ASYMPTOTIC FORMULAE FOR THE DISTRIBUTION OF HOTELLING'S GENERALIZED T_0^2 STATISTIC[1]

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1. Summary. In this paper the asymptotic expansion of a percentage point of Hotelling's generalized T_0^2 distribution is derived in terms of the corresponding percentage point of a χ^2 distribution. Our result generalizes Hotelling's and Frankel's asymptotic expansion for the generalized Student T [3], [4]. The technique used in this paper for obtaining the asymptotic expansion of T_0^2 is an extension of the previous methods of Welch [8] and of James [5], [6], who used them to solve the distribution problem of various statistics in connection with the Behrens-Fisher problem. An asymptotic formula for the cumulative distribution function (c.d.f.) of T_0^2 is also given together with an upper bound for the error committed when all but the first few terms are omitted in the series. This formula is a sort of multivariate analogue of Hartley's formula of "Studentization" [2].

2. Introduction. In the multivariate analysis of variance we use the following canonical probability law:

$$P(X_0, X_1) = \text{const.} \exp\left[-\tfrac{1}{2} \operatorname{tr} \Lambda(X_1 - \xi)(X_1' - \xi') - \tfrac{1}{2} \operatorname{tr} \Lambda X_0 X_0'\right] dX_0\, dX_1, \tag{2.1}$$

where X_1 and X_0 are $p \times m$ and $p \times m$ matrices, respectively, and $(1/m)X_1X_1' = S_1$ is the sample "between" dispersion matrix and $(1/n)\ X_0X_0' = S_0$ is the sample "within" dispersion matrix, the prime denoting the transpose of a matrix. ξ is a $p \times m$ matrix, $(1/m)\xi\xi'$ being the population "between" dispersion matrix, and Λ is a $p \times p$ symmetric positive definite matrix. It is assumed that m may be $\geqq p$ or $<p$, but $n \geqq p$. To test the null hypothesis $H_0: \xi = 0$, Hotelling [3] proposed a test based on the statistic:

$$T_0^2 = m \operatorname{tr} S_1 S_0^{-1} \tag{2.2}$$

and derived the exact distribution of this statistic when $p = 2$ and $\xi = 0$. For general values of p the exact distribution of T_0^2 is not available at present, even in the null case $\xi = 0$.

3. Derivation of asymptotic formula of T_0^2. For general values of p it is known that the statistic

$$\chi^2 = m \operatorname{tr} S_1 \Lambda \tag{3.1}$$

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has a χ^2 distribution with mp degrees of freedom. That is to say, we have

$$\Pr\{m \operatorname{tr} S_1\Lambda \leqq 2\theta\} = G_\rho(\theta), \tag{3.2}$$

where 2θ denotes the tabled value of χ^2 for a particular level of significance, $\rho = mp/2$, and

$$G_\rho(\theta) = [\Gamma(\rho)]^{-1} \int_0^\theta t^{\rho-1} e^{-t}\, dt.$$

Hence, if Λ is known, the statistic χ^2 given by (3.1) may be used to test H_0 exactly, and if Λ is unknown but if S_0 is based on a large number of degrees of freedom, i.e., if n is large, we may use as an approximation the result

$$\Pr\{m \operatorname{tr} S_1S_0^{-1} \leqq 2\theta\} = G_\rho(\theta). \tag{3.4}$$

This suggests that in the general case we try to find a function $h(S_0)$ of the elements of S_0 such that

$$\Pr\{m \operatorname{tr} S_1S_0^{-1} \leqq 2h(S_0)\} = G_\rho(\theta). \tag{3.5}$$

When n is large, $2h(S_0)$ will approach $2\theta \equiv \chi^2$, and we now expect to write $2h(S_0)$ as a series with χ^2 as its first term and successive terms of decreasing order of magnitude.

Now

$$\Pr\{m \operatorname{tr} S_1S_0^{-1} \leqq 2h(S_0)\} = \int_R \Pr\{m \operatorname{tr} S_1S_0^{-1} \leqq 2h(S_0) \mid S_0\} \Pr\{dS_0\}, \tag{3.6}$$

where the first expression on the right denotes the conditional probability of the relation indicated for fixed values of the elements of S_0, and the second denotes the probability element of S_0, which has a Wishart distribution with n degrees of freedom, and the domain of integration R is over all possible values of the elements of S_0. Now we may expand $\Pr\{m \operatorname{tr} S_1S_0^{-1} \leqq 2h(S_0) \mid S_0\}$ about an origin $(\sigma_{11}, \sigma_{22}, \cdots, \sigma_{pp}, \sigma_{12}, \cdots, \sigma_{p-1,p})$ in a Taylor series, where

$$\Lambda^{-1} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{pmatrix}.$$

Thus,

$$\begin{aligned} &\Pr\{m \operatorname{tr} S_1S_0^{-1} \leqq 2h(S_0) \mid S_0\} \\ &\quad = \left\{\exp\left[\sum_{i \leqq j=1}^{p} (s_{0ij} - \sigma_{ij}) \frac{\partial}{\partial \sigma_{ij}}\right]\right\} \Pr\{m \operatorname{tr} S_1\Lambda \leqq 2h(\Lambda^{-1})\} \\ &\quad = \{\exp[\operatorname{tr}(S_0 - \Lambda^{-1})\partial]\} \Pr\{m \operatorname{tr} S_1\Lambda \leqq 2h(\Lambda^{-1})\}, \end{aligned} \tag{3.7}$$

where s_{0ij} is the ith row, jth column element of S_0, and ∂ denotes the matrix of derivative operators:

$$\partial = \begin{bmatrix} \frac{\partial}{\partial \sigma_{11}} & \frac{1}{2}\frac{\partial}{\partial \sigma_{12}} & \cdots & \frac{1}{2}\frac{\partial}{\partial \sigma_{1p}} \\ \frac{1}{2}\frac{\partial}{\partial \sigma_{21}} & \frac{\partial}{\partial \sigma_{22}} & \cdots & \frac{1}{2}\frac{\partial}{\partial \sigma_{2p}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2}\frac{\partial}{\partial \sigma_{p1}} & \frac{1}{2}\frac{\partial}{\partial \sigma_{p2}} & \cdots & \frac{\partial}{\partial \sigma_{pp}} \end{bmatrix}, \tag{3.8}$$

its typical element being $\partial_{ij} = \frac{1}{2}(1 + \delta_{ij})(\partial/\partial\sigma_{ij})$, where δ_{ij} is the Kronecker delta. Whether uniformly convergent or not, the right-hand side of (3.7) is an asymptotic representation of $\Pr\{m \operatorname{tr} S_1 S_0^{-1} \leqq 2h(S_0) \mid S_0\}$, for sufficiently large values of n. Hence, substitution of (3.7) into (3.6) and term by term integration, which may be done legitimately, yields:

$$\begin{aligned} G_p(\theta) &= \int_R \exp\,[\operatorname{tr}\,(S_0 - \Lambda^{-1})\partial]\,\Pr\,\{m \operatorname{tr} S_1\Lambda \leqq 2h(\Lambda^{-1})\}\,\Pr\,\{dS_0\} \\ &= \Theta\,\Pr\,\{m \operatorname{tr} S_1\Lambda \leqq 2h(\Lambda^{-1})\}, \end{aligned} \tag{3.9}$$

where

$$\Theta = \int_R \exp\,[\operatorname{tr}\,(S_0 - \Lambda^{-1})\partial]\,\Pr\,\{dS_0\}.$$

Since S_0 has a Wishart distribution with n degrees of freedom, we have

$$\begin{aligned} \Theta &= \exp\,[-\operatorname{tr} \Lambda^{-1}\,\partial]\cdot\text{const.}\,|\,\Lambda\,|^{n/2}\int_R |\,S_0\,|^{(n-p-1)/2} \cdot\exp\left[\operatorname{tr}\left(S_0\partial - \frac{n}{2}\Lambda S_0\right)\right] d\,S_0 \\ &= \exp\,[-\operatorname{tr} \Lambda^{-1}\,\partial]\cdot\text{const.}\,|\,\Lambda\,|^{n/2}\int_R |\,S_0\,|^{(n-p-1)/2} \cdot\exp\left[-\frac{n}{2}\operatorname{tr}\left(\Lambda - \frac{2}{n}\partial\right)S_0\right] dS_0 \\ &= \exp\,[-\operatorname{tr} \Lambda^{-1}\,\partial]\cdot|\,\Lambda\,|^{n/2}\left|\Lambda - \frac{2}{n}\partial\right|^{-n/2} \\ &= \exp\,[-\operatorname{tr} \Lambda^{-1}\,\partial]\cdot\left|I - \frac{2}{n}\Lambda^{-1}\partial\right|^{-n/2}, \end{aligned}$$

where I is the $p \times p$ identity matrix. Now using [5],

$$-\log|I - Y| = \operatorname{tr} Y + \tfrac{1}{2}\operatorname{tr} Y^2 + \tfrac{1}{3}\operatorname{tr} Y^3 + \cdots, \tag{3.10}$$

we obtain

$$
\begin{aligned}
\Theta &= \exp\left[-\operatorname{tr}\Lambda^{-1}\partial - \frac{n}{2}\log\left|I - \frac{2}{n}\Lambda^{-1}\partial\right|\right] \\
&= \exp\left[-\operatorname{tr}\Lambda^{-1}\partial + \frac{n}{2}\left\{\operatorname{tr}\left(\frac{2}{n}\Lambda^{-1}\partial\right) + \tfrac{1}{2}\operatorname{tr}\left(\frac{2}{n}\Lambda^{-1}\partial\right)^2 \right.\right. \\
&\qquad\qquad \left.\left. + \tfrac{1}{3}\operatorname{tr}\left(\frac{2}{n}\Lambda^{-1}\partial\right)^3 + \cdots\right\}\right] \\
&= \exp\left[\frac{1}{n}\operatorname{tr}(\Lambda^{-1}\partial)^2 + \frac{4}{3n^2}\operatorname{tr}(\Lambda^{-1}\partial)^3 + \cdots\right] \\
&= 1 + \frac{1}{n}\operatorname{tr}(\Lambda^{-1}\partial)^2 + \frac{1}{n^2}\{\tfrac{4}{3}\operatorname{tr}(\Lambda^{-1}\partial)^3 + \tfrac{1}{2}(\operatorname{tr}(\Lambda^{-1}\partial)^2)^2\} + O(n^{-3}).
\end{aligned} \tag{3.11}
$$

It is to be noted here that in (3.11) the operator ∂ does not act on Λ^{-1} present in Θ itself, and it is more useful for our purpose to write (3.11) in suffix form:

$$
\begin{aligned}
\Theta &= 1 + \frac{1}{n}\sum \sigma_{rs}\sigma_{tu}\partial_{st}\partial_{ur} \\
&\quad + \frac{1}{n^2}\{\tfrac{4}{3}\sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\partial_{st}\partial_{uv}\partial_{wr} + \tfrac{1}{2}\sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}\partial_{st}\partial_{ur}\partial_{wx}\partial_{yv}\} \\
&\quad + O(n^{-3}),
\end{aligned} \tag{3.12}
$$

where $\sum$ denotes the summation over all suffixes $r, s, \cdots$, each of which ranges from 1 to p.

Now we represent $h(S_0)$ as

$$h(S_0) = \theta + h_1(S_0) + h_2(S_0) + \cdots, \tag{3.13}$$

$h_s(S_0)$ being of order n^{-s}; i.e., we write $h(S_0)$ as an asymptotic series such that

$$|n^s\{h(S_0) - \theta - h_1(S_0) - \cdots - h_s(S_0)\}|$$

is made arbitrarily small for sufficiently large values of n. Then (3.13) may be substituted into $\Pr\{m \operatorname{tr} S_1\Lambda \leqq 2h(\Lambda^{-1})\}$, and by Taylor's expansion we have

$$
\begin{aligned}
&\Pr\{m\operatorname{tr} S_1\Lambda \leqq 2h(\Lambda^{-1})\} \\
&\qquad = \exp[\{h_1(\Lambda^{-1}) + h_2(\Lambda^{-1}) + \cdots\}D]\Pr\{m\operatorname{tr} S_1\Lambda \leqq 2\theta\} \\
&\qquad = [1 + \{h_1(\Lambda^{-1}) + h_2(\Lambda^{-1}) + \cdots\}D \\
&\qquad\qquad + \tfrac{1}{2}\{h_1(\Lambda^{-1}) + h_2(\Lambda^{-1}) + \cdots\}^2 D^2 + \cdots] \\
&\qquad\qquad \times \Pr\{m\operatorname{tr} S_1\Lambda \leqq 2\theta\},
\end{aligned} \tag{3.14}
$$

where $D = \partial/\partial\theta$. By substituting (3.12) and (3.14) into (3.9), we obtain

$$G_p(\theta) = \Big[1 + \frac{1}{n}\sum \sigma_{rs}\sigma_{tu}\partial_{st}\partial_{ur} + \frac{1}{n^2}\{\tfrac{4}{3}\sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\partial_{st}\partial_{uv}\partial_{wr} + \tfrac{1}{2}\sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}\partial_{st}\partial_{ur}\partial_{wx}\partial_{yv}\} + O(n^{-3})\Big] \times [1 + h_1(\Lambda^{-1})D + \{h_2(\Lambda^{-1})D + \tfrac{1}{2}h_1^2(\Lambda^{-1})D^2\} + O(n^{-3})] \times \Pr\{m \operatorname{tr} S_1\Lambda \leqq 2\theta\}. \tag{3.15}$$

By equating terms of successive order in (3.15), we obtain

$$\Big\{h_1(\Lambda^{-1})D + \frac{1}{n}\sum \sigma_{rs}\sigma_{tu}\partial_{st}\partial_{ur}\Big\}\Pr\{m \operatorname{tr} S_1\Lambda \leqq 2\theta\} = 0, \tag{3.16}$$

$$\Big[h_2(\Lambda^{-1})D + \tfrac{1}{2}h_1^2(\Lambda^{-1})D^2 + \frac{1}{n}\sum \sigma_{rs}\sigma_{tu}\{h_1^{(st,ur)}(\Lambda^{-1})D + 2h_1^{(st)}(\Lambda^{-1})\partial_{ur}D + h_1(\Lambda^{-1})\partial_{st}\partial_{ur}D\} + \frac{4}{3n^2}\sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\partial_{st}\partial_{uv}\partial_{wr} + \frac{1}{2n^2}\sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}\partial_{st}\partial_{ur}\partial_{wx}\partial_{yv}\Big] \times \Pr\{m \operatorname{tr} S_1\Lambda \leqq 2\theta\} = 0, \tag{3.17}$$

and so on, where $h_1^{(st)}(\Lambda^{-1}) = \partial_{st}h_1(\Lambda^{-1})$ and $h_1^{(st,ur)}(\Lambda^{-1}) = \partial_{ur}\partial_{st}h_1(\Lambda^{-1})$.

It now remains to carry out the operations ∂ and D indicated in (3.16) and (3.17) in order to obtain $h_1(\Lambda^{-1})$, $h_2(\Lambda^{-1})$ and hence $h_1(S_0)$, $h_2(S_0)$. These operators will operate on $\Pr\{m \operatorname{tr} S_1\Lambda \leqq 2\theta\}$, which is a $p \times m$-fold integral, and the operations may be thought of as differentiations, with respect to the boundary only, of the integral of the probability density function of the X_1 throughout a region in the space of X_1. The method used to evaluate $\partial_{st}\partial_{ur}\Pr\{m \operatorname{tr} S_1\Lambda \leqq 2\theta\}$, $\partial_{st}\partial_{uv}\partial_{wr}\Pr\{m \operatorname{tr} S_1\Lambda \leqq 2\theta\}$, $\cdots$, is to change the boundary slightly, expand the integral in powers of the quantities specifying this change, and obtain the derivatives by comparison with Taylor's expansion. We consider

$$J = \Pr\{m \operatorname{tr} S_1(\Lambda^{-1} + \epsilon)^{-1} \leqq 2\theta\}, \tag{3.18}$$

where ϵ is a $p \times p$ symmetric matrix. Then by Taylor expansion we have

$$J = \Big\{1 + \sum \epsilon_{rs}\partial_{rs} + \frac{1}{2!}\sum \epsilon_{rs}\epsilon_{tu}\partial_{rs}\partial_{tu} + \frac{1}{3!}\sum \epsilon_{rs}\epsilon_{tu}\epsilon_{vw}\partial_{rs}\partial_{tu}\partial_{vw} + \frac{1}{4!}\sum \epsilon_{rs}\epsilon_{tu}\epsilon_{vw}\epsilon_{xy}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy} + \cdots\Big\}\Pr\{m \operatorname{tr} S_1\Lambda \leqq 2\theta\}. \tag{3.19}$$

On the other hand, J is, by definition, written as

$$J = \frac{|\Lambda|^{m/2}}{(2\pi)^{pm/2}} \int_{R'} \exp\left[-\tfrac{1}{2} \operatorname{tr} \Lambda X_1 X_1'\right] dX_1, \tag{3.20}$$

where $X_1 X_1' = mS_1$, and domain of integration R' ranges over all possible values of the elements of X_1 such that $m \operatorname{tr} S_1(\Lambda^{-1} + \epsilon)^{-1} \leqq 2\theta$. It is now easy to show that integration of (3.20) yields

$$J = \left(\frac{|I - D_\eta E|}{|I - D_\eta|}\right)^{-m/2} G_p(\theta), \tag{3.21}$$

where D_η is a diagonal matrix which satisfies

$$\begin{aligned} X_1(p \times m) &= \Gamma(p \times p) Z(p \times m), \\ \tfrac{1}{2}\Gamma'(\Lambda^{-1} + \epsilon)^{-1}\Gamma &= I(p), \end{aligned} \tag{3.22}$$

and

$$\tfrac{1}{2}\Gamma'\Lambda\Gamma = I(p) - D_\eta,$$

Γ being a nonsingular matrix, and E is an operator such that

$$EG_p(\theta) = G_{p+1}(\theta).$$

Now, letting $\Delta = E - 1$ and using (3.22), we have

$$\begin{aligned} \frac{|I - D_\eta E|}{|I - D_\eta|} &= \frac{|I - D_\eta - D_\eta \Delta|}{|I - D_\eta|} \\ &= \frac{|\tfrac{1}{2}\Gamma'\Lambda\Gamma - \{\tfrac{1}{2}\Gamma'(\Lambda^{-1} + \epsilon)^{-1}\Gamma - \tfrac{1}{2}\Gamma'\Lambda\Gamma\}\Delta|}{|\tfrac{1}{2}\Gamma'\Lambda\Gamma|} \\ &= \frac{|\Lambda - \{(\Lambda^{-1} + \epsilon)^{-1} - \Lambda\}\Delta|}{|\Lambda|} = |I - \{\Lambda^{-1}(\Lambda^{-1} + \epsilon)^{-1} - I\}\Delta| \\ &= |I - X\Delta|, \end{aligned}$$

where $X = \Lambda^{-1}(\Lambda^{-1} + \epsilon)^{-1} - I$. Hence, (3.21) becomes

$$J = |I - X\Delta|^{-(m/2)} G_p(\theta). \tag{3.23}$$

Now, using (3.10) again, we rewrite (3.23) as

$$\begin{aligned} J &= \exp\left\{-\frac{m}{2} \log |I - X\Delta|\right\} G_p(\theta) \\ &= \exp\left\{\frac{m}{2} \operatorname{tr} X\Delta + \frac{m}{4} \operatorname{tr} X^2\Delta^2 + \frac{m}{6} \operatorname{tr} X^3\Delta^3 + \frac{m}{8} \operatorname{tr} X^4\Delta^4 + \cdots\right\} G_p(\theta) \\ &= \left[1 + \frac{m}{2} \operatorname{tr} X\Delta + \left\{\frac{m}{4} \operatorname{tr} X^2 + \frac{m^2}{8} (\operatorname{tr} X)^2\right\} \Delta^2 \right. \end{aligned}$$

$$+\left\{\frac{m}{6}\operatorname{tr}X^3+\frac{m^2}{8}(\operatorname{tr}X)(\operatorname{tr}X^2)+\frac{m^3}{48}(\operatorname{tr}X)^3\right\}\Delta^3 \tag{3.24}$$
$$+\left\{\frac{m}{8}\operatorname{tr}X^4+\frac{m^2}{12}(\operatorname{tr}X)(\operatorname{tr}X^3)+\frac{m^2}{32}(\operatorname{tr}X^2)^2\right.$$
$$\left.+\frac{m^3}{32}(\operatorname{tr}X)^2(\operatorname{tr}X^2)+\frac{m^4}{384}(\operatorname{tr}X)^4\right\}\Delta^4+\cdots\Bigg]G_\rho(\theta);$$

X can be represented as

$$X=\Lambda^{-1}(\Lambda^{-1}+\epsilon)^{-1}-I=\Lambda^{-1}(\Lambda^{-1}+\sum\epsilon_{rs}\Lambda_{rs}^{-1})^{-1}$$
$$-I=(I+\sum\epsilon_{rs}\Lambda_{rs}^{-1}\Lambda)^{-1}-I$$
$$=-\sum\epsilon_{rs}(\Lambda_{rs}^{-1}\Lambda)+\sum\epsilon_{rs}\epsilon_{tu}(\Lambda_{rs}^{-1}\Lambda)(\Lambda_{tu}^{-1}\Lambda) \tag{3.25}$$
$$-\sum\epsilon_{rs}\epsilon_{tu}\epsilon_{vw}(\Lambda_{rs}^{-1}\Lambda)(\Lambda_{tu}^{-1}\Lambda)(\Lambda_{vw}^{-1}\Lambda)$$
$$+\sum\epsilon_{rs}\epsilon_{tu}\epsilon_{vw}\epsilon_{xy}(\Lambda_{rs}^{-1}\Lambda)(\Lambda_{tu}^{-1}\Lambda)(\Lambda_{vw}^{-1}\Lambda)(\Lambda_{xy}^{-1}\Lambda)-\cdots,$$

where Λ_{rs}^{-1} is a $p\times p$ matrix obtained by operating ∂_{rs} on Λ, i.e., Λ_{rs}^{-1} has its ith row, jth column element, $\frac{1}{2}(\delta_{ri}\delta_{sj}+\delta_{si}\delta_{rj})$. Writing

$$\operatorname{tr}(\Lambda_{rs}^{-1}\Lambda)=(rs),$$
$$\operatorname{tr}(\Lambda_{rs}^{-1}\Lambda)(\Lambda_{tu}^{-1}\Lambda)=(rs\mid tu),$$
$$\operatorname{tr}(\Lambda_{rs}^{-1}\Lambda)(\Lambda_{tu}^{-1}\Lambda)(\Lambda_{vw}^{-1}\Lambda)=(rs\mid tu\mid vw),$$
$$\operatorname{tr}(\Lambda_{rs}^{-1}\Lambda)(\Lambda_{tu}^{-1}\Lambda)(\Lambda_{vw}^{-1}\Lambda)(\Lambda_{xy}^{-1}\Lambda)=(rs\mid tu\mid vw\mid xy),$$

and substituting (3.25) into (3.24), we obtain

$$J=\Bigg[1+\sum\epsilon_{rs}\left\{-\frac{m}{2}(rs)\Delta\right\}+\frac{1}{2!}\sum\epsilon_{rs}\epsilon_{tu}\left\{(rs\mid tu)\left(m\Delta+\frac{m}{2}\Delta^2\right)\right.$$
$$\left.+\frac{m^2}{4}(rs)(tu)\Delta^2\right\}+\frac{1}{3!}\sum\epsilon_{rs}\epsilon_{tu}\epsilon_{vw}\left\{(rs\mid tu\mid vw)(-3m\Delta-3m\Delta^2-m\Delta^3)\right.$$
$$\left.+(rs)(tu\mid vw)\left(-\tfrac{3}{2}m^2\Delta^2-\tfrac{3}{4}m^2\Delta^3\right)-\frac{m^3}{8}(rs)(tu)(vw)\Delta^3\right\}$$
$$+\frac{1}{4!}\sum\epsilon_{rs}\epsilon_{tu}\epsilon_{vw}\epsilon_{xy}\left\{(rs\mid tu\mid vw\mid xy)(12m\Delta+18m\Delta^2+12m\Delta^3+3m\Delta^4)\right. \tag{3.26}$$
$$+(rs)(tu\mid vw\mid xy)(6m^2\Delta^2+6m^2\Delta^3+2m^2\Delta^4)$$
$$+(rs\mid tu)(vw\mid xy)(3m^2\Delta^2+3m^2\Delta^3+\tfrac{3}{4}m^2\Delta^4)$$
$$+(rs)(tu)(vw\mid xy)(\tfrac{3}{2}m^3\Delta^3+\tfrac{3}{4}m^3\Delta^4)$$
$$\left.+(rs)(tu)(vw)(xy)\frac{m^4}{16}\Delta^4\right\}+\cdots\Bigg]G_\rho(\theta).$$

Then term by term comparison between two expansions for J, (3.19) and (3.26), gives ∂_{rs} Pr $\{m \text{ tr } S_1\Lambda \leqq 2\theta\}$, $\partial_{rs}\partial_{tu}$ Pr $\{m \text{ tr } S_1\Lambda \leqq 2\theta\}$, etc., but in doing so we must take such a care that, for example,

$$\sum a_{ijk}\epsilon_i\epsilon_j\epsilon_k = \sum b_{ijk}\epsilon_i\epsilon_j\epsilon_k$$

implies $a_{ijk} = b_{ijk}$ if both a_{ijk} and b_{ijk} are completely symmetrical in their suffices. With this in mind and using the relation

$$\Delta G_\rho(\theta) = -Eg_\rho(\theta),$$

where $g_\rho(\theta) = D\, G_\rho(\theta)$, we obtain

$$\partial_{rs} \text{ Pr } \{m \text{ tr } S_1\Lambda \leqq 2\theta\} = \frac{m}{2}(rs)Eg_\rho(\theta), \tag{3.27}$$

$$\begin{aligned} &\partial_{rs}\partial_{tu} \text{ Pr } \{m \text{ tr } S_1\Lambda \leqq 2\theta\} \\ &\qquad = -\left\{\frac{m}{2}(rs \mid tu)(E^2 + E) + \frac{m^2}{4}(rs)(tu)(E^2 - E)\right\} g_\rho(\theta), \end{aligned} \tag{3.28}$$

$$\begin{aligned} \partial_{rs}\partial_{tu}\partial_{vw} \text{ Pr } \{m \text{ tr } S_1\Lambda \leqq 2\theta\} &= \Big\{m(rs \mid tu \mid vw)(E^3 + E^2 + E) + \frac{m^2}{4} \\ &\cdot [(rs)(tu \mid vw) + (tu)(rs \mid vw) + (vw)(rs \mid tu)](E^3 - E) \\ &+ \frac{m^3}{8}(rs)(tu)(vw)(E^3 - 2E^2 + E)\Big\} \cdot g_\rho(\theta), \end{aligned} \tag{3.29}$$

$$\begin{aligned} &\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy} \text{ Pr } \{m \text{ tr } S_1\Lambda \leqq 2\theta\} \\ &= -\Big\{m[(rs \mid tu \mid vw \mid xy) \\ &+ (rs \mid vw \mid xy \mid tu) + (rs \mid xy \mid tu \mid vw)]\,(E^4 + E^3 + E^2 + E) \\ &+ \frac{m^2}{2}[(rs)(tu \mid vw \mid xy) + (xy)(tu \mid vw \mid rs) + (vw)(tu \mid xy \mid rs) \\ &+ (tu)(vw \mid xy \mid rs)](E^4 - E) + \frac{m^2}{4}[(rs \mid tu)(vw \mid xy) \\ &+ (rs \mid vw)(tu \mid xy) + (rs \mid xy)(tu \mid vw)](E^4 + E^3 - E^2 - E) \\ &+ \frac{m^3}{8}[(rs)(tu)(vw \mid xy) + (rs)(vw)(tu \mid xy) \\ &+ (rs)(xy)(tu \mid vw) + (tu)(vw)(rs \mid xy) + (tu)(xy)(rs \mid vw) \\ &+ (vw)(xy)(rs \mid tu)](E^4 - E^3 - E^2 + E) \\ &+ \frac{m^4}{16}(rs)(tu)(vw)(xy)(E^4 - 3E^3 + 3E^2 - E)\Big\}\, g_\rho(\theta). \end{aligned} \tag{3.30}$$

Upon substituting (3.28) into (3.16), we obtain

$$h_1(\Lambda^{-1}) = \frac{1}{4n}\sum \sigma_{rs}\sigma_{tu}\left[2m(st \mid ur)\left\{\frac{\theta^2}{\rho(\rho+1)} + \frac{\theta}{\rho}\right\} + m^2(st)(ur)\left\{\frac{\theta^2}{\rho(\rho+1)} - \frac{\theta}{\rho}\right\}\right].$$

Now,

$$(st) = \operatorname{tr} \Lambda_{st}^{-1}\Lambda = \tfrac{1}{2}\textstyle\sum_{i,j} (\delta_{si}\delta_{tj} + \delta_{ti}\delta_{sj})\sigma^{ji} = \tfrac{1}{2}(\sigma^{ts} + \sigma^{st}) = \sigma^{st}$$

and also,

$$(st \mid ur) = \operatorname{tr} (\Lambda_{st}^{-1}\Lambda)(\Lambda_{ur}^{-1}\Lambda) = \tfrac{1}{2}(\sigma^{rs}\sigma^{tu} + \sigma^{su}\sigma^{tr}).$$

Hence we have

$$\textstyle\sum \sigma_{rs}\sigma_{tu}(st \mid ur) = \tfrac{1}{2}p(p+1)$$

and

$$\textstyle\sum \sigma_{rs}\sigma_{tu}(st)(ur) = p.$$

We also note that $2\theta = \chi^2$, $\rho = mp/2$. Therefore we finally obtain, after some simplification,

$$h_1(\Lambda^{-1}) = \frac{1}{4n}\left\{\frac{p+m+1}{mp+2}\chi^4 + (p-m+1)\chi^2\right\}. \tag{3.31}$$

In a similar way we substitute (3.29), (3.30), and (3.31) into (3.17) to evaluate $h_2(\Lambda^{-1})$. We note here that since $h_1(\Lambda^{-1})$ given by (3.31) is independent of Λ^{-1}, the terms involving $h_1^{(st)}(\Lambda^{-1})$ and $h_1^{(st,ur)}(\Lambda^{-1})$ in (3.17) do not appear. As before, it can be easily shown that

$$\textstyle\sum \sigma_{rs}\sigma_{tu}\sigma_{vw}(st \mid uv \mid wr) = \tfrac{1}{8}p(p^2+3p+4), \quad \sum \sigma_{rs}\sigma_{tu}\sigma_{vw}(st)(uv \mid wr)$$
$$= \textstyle\sum \sigma_{rs}\sigma_{tu}\sigma_{vw}(uv)(st \mid wr) = \sum \sigma_{rs}\sigma_{tu}\sigma_{vw}(wr)(st \mid uv) = \tfrac{1}{2}p(p+1),$$
$$\textstyle\sum \sigma_{rs}\sigma_{tu}\sigma_{vw}(st)(uv)(wr) = p, \quad \sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}(st \mid ur \mid wx \mid yv)$$
$$= \textstyle\sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}(st \mid wx \mid yv \mid ur) = \tfrac{1}{4}p(p+1)^2,$$
$$\textstyle\sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}(st \mid yv \mid ur \mid wx) = \tfrac{1}{4}p(p+3), \quad \sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}(st)(ur \mid wx \mid yv)$$
$$= \textstyle\sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}(yv)(ur \mid wx \mid st) = \sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}(wx)(ur \mid yv \mid st)$$
$$= \textstyle\sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}(ur)(wx \mid yv \mid st) = \tfrac{1}{2}p(p+1),$$
$$\textstyle\sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}(st \mid ur)(wx \mid yv) = \tfrac{1}{4}p^2(p+1)^2, \quad \sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}(st \mid wx)(ur \mid yv)$$
$$= \textstyle\sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}(st \mid yv)(ur \mid wx) = \tfrac{1}{2}p(p+1),$$
$$\textstyle\sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}(st)(ur)(wx \mid yv) = \sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}(wx)(yv)(st \mid ur) = \tfrac{1}{2}p^2(p+1),$$
$$\textstyle\sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}(st)(wx)(ur \mid yv) = \sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}(st)(yv)(ur \mid wx)$$
$$= \textstyle\sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}(ur)(wx)(st \mid yv) = \sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}(ur)(yv)(st \mid wx) = p,$$

and

$$\sum \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}(st)(ur)(wx)(yv) = p^2.$$

Using these results we obtain from (3.17), after some simplification,

$$(3.32)\quad \begin{aligned} h_2(\Lambda^{-1}) = \frac{1}{48n^2}\Bigg[& \frac{6(p-1)(p+2)(m-1)(m+2)}{(mp+2)^2(mp+4)(mp+6)}\chi^8 \\ & + \frac{4mp^3 + 2(3m^2+3m+10)p^2 + 2(2m^3+3m^2+17m+18)p + 4(5m^2+9m+2)}{(mp+2)^2(mp+4)}\chi^6 \\ & + \frac{13p^2+24p-11m^2+7}{mp+2}\chi^4 \\ & + \{7p^2 + (-12m+12)p + (7m^2-12m+1)\}\chi^2\Bigg], \end{aligned}$$

which is independent of Λ^{-1} just as $h_1(\Lambda^{-1})$.

Now we substitute (3.31) and (3.32) into (3.13) to obtain

$$(3.33)\quad \begin{aligned} T_0^2 &= 2h(S_0) = 2\theta + 2h_1(S_0) + 2h_2(S_0) + O(n^{-3}) \\ &= \chi^2 + \frac{1}{2n}\left\{\frac{p+m+1}{mp+2}\chi^4 + (p-m+1)\chi^2\right\} \\ &\quad + \frac{1}{24n^2}\Bigg\{\frac{6(p-1)(p+2)(m-1)(m+2)}{(mp+2)^2(mp+4)(mp+6)}\chi^8 \\ &\quad + \frac{4mp^3 + 2(3m^2+3m+10)p^2 + 2(2m^3+3m^2+17m+18)p + 4(5m^2+9m+2)}{(mp+2)^2(mp+4)}\chi^6 \\ &\quad + \frac{13p^2+24p-11m^2+7}{mp+2}\chi^4 \\ &\quad + [7p^2 + (-12m+12)p + (7m^2-12m+1)]\chi^2\Bigg\} + O(n^{-3}), \end{aligned}$$

which is the asymptotic expression of a percentage point of the T_0^2 distribution in terms of the corresponding percentage point of the χ^2 distribution with mp degrees of freedom.

If we put $m = 1$ in (3.33), we have

$$(3.34)\quad \begin{aligned} T^2 &= \chi^2 + \frac{1}{2n}\{\chi^4 + p\chi^2\} \\ &\quad + \frac{1}{24n^2}\{4\chi^6 + (13p-2)\chi^4 + (7p^2-4)\chi^2\} + O(n^{-3}), \end{aligned}$$

which is the asymptotic expression of a percentage point of the generalized Student T distribution. This result, (3.34), was previously obtained by Hotelling and Frankel [3], [4].

There is another check of (3.33) by putting $p = 1$ in the formula.[3] In this case we have

$$(3.35)\quad \begin{aligned} T^2 &= \chi^2 + \frac{1}{2n}\{\chi^4 - (m-2)\chi^2\} \\ &\quad + \frac{1}{24n^2}\{4\chi^6 - 11(m-2)\chi^4 + (m-2)(7m-10)\chi^2\} + O(n^{-3}), \end{aligned}$$

which is the correct expansion for the ordinary variance ratio F with m, n degrees of freedom in terms of χ^2 with m degrees of freedom [1].

4. Asymptotic formula for the c.d.f. of T_0^2. Let $F(2\theta_1)$ be the c.d.f. of T_0^2, i.e.,

$$(4.1)\quad F(2\theta_1) = \Pr\{m \operatorname{tr} S_1 S_0^{-1} \leqq 2\theta_1\}.$$

Then, as (3.6), we can write

$$(4.2)\quad \begin{aligned} \Pr\{m \operatorname{tr} S_1 S_0^{-1} \leqq 2\theta_1\} &= \int_R \Pr\{m \operatorname{tr} S_1 S_0^{-1} \leqq 2\theta_1 \mid S_0\} \Pr\{dS_0\} \\ &= \Theta \Pr\{m \operatorname{tr} S_1 \Lambda \leqq 2\theta_1\}, \end{aligned}$$

where Θ is given by (3.12). Upon substituting (3.28), (3.29), and (3.30) into (4.2) we obtain, after some simplification,

$$(4.3)\quad \begin{aligned} F(2\theta_1) &= G_p(\theta_1) - \frac{1}{2n}\left\{\frac{2(p+m+1)\theta_1^2}{mp+2} + (p-m+1)\theta_1\right\} g_p(\theta_1) \\ &\quad - \frac{1}{48n^2}\Bigg[\frac{24\{mp^3 + 2(m^2+m+4)p^2 + (m^3+2m^2+21m+20)p + 8m^2+20m+20\}\theta_1^4}{(mp+2)(mp+4)(mp+6)} \\ &\quad + \frac{4\{3mp^3 - 2(3m^2-3m-4)p^2 - 3(3m^3+2m^2+11m-4)p - 40m^2-36m-4\}\theta_1^3}{(mp+2)(mp+4)} \\ &\quad - \frac{2\{3mp^3 + 2(3m^2+3m-4)p^2 - 3(3m^3-2m^2-5m+4)p - 8m^2+12m+4\}\theta_1^2}{mp+2} \\ &\quad - \{3mp^3 - 2(3m^2-3m+4)p^2 + 3(m^3-2m^2+5m-4)p \\ &\quad - 8m^2+12m+4\}\theta_1\Bigg] g_p(\theta_1) + O(n^{-3}), \end{aligned}$$

[3] The author is indebted to the referee for pointing out this check of (3.33).

where

$$G_\rho(\theta_1) = [\Gamma(\rho)]^{-1} \int_0^{\theta_1} t^{\rho-1} e^{-t}\, dt,\ g_\rho(\theta_1) = \frac{\partial}{\partial\theta_1} G_\rho(\theta_1), \text{ and } \rho = mp/2.$$

(4.3) is a sort of multivariate analogue of Hartley's formula of "Studentization." In fact it can be shown that when $p = 1$, (4.3) coincides with Hartley's formula for the c.d.f. of the univariate analysis of variance F statistic. (See equation (28), p. 178, [2].)

5. Discussion of the error and remarks. In view of the methods used in Sections 3 and 4, it is rather difficult to set a bound for the error committed by omitting all terms after the first few terms in the asymptotic formula for T_0^2 (3.33) or in the asymptotic formula for the c.d.f. of T_0^2 (4.3). There is, however, a method to find lower and upper bounds to the c.d.f. of T_0^2 which is fairly good for large values of n, and they can be used to set a bound for $O(n^{-3})$, say, in the asymptotic expansion of the c.d.f. of T_0^2.

We shall first obtain lower and upper bounds for the c.d.f. of T_0^2. It is well known (e.g., see [7]) that the joint probability law of the characteristic roots $e_1, e_2, \cdots, e_s$ of $m\, S_1 S_0^{-1}$ under the null hypothesis H_0 is given by

$$\begin{aligned} &P(e_1, e_2, \cdots, e_s) \\ (5.1)\qquad &= C(s, t, p, n) \prod_{i=1}^{s} e_i^{(t-s-1)/2} \left(1 + \frac{e_i}{n}\right)^{-(m+n)/2} de_i \prod_{i<j=1}^{s-1} (e_i - e_j), \end{aligned}$$

where $0 \leqq e_s \leqq e_{s-1} \leqq \cdots \leqq e_1 < \infty$, $s = \min(p, m)$, $t = \max(p, m)$, and

$$C(s, t, p, n) = \frac{\pi^{s/2}}{n^{st/2}} \prod_{i=1}^{s} \frac{\Gamma\{\frac{1}{2}(n + t - p + i)\}}{\Gamma\{\frac{1}{2}(t - s + i)\}\Gamma\{\frac{1}{2}(n - p + i)\}\Gamma(i/2)}.$$

The statistic T_0^2 is expressed as

$$(5.2)\qquad T_0^2 = m \operatorname{tr} S_1 S_0^{-1} = \sum_{i=1}^{s} e_i,$$

and the c.d.f. of T_0^2 is given by

$$\begin{aligned} &F(2\theta_1) = C(s, t, p, n) \int \cdots \int_{R_1} \\ (5.3)\qquad &\prod_{i=1}^{s} e_i^{(t-s-1)/2} \left(1 + \frac{e_i}{n}\right)^{-(m+n)/2} de_i \prod_{i<j=1}^{s-1} (e_i - e_j), \end{aligned}$$

where R_1 is the domain of integration such that $0 \leqq e_s \leqq e_{s-1} \leqq \cdots \leqq e_1 < \infty$ and $0 \leqq \sum_{i=1}^{s} e_i \leqq 2\theta_1$. Now for any non-negative values of e_i and n, the following inequality holds:

$$\log\left(1 + \frac{e_i}{n}\right) \leqq \frac{e_i}{n}$$

for $i = 1, \cdots, s$, where equality holds when $e_i = 0$ or $n \to \infty$. Hence we have

$$\prod_{i=1}^{s} \left(1 + \frac{e_i}{n}\right)^{-(m+n)/2} \geqq \exp\left[-\frac{m+n}{2n} \sum_{i=1}^{s} e_i\right].$$

Therefore, the probability law (5.1) is bounded from below as follows:

$$P_1(e_1, \cdots, e_s) \leqq P(e_1, \cdots, e_s) \tag{5.4}$$

where

$$P_1(e_1, \cdots, e_s) = C(s, t, p, n) \prod_{i=1}^{s} e_i^{(t-s-1)/2} \, de_i \exp\left[-\frac{m+n}{2n} \sum_{i=1}^{s} e_i\right] \prod_{i<j=1}^{s-1} (e_i - e_j).$$

It must be noted here that $P_1(e_1, \cdots, e_s)$ is not a probability law, although it is non-negative for all e_i such that $0 \leqq e_s \leqq \cdots \leqq e_1 < \infty$. Now integrating both sides of (5.4) in R_1 we obtain

$$F_1(2\theta_1) \leqq F(2\theta_1), \tag{5.5}$$

where

$$F_1(2\theta_1) = C(s, t, p, n) \int_{R_1} \cdots \int \prod_{i=1}^{s} e_i^{(t-s-1)/2} \, de_i \exp\left[-\frac{m+n}{2n} \sum_{i=1}^{s} e_i\right] \prod_{i<j=1}^{s-1} (e_i - e_j),$$

and also integrating both sides of (5.4) in R_2 where $0 \leqq e_s \leqq \cdots \leqq e_1 < \infty$ and $2\theta_1 \leqq \sum_{i=1}^{s} e_i < \infty$ and subtracting each from 1, we have

$$F(2\theta_1) \leqq F_2(2\theta_1), \tag{5.6}$$

where

$$F_2(2\theta_1) = 1 - C(s, t, p, n) \int_{R_2} \cdots \int \prod_{i=1}^{s} e_i^{(t-s-1)/2} \, de_i \exp\left[-\frac{m+n}{2n} \sum_{i=1}^{s} e_i\right] \prod_{i<j=1}^{s-1} (e_i - e_j).$$

In order to evaluate $F_1(2\theta_1)$ and $F_2(2\theta_1)$, we observe that as n tends to ∞, $T_0^2 = \sum_{i=1}^{s} e_i$ has a χ^2 distribution with st degrees of freedom in the limit; i.e., we have

$$K(s, t, p) \int_{R_1} \cdots \int \prod_{i=1}^{s} e_i^{(t-s-1)/2} \, de_i \exp\left[-\tfrac{1}{2} \sum_{i=1}^{s} e_i\right] \prod_{i<j=1}^{s-1} (e_i - e_j) = G_{\rho_1}(\theta_1), \tag{5.7}$$

where

$$K(s, t, p) = \lim_{n \to \infty} C(s, t, p, n) = \frac{\pi^{s/2}}{2^{st/2}} \frac{1}{\prod_{i=1}^{s} \Gamma\{\frac{1}{2}(t - s + i)\}\Gamma\left(\frac{i}{2}\right)}$$

and $\rho_1 = st/2$. Hence integration of (5.5) yields

$$F_1(2\theta_1) = L(s, t, p, n) G_{\rho_1}\left(\frac{m + n}{n} \theta_1\right), \tag{5.8}$$

where

$$L(s, t, p, n) = \frac{C(s, t, p, n)}{K(s, t, p)} \left(\frac{n}{m + n}\right)^{st/2} = \left(\frac{2}{m + n}\right)^{st/2} \prod_{i=1}^{s} \frac{\Gamma\left(\frac{n + t - p + i}{2}\right)}{\Gamma\left(\frac{n - p + i}{2}\right)}$$

Similarly we obtain from (5.6)

$$F_2(2\theta_1) = 1 - L(s, t, p, n)\left\{1 - G_{\rho_1}\left(\frac{m + n}{n}\right)\right\}. \tag{5.9}$$

Now if we write (4.3) as

$$F(2\theta_1) = a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + R_3, \tag{5.10}$$

where R_3 is the error committed by omitting all terms except the first three terms in the asymptotic series of $F(2\theta_1)$, the absolute value of R_3 has the following upper bound:

$$| R_3 | \leqq \max\left\{\left| F_1(2\theta_1) - a_0 - \frac{a_1}{n} - \frac{a_2}{n^2}\right|, \left| F_2(2\theta_1) - a_0 - \frac{a_1}{n} - \frac{a_2}{n^2}\right|\right\}, \tag{5.11}$$

where $F_1(2\theta_1)$ and $F_2(2\theta_1)$ are given by (5.8) and (5.9), respectively.

The actual manner in which (3.33) converges to the true value T_0^2 or in which (4.3) converges to the true value $F(2\theta_1)$ is not known, but it is hoped that the use of the first few corrective terms may result in a test which is more accurate than the χ^2 approximation, at any rate for moderately large values of n. In the case of the asymptotic formula for the c.d.f. of T_0^2 (4.3), we may judge the magnitude of the error involved in using the first few terms of the series by (5.11), which turns out to be rather small numerically when n is sufficiently large.

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