. As a converse, we have the

THEOREM 5.1. *If $\varphi_y(x_1, \cdots, x_n)$ is a measurable function with $0 \leqq \varphi_y(x_1, \cdots, x_n) \leqq 1$, then there exists a tolerance region $S(x_1, \cdots, x_n)$ having $\varphi_y(x_1, \cdots, x_n)$ as its characteristic function.*

PROOF. Let Z be a random variable which has the uniform distribution on [0, 1] and define a randomized tolerance region by

$$S'(x_1, \cdots, x_n; z) = \{y \mid \varphi_y(x_1, \cdots, x_n) \geqq z\}.$$

Now we calculate the characteristic function of $S'(x_1, \cdots, x_n; z)$ and using (3.2) obtain

$$\begin{aligned}\varphi'(x_1, \cdots, x_n) &= E_Z\{\Phi'_y(x_1, \cdots, x_n; Z)\} \\ &= \Pr_Z\{\varphi_y(x_1, \cdots, x_n) \geqq Z\} \\ &= \varphi_y(x_1, \cdots, x_n).\end{aligned}$$

This proves the theorem.

We also state a theorem on similar β-expectation tolerance regions.

THEOREM 5.2. *A necessary and sufficient condition that $S(x_1, \cdots, x_n; z)$ be a similar β-expectation tolerance region is that $\varphi_y(x_1, \cdots, x_n) - \beta$ be an unbiased estimate of zero for the power product measure of P_X^θ over $\mathfrak{X}^{n+1}$.*

PROOF. Let $S(x_1, \cdots, x_n)$ be a tolerance region; then the expected content is

$$E_{WZ}^\theta\{P_X^\theta(S(X_1, \cdots, X_n; Z))\}. \tag{5.1}$$

From the definition of $\varphi_y(x_1, \cdots, x_n)$ this becomes

$$E_{WY}^\theta\{\varphi_y(X_1, \cdots, X_n)\}. \tag{5.2}$$

Obviously, then, a necessary and sufficient condition that (5.1) be equal to β is that $\varphi_y(x_1, \cdots, x_n) - \beta$ be an unbiased estimate of zero.

To introduce the notion of a good tolerance region, we need a function which gives us for each θ in Ω the relative merits of sets S in $\mathfrak{A}$. Let the "desirability" of a set S when the probability measure is P_X^θ be given by a probability measure $Q_\theta(S)$ defined for all $S \varepsilon \mathfrak{A}$. Then for a tolerance region S we define the power to be

$$E_W^\theta\{Q_\theta(S(X_1, \cdots, X_n))\}; \tag{5.3}$$

it is the average value of the "desirability" of the set S and in general is a function of θ. In terms of the characteristic function of $S(x_1, \cdots, x_n)$, the β-expectation condition is

$$\int_{\mathfrak{X}^{n+1}} \varphi_y(x_1, \cdots, x_n)\, dP_X^\theta(y) \prod_{i=1}^n dP_X^\theta(x_i) \leqq \beta, \tag{5.4}$$

and the power is

$$\int_{\mathfrak{X}^{n+1}} \varphi_y(x_1, \cdots, x_n)\, dQ_\theta(y) \prod_{i=1}^n dP_X^\theta(x_i). \tag{5.5}$$

The problem of finding a good tolerance region is then to find a characteristic function satisfying the size condition (5.4) and having good properties for the power (5.5). Obviously, this is equivalent to finding a good test function $\varphi_y(x_1, \cdots, x_n)$ for the hypothesis testing problem, over $\mathfrak{X}^{n+1}$,

$$\text{(5.6)} \qquad \begin{array}{ll} \text{Hypothesis:} \ (P_X^\theta, \cdots, P_X^\theta, P_X^\theta), & \theta \ \varepsilon \ \Omega; \\ \text{Alternative:} \ (P_X^\theta, \cdots, P_X^\theta, Q_\theta), & \theta \ \varepsilon \ \Omega; \end{array}$$

$(P_X^\theta, \cdots, P_X^\theta, Q_\theta)$, for example, designates the probability measure of $(X_1, \cdots, X_n, Y)$ over $\mathfrak{X}^{n+1}$ where $X_1, \cdots, X_n, Y$ are independent, each X_i has probability measure P_X^θ, and Y has probability measure Q_θ.

For the hypothesis testing problem there may exist a uniformly most powerful test. In this case we would call the corresponding tolerance region most powerful. Failing the existence of a most powerful test, we could look for one yielding a maximum value to the minimum power over the alternative. The corresponding tolerance region we would then call minimax.

6. β-expectation tolerance regions for normal distributions.

6.1. *Univariate normal.* Consider sampling from the univariate normal distribution with density function

$$(2\pi\sigma^2)^{-1/2} \exp\left[-\frac{1}{2\sigma^2}(x - \mu)^2\right],$$

where the parameter space Ω is given by $\mu \ \varepsilon \ R^1$, $\sigma^2 \ \varepsilon \ ]0, \infty[$. If a tolerance region is desired which tends to cover the center of the distribution more than the tails, then a reasonable choice of the measure $Q_{\mu\sigma^2}(A)$ on the real line might be the normal probability measure with mean μ and variance $\alpha_1^2\sigma^2$ with $0 < \alpha_1 < 1$. This measure obviously gives more measure to sets in the neighbourhood of μ and less to sets far from μ.

We now consider the analogous hypothesis testing problem. Let $X_1, \cdots, X_n$, Y be independent and let X_i have a normal distribution with mean μ and variance σ^2 and Y have a normal distribution with mean μ, variance $\alpha^2\sigma^2$. The hypothesis testing problem is of the form

$$\text{(6.1.1)} \qquad \begin{array}{lll} \text{Hypothesis:} & \alpha = 1 & (\mu, \sigma^2) \ \varepsilon \ \Omega; \\ \text{Alternative:} & \alpha = \alpha_1 & (\mu, \sigma^2) \ \varepsilon \ \Omega. \end{array}$$

If we define $\bar{x} = n^{-1} \sum x_i$ and $s_x^2 = (n-1)^{-1} \sum (x_i - \bar{x})^2$, then it is easily seen that this problem has the sufficient statistic $(\bar{x}, s_x^2, y)$.

We now apply the invariance method to the problem expressed in terms of the sufficient statistic. Consider the group G of transformations induced by the two groups

$$\text{(6.1.2)} \qquad G_1 = \left\{ \begin{pmatrix} \bar{x}' = \bar{x} + a \\ s_x'^2 = s_x^2 \\ y' = y + a \end{pmatrix} \middle| \ a \ \varepsilon \ R^1 \right\},$$

and

$$G_2 = \left\{ \begin{pmatrix} \bar{x}' = c\bar{x} \\ s_x'^2 = c^2 s_x \\ y' = cy \end{pmatrix} \middle| \, c \, \varepsilon \,] \, 0, \infty \, [\right\}. \tag{6.3.1}$$

Obviously, G_1 is a normal subgroup of G. For the group of transformations G_1, a maximal invariant function is $((y - \bar{x}), s_x^2)$. The group G_2 induces a group on this maximal invariant, and it has maximal invariant $\Upsilon = (y - \bar{x})/s_x$. By a theorem of Hunt and Stein [6] Υ is maximal invariant for G.

In accordance with the invariance principle we look for tests based on Υ. Since the variance of $(y - \bar{x})$ is $(\alpha^2 + 1/n)\, \sigma^2$, then under the hypothesis Υ has the distribution of $(1 + 1/n)^{1/2}\, t$, and under the alternative, the distribution of $(\alpha_1^2 + 1/n)^{1/2}\, t$, where t stands for a random variable with Student's t-distribution having $(n - 1)$ degrees of freedom. In terms of Υ, the hypothesis and alternative are simple. To find the most powerful invariant test, we now apply the Neyman-Pearson fundamental lemma. *Let* $c_0 = (1 + 1/n)^{1/2}$, $c_1 = (\alpha_1^2 + 1/n)^{1/2}$ *(clearly* $c_0 > c_1$*). Then, the most powerful test function* $\varphi(\Upsilon)$ *is based on the probability ratio*

$$\frac{\dfrac{1}{c_1[(n-1)\pi]^{1/2}} \dfrac{\Gamma\left(\dfrac{n}{2}\right)}{\Gamma\left(\dfrac{n-1}{2}\right)} \left[1 + \dfrac{\Upsilon^2}{c_1^2(n-1)}\right]^{-n/2}}{\dfrac{1}{c_0[(n-1)\pi]^{1/2}} \dfrac{\Gamma\left(\dfrac{n}{2}\right)}{\Gamma\left(\dfrac{n-1}{2}\right)} \left[1 + \dfrac{\Upsilon^2}{c_0^2(n-1)}\right]^{-n/2}}$$

or equivalently on $|\Upsilon|^{-1}$. Hence the most powerful invariant test function is

$$\begin{aligned} \varphi(W) &= 1 \quad \text{if } |\Upsilon| < a_\beta \\ &= 0 \quad \text{if } |\Upsilon| > a \end{aligned} \tag{6.1.4}$$

and to give the test size β, a_β is $(1 + 1/n)^{1/2}\, t_{1-\beta/2}$, where t_α is the point exceeded with probability α using the Student t-distribution with $(n - 1)$ degrees of freedom.

Since the alternative in (6.1) is a set of the maximal invariant partition of the parameter space, the envelope power function is constant valued over the alternative. By the theorem of Hunt and Stein [5], there is, for any non-invariant test, an invariant test for which the minimum power over the maximal invariant partitions of the parameter space is no smaller. Hence our most powerful invariant test maximizes the minimum power over the alternative among size α tests. Also, it is most stringent.

From the definition of Υ and $\varphi(\Upsilon)$ we have the test

$$\varphi_y(x_1, \cdots, x_n) = 1 \quad \text{if} \left| \frac{y - \bar{x}}{s_x} \right| < a_\beta$$

$$= 0 \quad \text{if} \left| \frac{y - \bar{x}}{s_x} \right| > a_\beta .$$

Thus, the minimax and most stringent tolerance region is

$$S(x_1, \cdots, x_n) = [\bar{x} - a_\beta s_x, \bar{x} + a_\beta s_x].$$

Values of a_β are given in Table 1. It is interesting to note that the tolerance region does not depend on the value of α_1, provided it is less than 1. Also, under the hypothesis, the test statistic has a fixed distribution; hence the test and therefore the tolerance region are similar, and we have a β-expectation similar tolerance region.

If we are interested in having our tolerance region cover the left tail of the distribution, we might choose $Q_{\mu\sigma^2}$ to be the normal distribution with mean $\mu - \epsilon$ and variance $\sigma^2(\epsilon > 0)$. An analysis similar to the above shows that a minimax and most stringent tolerance region is

$$S(x_1, \cdots, x_n) =]-\infty, \bar{x} + a'_\gamma s_x]$$

where a'_γ may be found from Table 1 by using $a'_\gamma = a_{2\gamma-1}$.

If σ^2 is known, our parameter space is given by $\mu \varepsilon R^1$. Using the same Q functions with σ^2 taking the given value, an analysis similar to that above shows that for ability to pick up the center of the distribution, the minimax and most stringent tolerance region is

$$S(x_1, \cdots, x_n) = [\bar{x} - b_\beta\sigma, \bar{x} + b_\beta\sigma]$$

where $b_\beta = (1 + 1/n)^{1/2} z_{(1-\beta)/2}$ and z_α is the point exceeded with probability α using the normal distribution with mean 0 and variance 1. Values of b_β are given in Table 2. Also, the minimax and most stringent tolerance region of size γ which tends to pick up the left-hand tail of the distribution is

$$S(x_1, \cdots, x_n) =]-\infty, \bar{x} + b'_\gamma\sigma]$$

where values of b'_γ may be found from Table 2 by using $b'_\gamma = b_{2\gamma-1}$.

If μ is known, the parameter space is given by $\sigma^2 \varepsilon]0, \infty[$. Using the same Q functions as before with μ taking the given value, a minimax and most stringent size β tolerance region for picking up the center of the distribution is

$$S(x_1, \cdots, x_n) = [\mu - t'_{(1-\beta)/2}s_x, \mu + t'_{(1-\beta)/2}s_x],$$

where s_x is here defined to be $n^{-1} \sum (x_i - \mu)^2$; t'_α is the point exceeded with probability α using Student's t-distribution with n degrees of freedom. Also, the minimax and most stringent size β tolerance region which tends to pick up

TABLE 1

Tolerance factors a_β for univariate normal distributions with unknown mean, unknown variance; sample size n

n	β					
	.995	.99	.975	.95	.90	.75
2	155.9	77.96	31.17	15.56	7.733	2.957
3	16.27	11.46	7.165	4.968	3.372	1.852
4	8.333	6.530	4.669	3.558	2.631	1.591
5	6.132	5.044	3.829	3.041	2.335	1.473
6	5.156	4.355	3.417	2.777	2.176	1.405
7	4.615	3.963	3.174	2.616	2.077	1.361
8	4.274	3.712	3.014	2.508	2.010	1.330
9	4.040	3.537	2.900	2.431	1.960	1.307
10	3.870	3.408	2.816	2.373	1.923	1.290
11	3.741	3.310	2.751	2.327	1.893	1.276
12	3.639	3.233	2.699	2.291	1.869	1.264
13	3.558	3.170	2.657	2.261	1.850	1.255
14	3.491	3.118	2.621	2.236	1.833	1.246
15	3.435	3.074	2.592	2.215	1.819	1.239
16	3.387	3.037	2.567	2.197	1.807	1.234
17	3.346	3.005	2.545	2.181	1.797	1.228
18	3.311	2.978	2.525	2.168	1.787	1.224
19	3.280	2.953	2.509	2.155	1.779	1.220
20	3.252	2.932	2.494	2.145	1.772	1.216
21	3.228	2.912	2.480	2.135	1.765	1.213
22	3.206	2.895	2.468	2.126	1.759	1.210
23	3.186	2.879	2.457	2.119	1.754	1.207
24	3.168	2.865	2.447	2.111	1.749	1.205
25	3.152	2.852	2.438	2.105	1.745	1.202
26	3.137	2.840	2.430	2.099	1.741	1.200
27	3.123	2.830	2.422	2.093	1.737	1.198
28	3.111	2.820	2.415	2.088	1.733	1.197
29	3.099	2.811	2.409	2.083	1.730	1.195
30	3.088	2.802	2.403	2.079	1.727	1.193
31	3.078	2.794	2.397	2.075	1.724	1.192
41	3.007	2.737	2.357	2.046	1.704	1.181
61	2.938	2.682	2.318	2.017	1.684	1.171
121	2.872	2.628	2.279	1.988	1.665	1.161
∞	2.807	2.576	2.241	1.960	1.645	1.150

the left-hand tail is

$$S(x_1, \cdots, x_n) =]-\infty, \mu + t'_{1-\beta} s_x],$$

where t' and s_x are defined immediately above.

6.2 *Multivariate normal.* Consider sampling from a multivariate normal distribution for which the density function is

$$K \exp [-\tfrac{1}{2}(w - \mu)\Lambda(w - \mu)'].$$

TABLE 2

Tolerance factors b_β for univariate normal distributions with unknown mean, known variance; sample size n

n	β					
	.995	.99	.975	.95	.90	.75
2	3.438	3.155	2.745	2.401	2.015	1.409
3	3.241	2.974	2.588	2.263	1.899	1.328
4	3.138	2.880	2.506	2.191	1.839	1.286
5	3.075	2.822	2.455	2.147	1.802	1.260
6	3.032	2.782	2.421	2.117	1.777	1.242
7	3.001	2.754	2.396	2.095	1.758	1.230
8	2.977	2.732	2.377	2.079	1.745	1.220
9	2.959	2.715	2.363	2.066	1.734	1.213
10	2.944	2.702	2.351	2.056	1.725	1.206
11	2.932	2.690	2.341	2.047	1.718	1.201
12	2.922	2.681	2.333	2.040	1.712	1.197
13	2.913	2.673	2.326	2.034	1.707	1.194
14	2.906	2.666	2.320	2.029	1.703	1.191
15	2.899	2.660	2.315	2.024	1.699	1.188
16	2.893	2.655	2.310	2.020	1.696	1.186
17	2.888	2.650	2.306	2.017	1.693	1.184
18	2.884	2.646	2.303	2.014	1.690	1.182
19	2.880	2.643	2.300	2.011	1.688	1.180
20	2.876	2.639	2.297	2.008	1.686	1.179
21	2.873	2.636	2.294	2.006	1.684	1.177
22	2.870	2.634	2.292	2.004	1.682	1.176
23	2.867	2.631	2.290	2.002	1.680	1.175
24	2.865	2.629	2.288	2.000	1.679	1.174
25	2.863	2.627	2.286	1.999	1.677	1.173
26	2.860	2.625	2.284	1.997	1.676	1.172
27	2.859	2.623	2.283	1.996	1.675	1.171
28	2.857	2.621	2.281	1.995	1.674	1.171
29	2.855	2.620	2.280	1.994	1.673	1.170
30	2.853	2.618	2.278	1.992	1.672	1.169
31	2.852	2.617	2.277	1.991	1.671	1.169
41	2.841	2.607	2.269	1.984	1.665	1.164
61	2.830	2.597	2.260	1.976	1.658	1.160
121	2.819	2.586	2.251	1.968	1.652	1.155
∞	2.807	2.576	2.241	1.960	1.645	1.150

Let the parameter space Ω be given by $\mu \varepsilon R^k$ and Λ belonging to the space of $k \times k$ symmetric positive definite matrices. If a tolerance region is wanted which tends to cover the center of the distribution rather than the extremities, then for the parameters μ, Λ a reasonable choice for the $Q_{\mu\Lambda}(A)$ measure over R^k is the normal distribution with mean μ and covariance matrix $\alpha_1^2\Lambda^{-1}$ with $0 < \alpha_1 < 1$.

We now formulate the hypothesis testing problem which corresponds to this problem in tolerance region construction. Let $W_1, \cdots, W_n, \Xi$ be independent and let each W_i have a normal distribution μ, Λ; and let Ξ have a normal distribution $\mu, \alpha^{-2}\Lambda$. Then the problem is to find a best size β test for the problem

$$\text{Hypothesis: } \alpha = 1, \qquad (\mu, \Lambda) \in \Omega,$$

$$\text{Alternative: } \alpha = \alpha_1, \qquad (\mu, \Lambda) \in \Omega.$$

Defining $\bar{w} = n^{-1} \sum_{\alpha=1}^{n} w_\alpha$ and $A = (n - 1)^{-1} \sum (w_\alpha - w)' (w_\alpha - w)$, we have as sufficient statistic for this problem, $(\bar{w}, A, \xi)$.

TABLE 3

Tolerance factors c_β for bi-variate normal distributions with unknown means, unknown variance-covariance matrix; sample size n

n	β					
	.995	.99	.975	.95	.90	.75
3	106,667	26,664	4,264	1,064	264.0	40.00
4	746.2	371.2	146.2	71.25	33.75	11.25
5	159.4	98.61	51.34	30.57	17.48	7.295
6	76.66	52.50	31.06	20.25	12.61	5.833
7	50.23	36.41	23.13	15.87	10.37	5.082
8	38.18	28.68	19.06	13.50	9.091	4.626
9	31.50	24.25	16.61	12.03	8.273	4.320
10	27.33	21.41	15.00	11.04	7.705	4.101
11	24.50	19.45	13.85	10.32	7.288	3.936
12	22.47	18.02	13.00	9.778	6.970	3.807
13	20.94	16.93	12.35	9.357	6.719	3.705
14	19.75	16.08	11.83	9.019	6.516	3.620
15	18.81	15.40	11.41	8.743	6.348	3.550
16	18.04	14.83	11.06	8.513	6.208	3.491
17	17.39	14.36	10.76	8.318	6.088	3.440
18	16.85	13.97	10.51	8.151	5.985	3.395
19	16.39	13.62	10.30	8.006	5.895	3.356
20	15.99	13.33	10.11	7.879	5.816	3.322
21	15.64	13.07	9.941	7.768	5.747	3.292
22	15.34	12.84	9.795	7.668	5.685	3.265
23	15.07	12.64	9.663	7.580	5.629	3.240
24	14.82	12.46	9.546	7.500	5.579	3.218
25	14.61	12.29	9.440	7.427	5.533	3.198
26	14.41	12.14	9.343	7.362	5.492	3.179
27	14.23	12.01	9.256	7.302	5.454	3.162
28	14.07	11.89	9.176	7.247	5.419	3.147
29	13.92	11.78	9.102	7.197	5.387	3.133
30	13.79	11.67	9.034	7.150	5.357	3.119
31	13.66	11.58	8.971	7.107	5.330	3.107
32	13.54	11.49	8.913	7.067	5.304	3.095
42	12.73	10.87	8.502	6.783	5.122	3.013
62	11.97	10.28	8.110	6.509	4.945	2.931
122	11.26	9.732	7.736	6.246	4.773	2.851
∞	10.60	9.210	7.378	5.991	4.605	2.773

We apply the invariance method. Consider the group G_1 of transformations on the sample space $R^{k(n+1)}$:

$$G_1 = \left\{ \begin{pmatrix} w'_\alpha = w_\alpha B + \zeta(\alpha = 1, \cdots, n) \\ \xi' = \xi B + \zeta \end{pmatrix} \middle| \begin{matrix} B \text{ belongs to the class of nonsing-} \\ \text{ular } k \times \text{ matrices, and } \zeta \, \varepsilon \, R^k \end{matrix} \right\}.$$

These transformations leave the problem unchanged. The induced group on the

TABLE 4

Tolerance factors c_β for tri-variate normal distributions with unknown means, unknown variance-covariance matrix; sample size n

n	β					
	.995	.99	.975	.95	.90	.75
4	243,169	60,787	9,722	2,427	602.9	92.25
5	1,434	714.0	282.0	138.0	65.96	22.70
6	276.9	171.8	90.06	54.11	31.45	13.74
7	124.8	85.85	51.32	33.90	21.55	10.53
8	78.10	56.98	36.68	25.56	17.10	8.903
9	57.41	43.46	29.33	21.14	14.62	7.931
10	46.17	35.86	24.99	18.44	13.04	7.285
11	39.26	31.05	22.16	16.63	11.96	6.825
12	34.63	27.77	20.17	15.34	11.17	6.481
13	31.33	25.40	18.71	14.38	10.58	6.214
14	28.87	23.62	17.59	13.63	10.11	6.001
15	26.98	22.22	16.70	13.03	9.727	5.827
16	25.47	21.11	15.99	12.54	9.416	5.683
17	24.25	20.20	15.40	12.14	9.156	5.560
18	23.24	19.44	14.90	11.80	8.936	5.456
19	22.39	18.80	14.48	11.51	8.746	5.366
20	21.67	18.25	14.12	11.25	8.581	5.286
21	21.05	17.78	13.81	11.03	8.437	5.216
22	20.51	17.37	13.53	10.84	8.309	5.155
23	20.03	17.00	13.29	10.67	8.196	5.099
24	19.61	16.68	13.07	10.52	8.094	5.049
25	19.24	16.39	12.88	10.38	8.003	5.004
26	18.90	16.14	12.70	10.25	7.920	4.963
27	18.60	15.90	12.54	10.14	7.844	4.926
28	18.33	15.69	12.40	10.04	7.775	4.892
29	18.08	15.50	12.26	9.943	7.712	4.860
30	17.85	15.32	12.14	9.857	7.654	4.831
31	17.64	15.16	12.03	9.777	7.600	4.804
32	17.45	15.01	11.93	9.703	7.550	4.779
33	17.27	14.87	11.83	9.635	7.504	4.756
43	16.04	13.90	11.16	9.150	7.175	4.590
63	14.89	12.99	10.53	8.686	6.857	4.426
123	13.83	12.14	9.922	8.241	6.549	4.266
∞	12.84	11.34	9.348	7.815	6.251	4.108

space of the statistic (w, A, ξ) is

$$G_2 = \left\{ \begin{pmatrix} \bar{w}' = \bar{w}B + Z \\ \xi' = \xi B + Z \\ A' = B'AB \end{pmatrix} \middle| \begin{array}{l} B \text{ nonsingular,} \\ Z \, \varepsilon \, R^k \end{array} \right\}.$$

It is straightforward to show that a maximal invariant statistic is

$$\Upsilon^2 = (\xi - \bar{w})A^{-1}(\xi - \bar{w})'.$$

TABLE 5

Tolerance factors c_β for quadri-variate normal distributions with unknown means, unknown variance-covariance matrix; sample size n

n	β					
	.995	.99	.975	.95	.90	.75
5	432,000	107,992	17,272	4,312	1,072	164.8
6	2,325	1,158	457.9	224.5	107.8	37.71
7	422.4	262.5	138.1	83.36	48.85	21.85
8	182.3	125.8	75.64	50.31	32.34	16.26
9	110.6	81.01	52.54	36.92	25.03	13.46
10	79.38	60.38	41.10	29.92	20.99	11.80
11	62.65	48.91	34.43	25.69	18.46	10.70
12	52.46	41.74	30.11	22.87	16.72	9.916
13	45.70	36.89	27.10	20.87	15.47	9.335
14	40.91	33.40	24.89	19.38	14.52	8.886
15	37.37	30.78	23.22	18.23	13.77	8.528
16	34.64	28.75	21.89	17.31	13.18	8.236
17	32.49	27.13	20.83	16.57	12.69	7.994
18	30.75	25.82	19.95	15.96	12.28	7.790
19	29.32	24.72	19.22	15.44	11.93	7.615
20	28.12	23.83	18.60	15.00	11.63	7.464
21	27.10	23.02	18.07	14.62	11.38	7.332
22	26.22	22.34	17.60	14.28	11.15	7.216
23	25.46	21.75	17.20	13.99	10.95	7.113
24	24.79	21.23	16.84	13.73	10.78	7.021
25	24.20	20.77	16.52	13.50	10.62	6.938
26	23.68	20.36	16.24	13.30	10.48	6.863
27	23.21	19.99	15.98	13.11	10.35	6.795
28	22.79	19.66	15.75	12.94	10.23	6.733
29	22.41	19.36	15.54	12.79	10.12	6.676
30	22.06	19.09	15.35	12.64	10.03	6.624
31	21.74	18.84	15.17	12.51	9.935	6.576
32	21.45	18.61	15.01	12.40	9.851	6.532
33	21.19	18.39	14.86	12.28	9.774	6.490
34	20.94	18.20	14.72	12.18	9.703	6.452
44	19.23	16.84	13.75	11.46	9.195	6.177
64	17.66	15.57	12.83	10.77	8.706	5.907
124	16.20	14.38	11.96	10.11	8.234	5.643
∞	14.86	13.28	11.14	9.488	7.780	5.385

The problem as interpreted for the induced distributions of the statistic Υ has a simple hypothesis and a simple alternative. By applying the Neyman-Pearson fundamental lemma, a short analysis shows that the most powerful invariant test is

$$\begin{aligned}\varphi(\Upsilon^2) &= 1 \qquad \text{if } \Upsilon^2 < c_\beta \\ &= 0 \qquad \text{if } \Upsilon^2 > c_\beta .\end{aligned}$$

Under the hypothesis, $\Upsilon^2\ (1 + 1/n)^{-1}$ has the distribution of Hotelling's T^2 with $(n - 1)$ degrees of freedom. The probability density function of T^2 with $(n - 1)$ degrees of freedom is

$$\text{(6.2.1)} \qquad \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{n-k}{2}\right)} \frac{(T^2/n - 1)^{(k-2)/2}}{(1 + T^2/n - 1)^{n/2}}\, d(T^2/n - 1).$$

But if we make the transformation $T^2 = (n - 1)\ k/(n - k)\ F$, (6.2.1) is easily seen to become the probability density function of Fisher's F-distribution with k, $n - k$ degrees of freedom. Hence, to give the test and consequently the tolerance region size β, we take

$$c_\beta = (1 + 1/n)\ (n - 1)\ (k/n - k)\ F_{1-\beta}\,,$$

where F_α is the point exceeded with probability α using the F-distribution with k, $n - k$ degrees of freedom.

Now, by the same argument used for the univariate case, the minimax and most stringent size β tolerance region for the k-variate normal distribution is the ellipsoidal region given by

$$\{\xi \mid (\xi - \bar{w})\ A^{-1}\ (\xi - \bar{w})' \leqq c_\beta\}.$$

Values of c_β for $k = 2, 3, 4$ are given in Tables 3, 4, and 5.

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