PÓLYA TYPE DISTRIBUTIONS, II[1]

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In a previous publication a specific smoothing property characterizing a class of distributions which we called Pólya Type (P.T.) distributions was introduced [1]. Most of the standard distributions occurring in statistical practice are of Pólya Type. For this class of distributions many of the usual decision theoretic questions were analyzed. Explicitly in the case of the two action problem, complete classes of statistical procedures were characterized and Bayes and admissible procedures were also determined. This paper continues the further development of statistical applications for Pólya type distributions. We are still principally concerned with the two action problem. In a subsequent publication the n-action and estimation problem for P.T. distributions will be presented.

Our investigation is divided into three main parts. Part I describes some new characterizations of P.T. distribution. Attention is called to Lemma 3 which is very useful in establishing the fundamental variation diminishing properties of P.T. distributions as described in Theorem 3. Finally, Part I closes with two further results about the sums of two random variables one of which has a P.T. distribution.

In Part II we examine in detail many of the standard Neyman-Pearson concepts for the case when the underlying distributions are known to be Pólya Type. Representative topics treated include the principle of unbiasedness, envelope power functions, likelihood ratio tests, etc. Specifically, it is shown that in any testing problem uniformly most powerful unbiased tests always exist and in fact can easily be explicitly constructed. Although we deal here with the case where there is only a single free parameter, for many examples a problem involving several parameters can be reduced to that of one parameter by using the principle of similarity or the principle of invariance. At this point our theory can be directly applied. Another interesting consequence of the theory is the result that the likelihood ratio test for a composite hypothesis versus a composite alternative when the underlying family of distributions are of P.T. is an admissible test.

A general minimax theorem for the two action decision problem is developed in Part III. Explicitly the game defined by the usual risk function is shown to have a value (under very mild conditions imposed on the loss functions). Furthermore, the optimal strategies for both the statistician and nature are characterized. Specific attention is directed to the one and two-sided testing problems. A discussion of the computational job for obtaining the minimax strategies is also given.

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Although the three chapters are basically related, they may be read separately with only few references to other parts.

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Part I. Definition and Properties of Pólya Type Distributions.

Sec. 1. Definitions and preliminaries.

Def. 1. A family of distributions $P(x, \omega)$

$$P(x, \omega) = \beta(\omega) \int_{-\infty}^{x} p(x, \omega) \, d\mu(x)$$

of a real random variable X depending on a real parameter ω is said to belong to the class $\mathcal{P}_n$ (Pólya Type n) if

$$\begin{vmatrix} p(x_1, \omega_1) & \cdots & p(x_1, \omega_m) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ p(x_m, \omega_1) & \cdots & p(x_m, \omega_m) \end{vmatrix} \geqq 0 \tag{1.1}$$

for every $1 \leqq m \leqq n$ and all $x_1 < x_2 < \cdots < x_m$, and $\omega_1 < \omega_2 < \cdots < \omega_m$. The family belongs strictly to $\mathcal{P}_n$ if strict inequality holds in (1.1). μ is a σ-finite measure on the real line and $p(x, \omega)$ is taken to be continuous in each variable. Most of the results can be easily extended to the case where we allow $p(x, \omega)$ to have a finite number of discontinuities of the first kind, in each variable separately

If the family of distributions $P(x, \omega)$ belongs to $\mathcal{P}_n$ for every n, then we say that the family belongs to $\mathcal{P}_\infty$. If it belongs strictly to $\mathcal{P}_n$ for every n, then it belongs strictly to $\mathcal{P}_\infty$. We shall sometimes say that $p(x, \omega)$ is Pólya Type $n(\infty)$ if $P(x, \omega)$ belongs to $\mathcal{P}_n(\mathcal{P}_\infty)$.

For $n = 1, 2$ the conditions of being Pólya Type n reduce to familiar ones. p is Pólya Type 1 (strictly Pólya Type 1) if and only if $p(x, \omega) \geqq 0 (> 0)$ for all x and ω. p is Pólya Type 2 if and only if it has a monotone likelihood ratio, i.e., for every $x_1 < x_2$, $[p(x_1, \omega)]/[p(x_2, \omega)]$ is nonincreasing in ω. It is strictly Pólya Type 2 if and only if it has a strict monotone likelihood ratio, i.e., $[p(x_1, \omega]/[p(x_2, \omega)]$ is decreasing in ω for $x_1 < x_2$.

The distributions that can be classified as Pólya Type include almost all of the principal distributions occurring in statistical practice. The exponential family, the noncentral t, the noncentral F, and the noncentral chi-square distributions all belong strictly to $\mathcal{P}_\infty$. For a proof of this the reader is referred to [1]. Other examples are given in [2]. The most notable example of a density which is not Pólya Type is the Cauchy, i.e.,

$$p(x, \omega) = \frac{1}{\pi} \frac{1}{1 + (x - \omega)^2}.$$

Sec. 2. Some characterizations of Pólya Type distributions. This section will be devoted to presenting some alternative characterizations and some analytic properties of Pólya Type distributions. Theorem 3 and its corollaries should be carefully noted because the decision theory for Pólya Type distributions developed alternately in [1] is based almost entirely on this theorem.

THEOREM 1. *If p is Pólya type 2 and the derivatives involved exist everywhere, then*

$$\begin{vmatrix} p(x_1, \omega) & \dfrac{\partial}{\partial \omega} p(x_1, \omega) \\ p(x_2, \omega) & \dfrac{\partial}{\partial \omega} p(x_2, \omega) \end{vmatrix} \geqq 0 \tag{2.2}$$

for all ω and all $x_1 < x_2$, and

$$\begin{vmatrix} p(x, \omega) & \dfrac{\partial}{\partial \omega} p(x, \omega) \\ \dfrac{\partial}{\partial x} p(x, \omega) & \dfrac{\partial^2}{\partial x \partial \omega} p(x, \omega) \end{vmatrix} \geqq 0 \tag{2.3}$$

for all ω and all x. Conversely, if $p(x, \omega) > 0$ for all x and ω, (2.3) *implies* (2.2), *which in turn implies that p is Pólya type* 2. *Strict inequality in* (2.3) *implies strict inequality in* (2.2) *and this implies that p is strictly Pólya Type* 2.

REMARK. The requirement that $p(x, \omega) > 0$ in the converse theorem can be greatly relaxed by use of a device which will be fully explained in connection with Theorem 2 below.

PROOF. $p \, \varepsilon \, \mathcal{P}_2$ implies that for all $x_1 < x_2$ and $\omega_1 < \omega_2$

$$\frac{1}{\omega_2 - \omega_1} \begin{vmatrix} p(x_1, \omega_1) & p(x_1, \omega_2) \\ p(x_2, \omega_1) & p(x_2, \omega_2) \end{vmatrix} = \begin{vmatrix} p(x_1 \omega_1) & \dfrac{p(x_1, \omega_2) - p(x_1, \omega_1)}{\omega_2 - \omega_1} \\ p(x_2, \omega_1) & \dfrac{p(x_2, \omega_2) - p(x_2, \omega_1)}{\omega_2 - \omega_1} \end{vmatrix} \geqq 0.$$

The limit as ω_2 approaches ω_1 gives (2.2). Also, (2.3) is obtained from (2.2) analogous to the preceding by operating on columns.

The converse is established by showing that p has a monotone likelihood ratio. Indeed, (2.3) can be written as

$$[p(x, \omega)]^2 \frac{\partial}{\partial x} \left\{ \frac{\dfrac{\partial}{\partial \omega} p(x, \omega)}{p(x, \omega)} \right\} \geqq 0 \tag{2.4}$$

for all x and ω. This implies that $(\partial/\partial\omega)p(x, \omega)/p(x, \omega)$ is nondecreasing in x for all ω. This yields (2.2) which in turn implies

$$[p(x_1, \omega)^2] \frac{\partial}{\partial \omega} \left\{ \frac{p(x_2, \omega)}{p(x_1, \omega)} \right\} \geqq 0 \tag{2.5}$$

for all $x_1 < x_2$. (2.5) implies $[p(x_2, \omega)]/[p(x_1, \omega)]$ is nondecreasing in ω for all $x_1 < x_2$; i.e., p has a monotone likelihood ratio.

The strict converse is obtained by replacing $\geqq$ by $>$, nondecreasing by increasing, and monotone by strictly monotone in the preceding paragraph.

COROLLARY 1. *Suppose* $(\partial^2/\partial x \partial \omega) \log p(x, \omega)$ *exists and* p *belongs strictly to* $\mathcal{P}_1$. *Then* p *belongs to* $\mathcal{P}_2$ *if and only if*

$$\frac{\partial^2}{\partial x \partial \omega} \log p(x, \omega) \geqq 0$$

for all x *and* ω.

Our attention is now directed to consider a generalization of Theorem 1 for density functions which are Pólya Type of arbitrary degree.

THEOREM 2. *If* p *is Pólya Type* m *and all the derivatives involved exist everywhere, then*

$$\begin{vmatrix} p(x_1, \omega) & \frac{\partial}{\partial \omega} p(x_1, \omega) & \cdots & \frac{\partial^{n-1}}{\partial \omega^{n-1}} p(x_1, \omega) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ p(x_n, \omega) & \frac{\partial}{\partial \omega} p(x_n, \omega) & \cdots & \frac{\partial^{n-1}}{\partial \omega^{n-1}} p(x_n, \omega) \end{vmatrix} \geqq 0 \tag{2.6}$$

for all $n \leqq m$, ω, *and* $x_1 < x_2 < \cdots < x_n$, *and*

$$\begin{vmatrix} p(x, \omega) & \frac{\partial}{\partial \omega} p(x, \omega) & \cdots & \frac{\partial^{n-1}}{\partial \omega^{n-1}} p(x, \omega) \\ \frac{\partial}{\partial x} p(x, \omega) & \frac{\partial^2}{\partial x \partial \omega} p(x, \omega) & \cdots & \frac{\partial^{n}}{\partial x \partial \omega^{n-1}} p(x, \omega) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{\partial^{n-1}}{\partial x^{n-1}} p(x, \omega) & \frac{\partial^{n}}{\partial x^{n-1} \partial \omega} p(x, \omega) & \cdots & \frac{\partial^{2n-2}}{\partial x^{n-1} \partial \omega^{n-1}} p(x, \omega) \end{vmatrix} \geqq 0 \tag{2.7}$$

for all $n \leqq m$, ω, *and* x. *Conversely, strict inequality in* (2.6) *for every* $1 \leqq n \leqq m$ *implies that* p *is strictly Pólya Type* m *and strict inequality in* (2.7) *for all* $1 \leqq n \leqq m$ *implies the same in* (2.6).

PROOF. We need the following lemma.

LEMMA 1. *If $f_1, f_2, \cdots, f_n$ are differentiable real-valued functions on the real line and $\xi_1 < \xi_2$, then there exists a $\bar{\xi}$, $\xi_1 < \bar{\xi} < \xi_2$, such that*

$$\begin{vmatrix} a_{11} & \cdots & a_{ij} & f_1(\xi_1) & f_1(\xi_2) & a_{1,j+3} & \cdots & a_{1n} \\ \cdot & & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & & \cdot \\ a_{n1} & \cdots & a_{nj} & f_n(\xi_1) & f_n(\xi_2) & a_{n,j+3} & \cdots & a_{nn} \end{vmatrix} = (\xi_2 - \xi_1) \begin{vmatrix} a_{11} & \cdots & a_{1j} & f_1(\xi_1) & f_1'(\bar{\xi}) & a_{1,j+3} & \cdots & a_{1n} \\ \cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot \\ a_{n1} & \cdots & a_{nj} & f_n(\xi_1) & f_n'(\bar{\xi}) & a_{1,j+3} & \cdots & a_{nn} \end{vmatrix},$$

where the a_{ij}'s are any real numbers.

The proof of this lemma is an easy application of the mean value theorem and will be omitted.

The proof of the first part of the theorem proceeds as follows. Let $p_i(\omega) = p(x_i, \omega)$ and $p_i^k(\omega) = (\partial^k/\partial\omega^k)p_i(\omega)$. Suppose n and $x_1 < x_2 < \cdots < x_n$ are given. For $\omega_1 < \omega_2 < \cdots < \omega_n$

$$(2.8) \quad 0 \leqq \operatorname{sgn} \begin{vmatrix} p_1(\omega_1) & \cdots & p_1(\omega_n) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ p_n(\omega_1) & \cdots & p_n(\omega_n) \end{vmatrix} = \operatorname{sgn} \begin{vmatrix} p_1(\omega_1) & p_1^1(\omega_2^1) & p_1^2(\omega_3^2) & p_1^3(\omega_4^3) & \cdots & p_1^{n-1}(\omega_n^{n-1}) \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ p_n(\omega_1) & p_n^1(\omega_2^1) & p_n^2(\omega_3^2) & p_n^3(\omega_4^3) & \cdots & p_n^{n-1}(\omega_n^{n-1}) \end{vmatrix},$$

where $\omega_{i-1}^{j-1} \leqq \omega_i^j \leqq \omega_i^{j-1}$. This equality is obtained by repeated application of Lemma 1. Sgn is the function which equals $+1$ if its argument is positive, -1 if its argument is negative, and 0 if its argument is zero. Letting $\omega_2 \to \omega_1$, $\omega_3 \to \omega_1, \cdots, \omega_n \to \omega_1$, the last determinant approaches the determinant in (2.6) and therefore (2.6) must hold. (2.7) is derived from (2.6) by applying the same operations on the rows of the determinants in (2.6) as were applied to the columns of the first determinant in (2.8).

The proof of the converse of this theorem depends on Lemma 2 below. Lemmas 3 and 4 which will be needed subsequently in proving Theorem 3 are also included at this point because of the similarity of their proofs with the proof of Lemma 2.

LEMMA 2. *If all the derivatives involved exist and are continuous and*

$$\text{(2.9)} \qquad \begin{vmatrix} p_1(w) & p_1^1(w) & \cdots & p_n^{n-1}(w) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ p_n(w) & p_n^1(w) & \cdots & p_n^{n-1}(w) \end{vmatrix} > 0$$

for all w and $n \leqq m$ where p_i^j is the jth derivative of p_i, then

$$\text{(2.10)} \qquad \begin{vmatrix} p_1(w_1) & \cdots & p_1(w_n) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ p_n(w_1) & \cdots & p_n(w_n) \end{vmatrix} > 0$$

for all $w_1 < \cdots < w_n$ and all $n \leqq m$.

PROOF. The proof proceeds by mathematical induction. Clearly the lemma holds for $m = 1$. Suppose it holds up to $m - 1$, and suppose $w_1 < w_2 < \cdots < w_n$ are given. Let $q_i(w) = p_i(w)/p_1(w)$. Then

$$\text{(2.11)} \qquad \operatorname{sgn} \begin{vmatrix} p_1(w_1) & \cdots & p_1(w_1) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ p_1(w_1) & \cdots & p_1(w_1) \end{vmatrix} = \operatorname{sgn} \begin{vmatrix} 1 & \cdots & 1 \\ q_2(w_1) & \cdots & q_2(w_n) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ q_n(w_1) & \cdots & q_n(w_n) \end{vmatrix}$$

$$= \operatorname{sgn} \begin{vmatrix} 1 & 0 & \cdots & 0 \\ q_2(w_1) & q_2^1(u_2) & \cdots & q_2^1(u_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ q_n(w_1) & q_n^1(u_2) & \cdots & q_n^1(u_n) \end{vmatrix},$$

where $w_1 < u_2 < w_2 < u_3 < \cdots < u_n < w_n$. But

$$\frac{d^j}{dw^j}\frac{f(w)}{p_1(w)} = \frac{1}{p_1(w)}\frac{d^j}{dw^j}f(w) + \sum_{k=1}^{j-1} a_k(w)\frac{d^k}{dw^k}f(w)$$

where the $a_k(w)$'s do not depend on f. Therefore, for all w

$$\text{(2.12)}\quad \operatorname{sgn}\begin{vmatrix} p_1(w) & p_1^1(w) & \cdots & p_1^{n-1}(w) \\ p_2(w) & p_2^1(w) & \cdots & p_2^{n-1}(w) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ p_n(w) & p_n^1(w) & \cdots & p_n^{n-1}(w) \end{vmatrix} = \operatorname{sgn}\begin{vmatrix} p_1(w) & 0 & \cdots & 0 \\ p_2(w) & q_2^1(w) & \cdots & q_2^{n-1}(w) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ p_n(w) & q_n^1(w) & \cdots & q_n^{n-1}(w) \end{vmatrix}.$$

Since the first determinant in (2.12) is positive by assumption, for all w and $n \leqq m$

$$\begin{vmatrix} q_2^1(w) & \cdots & q_2^{n-1}(w) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ q_n^1(w) & \cdots & q_n^{n-1}(w) \end{vmatrix} > 0$$

By the induction assumption this implies

$$\begin{vmatrix} q_2^1(u_2) & \cdots & q_2^1(u_n) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ q_n^1(u_2) & \cdots & q_n^1(u_n) \end{vmatrix} > 0$$

for all $u_2 < \cdots < u_n$ and all $n \leqq m$. But this determinant equals the last determinant in (2.11) so (2.10) follows.

LEMMA 3. *If p is strictly Pólya Type ∞ and all the derivatives involved exist and are continuous, then*

$$\operatorname{sgn}\det \| p(x_j, \omega_i) \| = \operatorname{sgn} q_n(\xi, \omega_{n+1}, \omega_n, \cdots, \omega_1),$$

$x_1 < x_2 < \cdots < x_{n+1}$, for some appropriate ξ satisfying $x_1 < \xi < x_{n+1}$,

where

$$q_1(\xi, \omega, \omega_1) = \frac{d}{dx}\left\{\frac{p(\xi, \omega)}{p(\xi, \omega_1)}\right\}, \quad q_2(\xi, \omega, \omega_2, \omega_1) = \frac{d}{dx}\left\{\frac{\frac{d}{dx}\frac{p(\xi, \omega)}{p(\xi, \omega_1)}}{\frac{d}{dx}\frac{p(\xi, \omega_2)}{p(\xi, \omega_1)}}\right\},$$

and

$$q_k(\xi, \omega, \omega_k, \omega_{k-1}, \cdots, \omega_1) = \frac{d}{dx}\left\{\frac{q_{k-1}(\xi, \omega, \omega_{k-1}, \cdots, \omega_1)}{q_{k-1}(\xi, \omega_k, \omega_{k-1}, \cdots, \omega_1)}\right\},$$

where $\omega_1 < \omega_2 < \cdots < \omega_n$ *but* ω *and* ω_{n+1} *are allowed to occur anywhere. (The notation means that the derivatives are taken with respect to* x *and evaluated at* $x = \xi$.)

PROOF. The proof proceeds by induction. Let $p(x_j, \omega_i) = p_i(x_j)$. For $n = 1$

$$\operatorname{sgn}\begin{vmatrix} p_1(x_1) & p_1(x_2) \\ p_2(x_1) & p_2(x_2) \end{vmatrix} = \operatorname{sgn}\begin{vmatrix} 1 & 1 \\ \dfrac{p_2(x_1)}{p_1(x_1)} & \dfrac{p_2(x_2)}{p_1(x_2)} \end{vmatrix} = \operatorname{sgn}\begin{vmatrix} 1 & 0 \\ \dfrac{p_2(x_1)}{p_1(x_1)} & \dfrac{d}{dx}\dfrac{p_2(\xi)}{p_1(\xi)} \end{vmatrix}$$

by Lemma 1. Assume the theorem is true for $n - 1$. Then

$$\operatorname{sgn}\begin{vmatrix} p_1(x_1) & \cdots & p_1(x_{n+1}) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ p_{n+1}(x_1) & \cdots & p_{n+1}(x_{n+1}) \end{vmatrix} = \operatorname{sgn}\begin{vmatrix} 1 & 0 & & 0 \\ \dfrac{p_2(x_1)}{p_1(x_1)} & q_1(\xi_2^1, \omega_2, \omega_1) & \cdots & q_1(\xi_{n+1}^1, \omega_2, \omega_1) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \dfrac{p_{n+1}(x_1)}{p_1(x_1)} & q_1(\xi_2^1, \omega_{n+1}, \omega_1) & \cdots & q_1(\xi_{n+1}^1, \omega_{n+1}, \omega_1) \end{vmatrix}$$

$$= \operatorname{sgn}\begin{vmatrix} q_1(\xi_2^1, \omega_2, \omega_1) & \cdots & q_1(\xi_{n+1}^1, \omega_2, \omega_1) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ q_1(\xi_2^1, \omega_{n+1}, \omega_1) & \cdots & q_1(\xi_{n+1}^1, \omega_{n+1}, \omega_1) \end{vmatrix},$$

where $x_{i-1} < \xi_i^1 < x_i$, $i = 2, \cdots, n + 1$. By the induction hypothesis

$$q_1(\xi_j^1, \omega_2, \omega_1) > 0 \text{ for } j = 2, \cdots, n + 1.$$

Therefore dividing each row by the first and applying Lemma 1,

$$\operatorname{sgn}\begin{vmatrix} q_1(\xi_2^1, \omega_2, \omega_1) & \cdots & q_1(\xi_{n+1}^1, \omega_2, \omega_1) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ q_1(\xi_2^1, \omega_{n+1}, \omega_1) & \cdots & q_1(\xi_{n+1}^1, \omega_{n+1}, \omega_1) \end{vmatrix}$$

$$= \operatorname{sgn} \begin{vmatrix} q_2(\xi_3'', \omega_3, \omega_2, \omega_1) & \cdots & q_2(\xi_{n+1}'', \omega_3, \omega_2, \omega_1) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ q_2(\xi_3'', \omega_{n+1}, \omega_2, \omega_1) & \cdots & q_2(\xi_{n+1}'', \omega_{n+1}, \omega_2, \omega_1) \end{vmatrix},$$

where $\xi_{i-1}^1 < \xi_i'' < \xi_i^1$, $i = 3, \cdots, n + 1$. Continuing in this manner and at each step using the fact that the first row is positive by the induction assumption the sequence terminates with the expression $q_n(\xi, \omega_{n+1}, \omega_n, \cdots, \omega_1)$, $x_1 < \xi < x_{n+1}$, which has the same sign as the original determinant.

ξ is a real number which occurs between x_1 and x_{n+1} and depends on $x_1, x_2, \cdots, x_{n+1}$. Det $\| p(x_j, \omega_i) \|$ will always have the same sign, regardless of the values of the x_j's and ω_i's, just so long as the same order relation exists between them. By the continuity assumptions in Lemma 3, ξ can be found to take on any arbitrary real value by varying $x_1, x_2, \cdots, x_{n+1}$. Hence we have the following lemma.

LEMMA 3a. *If the conditions of Lemma* 3 *are satisfied, then*

$$\operatorname{sgn} \det \| p(x_j, \omega_i) \| = \operatorname{sgn} q_n(x, \omega_{n+1}, \omega_n, \cdots, \omega_1),$$

where $x_1 < x_2 < \cdots < x_{n+1}$, $\omega_1 < \omega_2 < \cdots < \omega_n$, *and where* ω_{n+1} *is allowed to occur anywhere, and* x *is any real number.*

The proof of the converse to Theorem 2 follows readily from Lemma 2. In fact, strict inequality in (2.6) for every $n \leqq m$ implies that p is strictly Pólya Type m by Lemma 2 with $p_i(w) = p(x_i, \omega)$, ω playing the role of w in Lemma 2. Strict inequality in (2.7) implies strict inequality in (2.6) by Lemma 2 with $p_i(w) = (\partial^{i-1}/\partial\omega^{i-1})\, p(x, \omega)$, ω being fixed and x playing the role of w in Lemma 2.

The converse statement in Theorem 2 involves strict inequality in (2.6) and (2.7). What can be said if just (2.6) and (2.7) hold? A positive result can be achieved if the following slight condition holds. If relations (2.6) and (2.7) are valid and if for every ω and $n \leqq m$ they hold with strict inequality for some $x_1 < x_2 < \cdots < x_n$, which may depend upon ω, then we can generally still prove that p is Pólya Type m by use of the following device. Let

$$p_\sigma(x, \omega) = \frac{\int_{-\infty}^{\infty} \phi(x - u, \sigma)\, p(u, \omega)\, d\mu(u)}{\int_{-\infty}^{\infty} \phi(x - u, \sigma)\, d\mu(u)}, \tag{2.13}$$

where ϕ is the normal density function with mean 0 and variance σ. The measure μ is chosen so that the integrals exist and are positive with μ possessing positive measure everywhere. As $\sigma \to 0$, $p_\sigma(x, \omega) \to p(x, \omega)$ uniformly in any finite interval and

$$\frac{\partial}{\partial \omega} p_\sigma(x, \omega) = \frac{\int_{-\infty}^{\infty} \phi(x - u, \sigma) \frac{\partial}{\partial \omega} p(u, \omega)\, d\mu(u)}{\int_{-\infty}^{\infty} \phi(x - u, \sigma)\, d\mu(u)} \to \frac{\partial}{\partial \omega} p(x, \omega), \tag{2.14}$$

where we have assumed that (2.14) is valid, i.e., the integral can be differentiated inside the integral sign. But[2]

$$\begin{vmatrix} p_\sigma(x_1, \omega) & \frac{\partial}{\partial \omega} p_\sigma(x_1, \omega) & \cdots & \frac{\partial^{n-1}}{\partial \omega^{n-1}} p_\sigma(x_1, \omega) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ p_\sigma(x_n, \omega) & \frac{\partial}{\partial \omega} p_\sigma(x_n, \omega) & \cdots & \frac{\partial^{n-1}}{\partial \omega^{n-1}} p_\sigma(x_n, \omega) \end{vmatrix}$$

$$= \frac{1}{\left(\int_{-\infty}^{\infty} \phi(x - u, \sigma)\, d\mu(u)\right)^n} \int \cdots \int_{u_1 < \cdots < u_n} \begin{vmatrix} \phi(u_1 - x_1, \sigma) & \cdots & \phi(u_1 - x_n, \sigma) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \phi(u_n - x_1, \sigma) & \cdots & \phi(u_n - x_n, \sigma) \end{vmatrix}$$

$$\cdot \begin{vmatrix} p(u_1, \omega) & \frac{\partial}{\partial \omega} p(u_1, \omega) & \cdots & \frac{\partial^{n-1}}{\partial \omega^{n-1}} p(u_1, \omega) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ p(u_n, \omega) & \frac{\partial}{\partial \omega} p(u_n, \omega) & \cdots & \frac{\partial^{n-1}}{\partial \omega^{n-1}} p(u_n, \omega) \end{vmatrix} d\mu(u_1) \cdots d\mu(u_n).$$

Since ϕ is strictly Pólya Type ∞ for each σ, the first determinant in the integrand is always positive. By assumption the second determinant is not identically zero. Therefore p_σ satisfies the determinant criterion for strictly Pólya Type m densities by Theorem 2, and p is Pólya Type m since $\det \| p(x_i, w_j) \| = \lim_{\sigma \to 0} \det \| p_\sigma(x_i, \omega_j) \|$.

[2] See G. Pólya and G. Szegö, *Aufgaben und Lehrsatze aus der Analysis*, Vol. 1, p. 48, Problem 68.

This completes the various characterizations of Pólya Type distributions that will be given here. The remaining theorems and corollaries summarize the main properties of Pólya Type distributions. Theorem 3 and its corollaries are crucial to the decision theory in Parts II and III.

Sec. 3. Basic oscillation theorem for P. T. distribution. The following definition is needed to make the concepts in the theorem precise.

Def. 2: The number of sign changes $V(h)$ of a function $h(\omega)$ is taken to be $\sup_{\omega_1,\cdots\omega_m} N(h(\omega_i))$, where $N(h(\omega_i))$ is the number of changes of sign of the sequence $h(\omega_1), h(\omega_2), \cdots, h(\omega_n)$, $\omega_i < \omega_{i+1}$. A point ω_0 is called a change point for $h(\omega)$ if $h(\omega)\, h(\omega') \leqq 0$ whenever $\omega \leqq \omega_0 \leqq \omega'$ with $\omega \neq \omega'$ (ω, ω' essentially near ω_0) and definite inequality occurs for some specific choice of ω and ω' *or* $h(\omega_0)\, h(\omega)\, h(\omega') \leqq 0$ for $\omega < \omega_0 < \omega'$.

THEOREM 3. *Let p be strictly Pólya Type ∞ and assume that p can be differentiated n times with respect to x for all ω. Let F be a measure on the real line, and let h be a function of ω which changes sign n times. If*

$$g(x) = \int p(x, \omega) h(\omega)\, dF(\omega)$$

can be differentiated n times with respect to x inside the integral sign, then g changes sign at most n times and has at most n zeros counting multiplicities or is identically zero. The function g is identically zero if and only if the spectrum of F is contained in the set of zeros of h.

PROOF. Let $\omega_1, \omega_2, \cdots, \omega_n$ be the change points of h. $\omega_1 < \omega_2 < \cdots < \omega_n$. Form

$$\frac{d}{dx}\left\{\frac{g(x)}{p(x, \omega_1)}\right\} = \int \frac{d}{dx} \frac{p(x, \omega)}{p(x, \omega_1)}\, h(\omega)\, dF(\omega)$$

$$\begin{matrix} \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \end{matrix} \tag{2.15}$$

$$q_n^*(x, \omega_n, \omega_{n-1}, \cdots, \omega_1) = \int q_n(x, \omega, \omega_n, \cdots, \omega_1) h(\omega)\, dF(\omega).$$

The function $q_n^*(x, \omega_n, \cdots, \omega_1)$ is the function $q_n(x, \omega, \omega_n, \cdots, \omega_1)$ with $p(x, \omega)$ replaced by $g(x)$. All the above integrands are well-defined since Lemma 3 can be applied.

Suppose for definiteness $h(\omega) > 0$ for $\omega < \omega_1$ and n even. Then det $\| p(x_j, \omega_i) \|$, $i, j = 1, 2, \cdots, n + 1$, with $\omega_{n+1} = \omega$, and $x_1 < x_2 < \cdots < x_{n+1}$ has the same sign as the determinant obtained from the above with first and last rows interchanged. This last determinant is positive as p is assumed to be Pólya Type ∞. Hence, by Lemma 3 $q_n(x, \omega, \omega_n, \cdots, \omega_1) > 0$ for $\omega < \omega_1$ so the integrand in (2.15) is positive. For $\omega_1 < \omega < \omega_2$ with $\omega = \omega_{n+1}$ the original det $\| p(x_j, \omega_i) \|$ has the opposite sign of the determinant which has the last row inserted between the first and second rows in this determinant. This second determinant is positive so det $\| p(x_j, \omega_i) \| < 0$. Applying Lemma 3 again we see that the integrand is positive for $\omega_1 < \omega < \omega_2$. Repeating this line of argument we find that $h(\omega)$ and $q_n(x, \omega, \omega_n, \cdots, \omega_1)$ have the same sign so the integrand in (2.15) is always positive. Therefore $q_n^*(x, \omega_n, \omega_{n-1}, \cdots, \omega_1) > 0$ for all x. Now, $q_{n-1}(x, \omega_{n-1}, \omega_{n-2}, \cdots, \omega_1)$ is likewise positive for all x, by Lemma 3. Since $q_n^*(x, \omega_n, \cdots, \omega_1) > 0$ and $q_{n-1}(x, \omega_{n-1}, \cdots, \omega_1) > 0$ for all x, from the definition of $q_{n-1}^*(x, \omega_{n-1}, \cdots, \omega_1)$ we deduce that this function changes sign at most once and has at most one zero. Similarly, since $q_{n-2}(x, \omega_{n-2}, \cdots, \omega_1) > 0$ for all x, this implies that $q_{n-2}^*(x, \omega_{n-2}, \cdots, \omega_1)$ changes sign at most twice and has at most two zeros counting multiplicities. The end of this sequence of implications is that $g(x)$ changes sign at most n times and has at most n zeros counting multiplicities.

Suppose $h(\omega) > 0$ for $\omega < \omega_1$, but n is odd. Then reasoning analogous to that used in the even case shows that the integrand in (2.15) is always negative. Thus $q_n^*(x, \omega, \omega_n, \cdots, \omega_1) < 0$ for all x, and this implies the desired conclusion. A similar argument proves the result when $h(\omega) < 0$ for $\omega < \omega_1$.

By following the sequence of implications in reverse order it can be checked that if g changes sign n times, then it changes sign in the same order as $h(\omega)$. This gives us Corollary 2.

COROLLARY 2. *If the number of sign changes of* g *is* $n = V(h)$, *then* g *and* h *change signs in the same order.*

COROLLARY 3. *If* p *is Pólya Type* ∞ *but not strictly so, the results of Theorem* 3 *still hold if for any* n *and any prescribed* $\omega_1 < \cdots < \omega_n$ *there exists a set of* $x_1 < \cdots < x_n$ *(which may depend on* $\omega_1, \cdots, \omega_n$*) such that* det $\| p(x_i, \omega_j) \| > 0$.

This can be established by approximating $p(x, \omega)$ by $p_\sigma(x, \omega)$ as in (2.13).

The condition that p be strictly Pólya Type ∞ can be weakened also in another manner different from Corollary 3. The results of the theorem still hold if p is strictly Pólya Type $n + 1$, one more than the number of sign changes of h.

Completely analogous results can also be proved about the function

$$g(\omega) = \int p(x, \omega)h(x)\, d\mu(x).$$

Sec. 4. Addition theorem for P. T. distribution. The following two theorems present results which are interesting per se but which will not be of any use in the subsequent sections. These theorems illustrate some other nice smoothening properties possessed by Pólya frequency functions.

THEOREM 4. *Let* X *and* Y *be independent real random variables having continuously differentiable densities* f *and* g, *and let* $f(x - \omega) = f^*(x, \omega)$ *be strictly Pólya Type* ∞. *If* g *has* k *modes, then the density of* $z = x + y$,

$$h(z) = \int_{-\infty}^{\infty} f(t)g(z - t)\, dt,$$

has at most k *modes. Furthermore, for any constant* c, $h - c$ *changes sign no more often than* $g - c$.

To expedite the discussion we assume that differentiation can be performed underneath the integral.

PROOF.

$$\frac{d}{dz} h(z) = \int_{-\infty}^{\infty} f(t)g'(x - t)\, dt = \int_{-\infty}^{\infty} f(x - y)g'(y)\, dy.$$

Since the number of modes of a density is bounded above by the number of changes of sign of its derivative, the first conclusion follows from Theorem 3.

The second conclusion also follows from Theorem 3 since

$$h(z) - c = \int_{-\infty}^{\infty} f(t)[g(z - t) - c]\, dt = \int_{-\infty}^{\infty} f(x - y)[g(y) - c]\, dy.$$

Z. W. Birnbaum calls a real random variable X less peaked than another real random variable Y if $\Pr\{ |X| \leqq u\} \leqq \Pr\{ |Y| \leqq u\}$ for all $u > 0$. He proves that if X is less peaked than Y and Z is independent of X and Y and has a symmetric unimodal density, then $X + Z$ is less peaked than $X + Y$ (Ref. [3]). We can generalize this definition and with the aid of Theorem 3 generalize the result.

Def. 3. A real random variable X is less peaked of order n than another real random variable Y if $q(u) = \Pr\{ |X| \leqq u\} - \Pr\{ |Y| \leqq u\}$ changes sign n times and is nonpositive for sufficiently large u or else changes sign less than n times.

Birnbaum's definition of less peaked corresponds to less peaked of order 0.

THEOREM 5. *Let X be less peaked of order n than Y. If Z is independent of X and Y and has a density h which is symmetric and is such that $h(z - \omega) = h^*(z, \omega)$ is strictly Pólya Type ∞, then $X + Z$ is less peaked of order n than $Y + Z$.*

PROOF. If F and G are the cdf's of X and Y respectively, then

$$\Pr\{ |X + Z| \leqq u\} - \Pr\{ |Y + Z| \leqq u\}$$
$$= \int_{-\infty}^{\infty} [F(s) - F(-s) - G(s) + G(-s)]h(u - s)\, ds.$$

The first factor in the integrand is an odd function of s which changes sign at most n times for positive s and hence at most $2n + 1$ times altogether. The second factor is a symmetric, strict Pólya Type ∞ density function. By Theorem 3 the integral is an antisymmetric function of u which changes sign at most $2n + 1$ times and hence at most n times for positive u. Furthermore, if it changes sign n times for positive u, it changes $2n + 1$ times altogether and must therefore have the same sign for very large u that $F(s) - F(-s) - G(s) + G(-s)$ does for very large s.

Part II. Application of Pólya Type Distributions to Classical Results of the Neyman Pearson Variety.

Sec. 1. Preliminaries. A number of classical results can be derived when the underlying distribution is Pólya Type. These results concern Type A regions, uniformly most powerful tests, unbiased tests, the likelihood ratio test, etc. They unify and strengthen essentially all previously known results. A great deal of the literature on the theory of testing statistical hypotheses deals with special cases [4], whereas this approach is of a more general nature and yields much stronger results and at the same time constructive methods in determining the specialized tests.

The general situation we are dealing with is that of testing a null hypothesis against its alternative hypothesis, i.e., a 2-action problem. The parameter space Ω is the real line. There exist two measurable loss functions L_1 and L_2 on Ω where $L_i(\omega)$ is the loss incurred if action i is taken and ω is the true parameter point. The set in which $L_1(\omega) < L_2(\omega)$ is the set in which action 1 is preferred when ω is the true state of nature, and the set in which $L_2(\omega) < L_1(\omega)$ is the set in which action 2 is preferred. The two actions are indifferent at all other points. We shall assume that $L_1 - L_2 = h$ changes sign exactly n times where n will vary according to the problem we are considering but will remain constant within each problem. The points where $L_1 - L_2$ changes sign are assumed isolated and are $\omega_1^0, \omega_2^0, \cdots, \omega_n^0$. For the sake of definiteness we shall assume that $L_1(\omega) - L_2(\omega)$ is positive for $\omega < \omega_1^0$. Two successive ω_i^0's may be equal but not more than two. In fact, if $\omega_i^0 = \omega_{i+1}^0$, then $[L_1(\omega) - L_2(\omega)][L_1(\omega') - L_2(\omega')] > 0$ for $\omega < \omega_i^0 < \omega'$ (ω, ω' near ω_i^0) and $[L_1(\omega) - L_2(\omega)]. [L_1(\omega_i^0) - L_2(\omega_i^0)] < 0$ for the same choice of ω. This corresponds to the case where one action is preferred in a neighborhood of ω_i^0 except for $\omega = \omega_i^0$ where the other action is preferred.

Let ϕ be a randomized decision procedure. ϕ is a measurable function on the real line, and $\phi(x)$ is the probability of taking action 2 (accepting the alternative hypothesis) if x is the observed value of the real random variable X. (x is usually a sufficient statistic based on several observations.) Consider decision procedures ϕ of the form

$$\phi(x) = \begin{cases} 1 \text{ for } x_{2i} < x < x_{2i+1}, & i = 0, 1, \cdots, \left[\dfrac{n}{2}\right] \\ \lambda_j \text{ for } x = x_j, 0 \leqq \lambda_j \leqq 1, & j = 1, 2, \cdots, n \\ 0 \text{ elsewhere} \end{cases}$$

$[a]$ denotes the greatest integer $\leqq a$. $x_0 = -\infty$. All randomized decision procedures of this form will be said to belong to the class $\mathfrak{M}_n$ of monotone procedures. If the x_j's are all distinct, then action 2 is preferred in n intervals, action 1 in n or $n - 1$, and at n points there is possible randomization. Strategies with fewer intervals but essentially the same form also belong to $\mathfrak{M}_n$; this corresponds to the case where the x_j's are not all distinct.

The following theorem and lemma will be used in the subsequent discussion. For proofs and greater detail the reader is referred to [1]. It should be remarked that the proofs of Theorem 6 and Lemma 4 can be based essentially on Theorem 3.

THEOREM 6. *If $p(x, \omega)$ belongs strictly to $\mathcal{P}_{n+1}$, then for any randomized decision* *procedure* ϕ *not in* $\mathfrak{M}_n$ *there exists a unique* ϕ^0 *in* $\mathfrak{M}_n$ *such that* $\rho(\omega, \phi^0) \leqq \rho(\omega, \phi)$ *with inequality everywhere except for* $\omega = \omega_0^1, \omega_2^0, \cdots, \omega_n^0$. ρ *is given by* $\rho(\omega, \phi) = \int[(1 - \phi(x))L_1(\omega) + \phi(x) L_2(\omega)]\, p(x, \omega)\, d\mu(x)$. *Moreover, the set* $\mathfrak{M}_n$ *constitutes a minimal complete class of strategies.*

If the underlying distribution $p(x, \omega)$ does not strictly belong to $\mathcal{P}_{n+1}$, then the strategies of $\mathfrak{M}_n$ still constitute a complete class but the uniqueness and minimality conclusion of Theorem 6 is not valid in general. However, by a general device of approximating non strict Pólya Type distributions by strict Pólya Type (see Part I), many of the foregoing results can be extended. This shall be left as an exercise for the reader.

Lemma 4. *If* ϕ^1 *and* ϕ^2 *are two strategies in* $\mathfrak{M}_n$ *and* p *is strictly Pólya Type* $n + 1$, *then*

$$\int [\phi^1(x) - \phi^2(x)]p(x, \omega)\, d\mu(x)$$

has less than n *zeros counting multiplicities.*

In the future, when we say assume strictly Pólya Type n, we mean that the underlying distribution belongs strictly to $\mathcal{P}_n$.

Sec. 2. Uniformly most powerful one-sided tests. The case of uniformly most powerful tests for the classical exponential family of distributions and other specific examples was treated in Lehman's notes [4]. This represents a slight extension to the situation of Pólya Type distributions.

Assume strictly $\mathcal{P}_2$. For a one-sided testing problem, a uniformly most powerful level α test exists.

A one-sided testing problem occurs when

$$L_1(\omega) = \begin{cases} 1 & \omega < \omega_1 \\ 0 & \omega \geqq \omega_1 \end{cases} \quad \text{and} \quad L_2(\omega) = \begin{cases} 0 & \omega < \omega_1 \\ 1 & \omega \geqq \omega_1 \end{cases}$$

for some ω_1. Then $\rho(\omega, \phi) = \int\phi(x)p(x, \omega)\, d\mu(x)$ for $\omega \geqq \omega_1$ and $\rho(\omega, \phi) = \int (1 - \phi(x))\, p(x, \omega)\, d\mu(x)$ for $\omega < \omega_1$. Consider the function $f_\phi(\omega) = \int\phi(x)p(x, \omega)\, d\mu(x)$ where $\phi \varepsilon \mathfrak{M}_1$. $f_\phi(\omega) - c = \int(\phi(x) - c)\, p(x, \omega)\, d\mu(x)$ where c is an arbitrary positive constant. Since $\phi \varepsilon \mathfrak{M}_1$, $\phi - c$ changes sign at most once and in the direction from $+$ to $-$ if at all. Therefore by Theorem 3 and Corollary 2, $f_\phi - c$ changes sign at most once and in the same direction if at all. This implies that f_ϕ is a monotone decreasing function of ω. Consider that unique monotone test ϕ^* (unique $[\mu]$) for which $f_{\phi^*}(\omega_1) = \int\phi^*(x)p(x, \omega_1)\, d\mu(x) = \alpha$. For any other level α monotone test ϕ_1 the corresponding f_{ϕ_1} is uniformly smaller than f_{ϕ^*} by Lemma 4 so that ϕ^* is best among the monotone tests. Now consider any non-monotone level α test ϕ. By Theorem 6 there is a unique monotone test ϕ_2 which is better than ϕ except at ω_1, where equality holds. But since $f_{\phi^*} \geqq f_{\phi_2}$, ϕ^* also improves on ϕ.

Sec. 3. Nonexistence of uniformly most powerful two-sided tests. Assume strictly $\mathcal{P}_3$. For the two-sided testing problem uniformly most powerful level α tests do not exist in general. We discuss this now in greater detail.

A two-sided testing problem is determined by

$$L_1(\omega) = \begin{cases} 0 & \omega_1 \leqq \omega \leqq \omega_2 \\ 1 & \text{elsewhere} \end{cases} \quad \text{and} \quad L_2(\omega) = \begin{cases} 1 & \omega_1 \leqq \omega \leqq \omega_2 \\ 0 & \text{elsewhere} \end{cases}$$

for some $\omega_1 \leqq \omega_2$. By virtue of Theorem 6 we can restrict our consideration exclusively to monotone tests, i.e., tests in $\mathfrak{M}_2$. Let ϕ_3 be a monotone test and $f_{\phi_3}(\omega) = \int \phi_3(x)\, p(x, \omega)\, d\mu(x)$ be the corresponding power function. Consider the one-sided testing problem obtained from the two-sided problem above by removing one tail.

$$L_1^*(\omega) = \begin{cases} 0 & \omega \geqq \omega_1 \\ 1 & \omega < \omega_1 \end{cases} \quad \text{and} \quad L_2^*(\omega) = \begin{cases} 1 & \omega \geqq \omega_1 \\ 0 & \omega < \omega_1 \end{cases}$$

The existence of a u.m.p. level α test ϕ^* for this problem was shown in section 2. Suppose $\phi_3 \notin \mathfrak{M}_1$. Then $f_{\phi^*}(\omega) > f_{\phi_3}(\omega)$ for $\omega < \omega_1$, and ϕ_3 is not u.m.p. for the two-sided test. The strict inequality is assured by Theorem 6. Suppose $\phi_3 \in \mathfrak{M}_1$. Then f_{ϕ_3} is monotone decreasing which means that for $\omega > \omega_2$ the test $\phi \equiv \alpha$ is better.

A word should be said about what happens when p is not strictly Pólya Type 3. When $\omega_1 = \omega_2$ and $P(x, \omega)$ is the rectangular distribution on the interval $[0, \omega]$, a u.m.p. test exists. The acceptance region is $[x', \omega_1]$ where $x' = \alpha\omega_1$. When $P(x, \omega)$ is the rectangular distribution on $[0, \omega]$ but if $\omega_1 < \omega_2$, no u.m.p. test exists; i.e., when the null hypothesis is an interval no u.m.p. test exists. The rectangular distribution on $[0, \omega]$ is Pólya Type 3, but it is not strictly so. It even satisfies the condition that for every $\omega_1 < \omega_2 < \omega_3$ there exists a set $x_1 < x_2 < x_3$ such that $\det \| p(x_i, \omega_j) \| > 0$. Thus the condition of strictness in this result seems very essential.

Sec. 4. Uniformly most powerful unbiased tests.

(a) Assume strictly $\mathcal{P}_3$. For a two-sided testing problem a u.m.p. unbiased test exists. For this special testing problem the result is known for scattered examples.

A test ϕ is unbiased if and only if $f_\phi(\omega) \leqq \alpha$ for $\omega_1 \leqq \omega \leqq \omega_2$ and $f_\phi(\omega) \geqq \alpha$ for $\omega \leqq \omega_1$ and $\omega \geqq \omega_2$. Consider the test $\phi \equiv \alpha$. By Theorem 6 there exists a unique test $\phi^* \in \mathfrak{M}_2$ which uniformly improves in terms of risk over $\phi \equiv \alpha$ except at ω_1 and ω_2. Clearly ϕ^* is unbiased. ϕ^* is determined by x_1^*, x_2^*, λ_1^* and λ_2^* which are the values satisfying $\int \phi^*(x)\, p(x, \omega_i)\, d\mu(x) = \alpha$, $i = 1, 2$, where $\omega_1 < \omega_2$ and x_1^*, x_2^*, λ_1^*, and λ_2^*, determined satisfying $\int \phi^*(x)\, p(x, \omega_1)\, d\mu(x) = \alpha$ and $(d/d\omega) \int \phi^*(x)\, p(x, \omega)\, d\mu(x) \mid_{\omega_1} = 0$ if $\omega_1 = \omega_2$. When $\omega_1 = \omega_2$ the null hypothesis consists of a single point. Lemma 4 shows that fixing the value of f_{ϕ^*} at two points or fixing f_{ϕ^*} and its derivative at one point is sufficient to determine ϕ^* uniquely. Suppose there were an unbiased level α test ϕ for which $f_\phi(\omega) > f_{\phi^*}(\omega)$ for some $\omega < \omega_1$ or $\omega > \omega_2$. Then there would have to be a monotone test ϕ_4 which improves on ϕ except at ω_1 and ω_2. But this contradicts the fact that ϕ^* is the unique monotone test uniformly better than $\phi \equiv \alpha$ except for ω_1 and ω_2. Thus ϕ^* is the u.m.p. unbiased level α test.

(b) Assume strictly $\mathcal{P}_{n+1'}$. For any preference pattern for the two-action testing problem say involving $n + 1$ distinct regions where action 1 is favored, a u.m.p. unbiased test exists.

The above argument generalizes easily to any preference pattern. The unique test $\phi^* \varepsilon \mathfrak{M}_n$ which is uniformly better than $\phi \equiv \alpha$ except at $\omega_1, \omega_2, \cdots, \omega_n$ is the u.m.p. unbiased level α test where ω_i corresponds to the change points of $L_1 - L_2$. ϕ^* is uniquely determined by solving the system of equations $\int \phi^*(x)\, p(x, \omega_i)\, d\mu(x) = \alpha, i = 1, 2, \cdots, n$ for $x_1^*, x_2^*, \cdots, x_n^*, \lambda_1^*, \lambda_2^*, \cdots, \lambda_n^*$. For the case where $\omega_i = \omega_{i+1}$ for some i replace the equation $\int \phi_*(x)\, p(x, \omega_{i+1})\, d\mu(x) = \alpha$ by $(d/d\omega) \int \phi^*(x)\, p(x, \omega)\, d\mu(x) \mid_{\omega_i} = 0$. Lemma 4 shows that the system of equations in this latter case is still sufficient to determine ϕ^* uniquely.

Sec. 5. Generalization of unbiased tests. Assume strictly $\mathcal{P}_{n+1}$, and assume the two-action testing problem under consideration involves $n + 1$ preference regions. Let ϕ^0 be an arbitrary but fixed test. There exists a test u.m.p. with respect to the class Φ_{ϕ_0} of all tests which improve on ϕ^0.

This generalizes the concept of unbiased tests because the class of unbiased tests can be defined as the class of all tests which improve on the test $\phi \equiv \alpha$.

There exists a unique monotone test ϕ^* which improves on ϕ^0. f_{ϕ^*} lies above f_{ϕ^0} in those intervals in which action 2 is preferred to action 1 and it lies below in the other intervals. Some intervals may be degenerate and consist of a single point. Two tests in $\mathfrak{M}_n$ cannot improve on ϕ^0 since they both must have the same power as ϕ^0 at $\omega_1, \omega_2, \cdots, \omega_n$ and this is impossible by Lemma 4. Any nonmonotone test improving on ϕ^0 has a monotone test improving on it by Theorem 6, and this monotone test must be ϕ^*.

Sec. 6. Nature of Type A critical regions. Assume strictly $\mathcal{P}_3$ and assume that for any power function differentiation inside the integral sign with respect to ω is valid. For testing a single point ω_1 against all alternatives a Type A region can be characterized as the union of at most two semi-infinite intervals, i.e., its complement is a single interval.

A Type A critical region is the critical region for any test ϕ which maximizes the curvature $\int \phi(x)\, (\partial^2/\partial\omega^2)\, p(x, \omega) \mid_{\omega_1} d\mu(x)$ subject to the constraints $\int \phi(x)\, p(x, \omega_1)\, d\mu(x) = \alpha$ and $\int \phi(x)\, (\partial/\partial\omega)\, p(x, \omega) \mid_{\omega_1} d\mu(x) = 0$. We know from section 4 that a u.m.p. unbiased level α test exists. In fact it is the unique test $\phi^* \varepsilon \mathfrak{M}_2$ for which $\int \phi^*(x)\, p(x, \omega_1)\, d\mu(x) = 0$ and $(\partial/\partial\omega) \int \phi^*(x)\, p(x, \omega)\, d\mu(x) \mid_{\omega_1} = 0$. By Theorem 6 any nonmonotone test satisfying the constraints has a unique monotone test improving on it and this test must be ϕ^* because of the uniqueness.

Thus ϕ^* defines the Type A critical region, and since ϕ^* belongs to $\mathfrak{M}_2$ the critical region is the union of at most two semi-infinite intervals.

Two remarks should be made. First, all of the known Pólya Type distributions can be differentiated inside the integral sign. Second, for general distributions it is not true that a Type A critical region is the union of two semi-infinite intervals.

Sec. 7. Type A critical regions as a function of the level of significance. Assume strictly $\mathcal{P}_3$ and assume that every power function can be differentiated inside the integral sign with respect to ω. Further, suppose $p(x, \omega) > 0$ for all x and $(\partial/\partial\omega)\, p(x, \omega_1)$ is continuous in x. The assertion is that the complement of the Type A critical region for testing ω_1 against all alternative at level α contains the complement of the Type A critical region for testing ω_1 at level α^1 whenever $\alpha < \alpha^1$. In other words, whenever the hypothesis is rejected for the level of significance α, then it should be rejected also for level α^1 where $\alpha^1 > \alpha$.

This property is not true for Type A regions in general. (See [5].) In order to establish the above result, we need to use the following lemma.

Lemma 5. *Assume $\mathcal{P}_2$. If $p(x, \omega_1) > 0$ for all x and $(\partial/\partial\omega)p(x, \omega_1)$ exists and has at most isolated zeros, then there exists an x_0 such that*

$$\frac{\partial}{\partial\omega}\, p(x, \omega_1) \begin{cases} \leqq 0 & x < x_0 \\ \geqq 0 & x > x_0 \end{cases}$$

Proof. By Theorem 1

$$\begin{vmatrix} p(x_1, \omega_1) & \dfrac{\partial}{\partial\omega}\, p(x_1, \omega_1) \\ p(x_2, \omega_1) & \dfrac{\partial}{\partial\omega}\, p(x_2, \omega_1) \end{vmatrix} \geqq 0 \tag{3.1}$$

for $x_1 < x_2$. If $(\partial/\partial\omega)\, p(x, \omega_1)$ has no zeros, then the lemma is true with $x_0 = +\infty$. If $(\partial/\partial\omega)\, p(x, \omega_1)$ has zeros, take x_0 to be any one of them. Choose $x_1 = x < x_0 = x_2$. (3.1) reduces to $-p(x_0, \omega_1)\, (\partial/\partial\omega)\, p(x, \omega_1) \geqq 0$. Since $p(x_0, \omega_1) > 0$, we deduce that $(\partial/\partial\omega)\, p(x, \omega_1) \leqq 0$ for $x < x_0$. Now, select $x_1 = x_0 < x = x_2$. (3.1) reduces to $p(x_0, \omega_1)\, (\partial/\partial\omega)\, p(x, \omega_1) \geqq 0$. Since $p(x_0, \omega_1) > 0$, $(\partial/\partial\omega)p(x, \omega_1) \geqq 0$ for $x > x_0$.

From section 6 we know that the Type A critical region for level α is given by a test $\phi_\alpha \varepsilon \mathfrak{M}_2$. For simplicity of exposition let us suppose that no randomization is involved and that ϕ_α is defined by the points x_1 and x_2 which must satisfy

$$\int_{x_1}^{x_2} p(x, \omega_1)\, d\mu(x) = 1 - \alpha \text{ and } \int_{x_1}^{x_2} \frac{\partial}{\partial\omega}\, p(x, \omega_1)\, d\mu(x) = 0.$$

We assert that $x_1 < x_0 < x_2$, for otherwise the integrand $\int_{x_1}^{x_2} (\partial/\partial\omega)p(x, \omega_1)\, d\mu(x)$ would be of one sign. (The hypothesis implies that $(\partial/\partial\omega)p(x_1, \omega_1)$ is not identically zero in the interval (x_1, x_2).) Let x_1' and x_2' be the two points defining the level α' test, where $x_1' < x_0 < x_2'$ with α' near α ($\alpha' > \alpha$) so that x_1' and x_2' are near x_1 and x_2 respectively. Clearly the first constraint prevents the interval (x_1', x_2') from containing the interval (x_1, x_2). Suppose $x_1' \geqq x_1$ and $x_2' > x_2$. Subtracting the second constraint for x_1, x_2 from the second constraint from x_1', x_2' yields

$$\int_{x_2}^{x_2'} \frac{\partial}{\partial \omega} p(x, \omega_1)\, d\mu(x) - \int_{x_1}^{x_1'} \frac{\partial}{\partial \omega} p(x, \omega_1)\, d\mu(x) = 0. \tag{3.2}$$

Between x_2 and x_2' $(\partial/\partial\omega)p(x, \omega_1) > 0$ and between x_1 and x_1' $(\partial/\partial\omega)p(x, \omega_1) < 0$. Hence (3.2) is impossible. A similar argument excludes the case $x_1' < x_1, x_2' \leqq x_2$. Thus $x_1 \geqq x_1'$ and $x_2' \leqq x_2$. The reader can furnish the modifications necessary for the argument when randomization is required at the end points.

Sec. 8. Envelope power function. Assume strictly $\mathcal{P}_3$. For the problem of testing at level α a single point ω_1 against all alternatives the envelope power function $\rho(\omega)$ decreases monotonically away from ω_1 in both directions. Let $\mathcal{U}_\rho$ be the class of tests ϕ such that if $\rho(\omega') > \rho(\omega'')$ where $\omega_1 \leqq \omega' < \omega''$ or $\omega'' < \omega' \leqq \omega_1$, then $\rho(\omega', \phi) > \rho(\omega'', \phi)$. It will now be established that there exists a test u.m.p. with respect to the class $\mathcal{U}_\rho$.

Theorem 6 shows that in obtaining the envelope power function the only tests that need be considered are those in $\mathfrak{M}_2$. For $\omega \geqq \omega_1$, $\rho(\omega) = \rho(\omega, \phi^*)$ where ϕ^* is the u.m.p. level α one-sided test of $\omega \leqq \omega_1$ against $\omega > \omega_1$. For $\omega \leqq \omega_1$ $\rho(\omega) = \rho(\omega, \phi^{**})$ where ϕ^{**} is the u.m.p. level α one-sided test of $\omega \geqq \omega_1$ against $\omega < \omega_1$. Lemma 4 shows that no other monotone test can improve on $\rho(\omega, \phi^*)$ for $\omega \geqq \omega_1$ or on $\rho(\omega, \phi^{**})$ for $\omega < \omega_1$. Thus the ρ so defined is actually the envelope power function, and from its definition it is clear that it decreases monotonically away from ω_1 in both directions.

By Section 4 there exists a u.m.p. unbiased level α test. Now any test of $\mathfrak{M}_2$ has the form that the power function can have at most one relative maximum. Indeed, it is sufficient to show that the set of points ω where $\int \phi(x)\, p(x, \omega)\, d\mu(x)$ exceeds any given constant $0 < K < 1$ consists of a single interval. As $\phi(x) - K$ changes sign twice in the order $- + -$, we deduce that the same holds for $\int (\phi(x) - K)\, p(x, \omega)\, d\mu(x) = \int \phi(x)\, p(x, \omega)\, d\mu(x) - K$ from which the conclusion follows. Any unbiased test in $\mathfrak{M}_2$ must therefore also be in $\mathcal{U}_\rho$, and conversely. The only other possible competing tests that need be worried about are the u.m.p. unbiased level α' tests where $\alpha' < \alpha$. By Section 7 the acceptance region for $\phi_{\alpha'}$ contains the acceptance region for ϕ_α. Thus the probability of rejection at any ω for $\phi_{\alpha'}$ is at least as small as that for ϕ_α. Hence $\rho(\omega, \phi_{\alpha'}) \geqq \rho(\omega, \phi_\alpha)$ for all $\omega \neq \omega_1$.

Sec. 9. The nature of the likelihood ratio test. Assume strictly $\mathcal{P}_2$, and assume $p(x, \omega)$ is a continuous function of x and ω. We prove that the likelihood ratio test is a monotone test.

More explicitly what we mean by the likelihood ratio test being monotone is the following. Let the null hypothesis be that $\omega \varepsilon \Lambda$ and the alternative hypothesis be that $\omega \varepsilon \Omega$. $\Lambda \cap \Omega = \phi$ and $\Lambda \cup \Omega = R^1$. Suppose Λ is the union of n disjoint intervals some of which may be degenerate, i.e., points. Then

$$I(c) = \left\{x \,\middle|\, \frac{\sup_{\omega \varepsilon \Lambda} p(x, \omega)}{\sup_{\omega \varepsilon \Omega} p(x, \omega)} \geqq c\right\}$$

is the union of at most n disjoint intervals and hence belongs to $\mathfrak{M}_n$.

Consider a point ω_1 in one of the intervals of Λ. Let $I_{\omega_1} =$

$$\{x \mid p(x, \omega_1) \geqq c \sup_{\omega \varepsilon \Omega} p(x, \omega)\}.$$

That I_{ω_1} depends on c will be understood. $I_{\omega_1} = \bigcap_{\omega \varepsilon \Omega} \{x \mid p(x, \omega_1) \geqq cp(x, \omega)\}$. Since $p \varepsilon \mathcal{P}_2$, $[p(x, \omega_1)]/[p(x, \omega)]$ is a monotone decreasing (increasing) function of x for $\omega > \omega_1$ $(\omega < \omega_1)$. Thus $\{x \mid p(x, \omega_1) \geqq cp(x, \omega)\}$ is a semi-infinite interval either to the left or right so I_{ω_1} is an interval. If ω_1 is not a degenerate interval of Λ, consider another point ω_2 (for definiteness $\omega_1 < \omega_2$) which is in the same interval of Λ as ω_1. $I_{\omega_2} = \{x \mid p(x, \omega_2) \geqq c \sup_{\omega \varepsilon \Omega} p(x, \omega)\}$ is an interval. Either I_{ω_1} is contained in I_{ω_2}, or I_{ω_2} contains points of I_{ω_1} and points greater than those in I_{ω_1}. It cannot happen that I_{ω_2} contains points less than those in I_{ω_1} without containing all points in I_{ω_1}. Suppose the contrary that this did happen. There then exist two points $x_1 > x_2$ such that $x_2 \varepsilon I_{\omega_2}$, $x_2 \not\varepsilon I_{\omega_1}$, $x_1 \varepsilon I_{\omega_1}$ and $x_1 \not\varepsilon I_{\omega_2}$. Since $x_2 \varepsilon I_{\omega_2}$ $[p(x_2, \omega_2)]/[p(x_2, \omega) \geqq c$ for all ω in Ω, and since $x_2 \not\varepsilon I_{\omega_1}$ there exists a $\omega' \varepsilon \Omega$ such that $[p(x_2, \omega_1)]/[p(x_2, \omega')] < c$. Thus $p(x_2, \omega_2) > p(x_2, \omega_1)$. By a similar argument $p(x_1, \omega_1) > p(x_1, \omega_2)$. This gives $p(x_1, \omega_1) p(x_2, \omega_2) - p(x_1, \omega_2)\, p(x_2, \omega_1) > 0$ which is impossible by assumption since $x_1 > x_2$. Thus the assertion is true.

The continuity of $p(x, \omega)$ in both variables simultaneously implies the following continuity property between I_ω and ω. The proof is standard and shall be omitted.

Property 1. Let ω_0 be a fixed point in a nondegenerate interval J of Λ. For every open interval U properly containing I_{ω_0} there exists an $\epsilon > 0$ such that I_ω is contained in the open interval U for all $\omega \varepsilon J$ satisfying $| \omega - \omega_0 | < \epsilon$.

Consider any nondegenerate interval $I = (a, b) \varepsilon \Lambda$. It will now be shown that $\bigcup_{\omega \varepsilon I} I_\omega$ is an interval. Suppose to the contrary that there is a point x^* such that $x^* \not\varepsilon \bigcup_{\omega \varepsilon I} I_\omega$ and there exist I_ω for $\omega \varepsilon I$ above and below x^*. Property 1 and the fact that if $\omega_1 < \omega_2$ I_{ω_2} contains points less than those in I_{ω_1} only if $I_{\omega_1} \subset I_{\omega_2}$ show that the set of $\omega \varepsilon I$ for which I_ω lies above x^* is an open interval if $b \not\varepsilon I$ and a half-open interval if $b \varepsilon I$. Similarly the set of $\omega \epsilon I$ for which I_ω lies below x^* is an open interval if $a \not\varepsilon I$, and a half-open interval if $a \varepsilon I$. But if $a, b \not\varepsilon I$ it is impossible for the interval $I = (a, b)$ to be the union of two disjoint nonempty open intervals. Similar contradictions hold when $a \varepsilon I$, $b \varepsilon I$, and $a, b \varepsilon I$. Therefore $\bigcup_{\omega \varepsilon I} I_\omega$ is an interval. The next interval in Ω to the right of I will produce an interval in the x-space to the right of $\bigcup_{\omega \varepsilon I} I_\omega$ or including $\bigcup_{\omega \varepsilon I} I_\omega$. Repeating this reasoning we see that the proof of our proposition is complete.

It has thus been shown that the likelihood ratio test is a monotone test and hence admissible by virtue of Theorem 6.

Part III. Minimax Strategies for Nature and the Statistician in the General Two-action Problem. A brief summary of the results already obtained in [2] for the one-sided testing problem will first be given for the sake of completeness. The parameter space Ω is an interval (c, d) of the real line. c may be $-\infty$, d may be $+\infty$, and the interval may be open or closed at c or d if either or both are finite. There is a point $\omega_0 \in (c, d)$ such that action 1 is preferred for $\omega \leqq \omega_0$ and action 2 is preferred for $\omega \geqq \omega_0$. The two loss functions $L_1(\omega)$ and $L_2(\omega)$ are continuous, and $L_1(\omega) = 0$ for $\omega \leqq \omega_0$; > 0 for $\omega > \omega_0$ and $L_2(\omega) = 0$ for $\omega \geqq \omega_0$; > 0 for $\omega < \omega_0$. The risk function ρ is given by

$$\rho(F, \phi) = \iint [L_1(\omega)\phi(x) + L_2(\omega)(1 - \phi(x))]p(x, \omega)\, d\mu(x)\, dF(\omega), \tag{4.1}$$

where F is the randomized strategy (a priori distribution) for nature and ϕ is the randomized strategy for the statistician. p is assumed to be strictly Pólya Type 2. The following two conditions were required:

Condition 1. If Ω is open at d and a is in the interior of the convex hull of the spectrum of μ, then as $\omega \to d$

$$L_1(\omega) \int_{-\infty}^{a} P(x, \omega)\, d\mu(x) \to 0.$$

Condition 2. If Ω is open at c and b is in the interior of the convex hull of the spectrum of μ, then as $\omega \to c$

$$L_2(\omega) \int_{b}^{\infty} P(x, \omega)\, d\mu(x) \to 0.$$

Under the above assumptions it was shown in [2] that the game $G = (\{F\}, \{\phi\}, \rho)$ has a value and both nature and the statistician have minimax strategies. Moreover the statistician has a monotone minimax strategy and nature has a minimax strategy which concentrates at just two points.

Sec. 1. Minimax theorem in the case when Ω is closed. Our first objective is to present the basic minimax theorem for the general two-action problem in the case where the parameter space Ω is closed. We deal with the situation where there exists $n + 1$ distinct intervals arranged in order in which actions 1 and 2 are successively preferred. When $n = 1$ then our general preference pattern reduces principally to the classical one-sided test of hypothesis. For $n = 2$, we are treating the classical two-sided testing problem. We assume throughout that $L_1(\omega)$ and $L_2(\omega)$ are both continuous. The fundamental preliminary minimax theorem becomes:

THEOREM 7. *If the parameter space Ω is closed, then the game defined by the risk function $\rho(F, \phi)$ is determined (has a value) and the statistician possesses a monotone minimax strategy while nature has a minimax strategy involving at most $n + 1$ points of increase, i.e. the nature's minimax distribution concentrates at most at $n + 1$ points. $n + 1$ is the total number of disconnected preference regions of both actions.*

PROOF. As Ω is closed we know that the space of distributions F over Ω is compact in the weak* topology with respect to the continuous functions on Ω. This is essentially the Helly selection theorem. Also, the space of strategies ϕ in the two-action problem is also compact in the weak* topology over the integrable functions on X. Obviously, $\rho(F, \theta)$ is linear and continuous with respect to the appropriate weak* topologies and thus optimal strategies F^0 and ϕ^0 exist and the game $\rho(F, \phi)$ has a value (see [6]).

As the class of all monotone strategies constitute a complete class [1], there exists a monotone strategy ϕ^* which improves uniformly on ϕ^0 in terms of risk and hence ϕ^* is minimax. Let

$$T = \{\omega \mid \rho(\omega, \phi^*) = \max_\omega \rho(\omega, \phi^*) = v\},$$

where v is the value of the game. We must distinguish between n odd or even. The analysis will be made for n odd and the details for n even are left for the reader to supply. Suppose for definiteness that the monotonic strategy ϕ^* has the form

$$\phi^*(x) = \begin{cases} 1 & x_{2i-1} < x < x_{2i} \\ \\ 0 & x_{2i} < x < x_{2i+1} \end{cases} \qquad i = 1, \cdots, m,$$

with $x_1 = -\infty$ and $x_{2m+1} = +\infty$ and where the x_j are distinct. In other words there are $2m$ disconnected intervals where different preferences of actions 1 and 2 are desired. Of course, m is limited such that $2m \leqq n + 1$ (see Theorem 6).

We now assert that T meets at least $2m$ alternate intervals where actions 1 and 2 are successively preferred. Suppose the contrary: let us consider

$$\int p(x, \omega)[L_1(\omega) - L_2(\omega)]\, dF^0(\omega). \tag{4.2}$$

As $F^0(\omega)$ must concentrate its full measure in T and the only sign changes of $L_1(\omega) - L_2(\omega)$ occur as we pass from one preference region to another, we infer that $[L_1(\omega) - L_2(\omega)]\, dF^0(\omega)$ changes sign less than $2m - 1$ times. Thus, (4.2) by Theorem 3 must change signs fewer than $2m - 1$ times. However, ϕ^* is Bayes against F^0 and must therefore take the values $+1$ or 0 according as (4.2) is negative or positive. Thus ϕ^* cannot have the form as indicated. This contradiction implies the assertion made above about T.

Select $2m$ points T^* in T each belonging to a different preference region such that $L_1(\omega) - L_2(\omega)$ traversing these points changes sign $2m - 1$ times. By Theorem 5 of [1] there exists a distribution F^* which fully concentrates on T^* against which ϕ^* is Bayes. We now show that F^* is minimax. As F^* concentrates in $T^* \subset T$, we get $v = \rho(F^*, \phi^*)$. Using the Baysian nature of ϕ^* for F^*, we obtain

$$v = \rho(F^*, \phi^*) \leqq \rho(F^*, \phi), \qquad \text{for all strategies } \phi.$$

The proof of the theorem is thus complete.

Sec. 2. Two-sided minimax theorem. Our next task is to eliminate the restriction that Ω is closed. For this purpose we need to impose some further conditions on the family of densities $p(x, \omega)$. To expedite and clarify the reasoning, we restrict ourselves to the two-sided problem. Similar analysis would apply to the general two-action problem.

Reviewing the basic assumptions, we have that the parameter space Ω is an interval (c, d) of the real line. c may be $-\infty$, d may be $+\infty$, and the interval may be open or closed at c or d if either or both are finite. There exist two points $\omega_1, \omega_2 \, \varepsilon \, (c, d)$ such that action 1 is preferred for $\omega \leqq \omega_1$ and $\omega \geqq \omega_2$ and action 2 is preferred for $\omega_1 \leqq \omega \leqq \omega_2$. The two loss functions L_1 and L_2 are continuous, and $L_1(\omega) = 0$ for $\omega \leqq \omega_1, \omega \geqq \omega_2$; > 0 for $\omega_1 < \omega < \omega_2$, and $L_2(\omega) = 0$ for $\omega_1 \leqq \omega \leqq \omega_2$; > 0 for $\omega < \omega_1, \omega > \omega_2$. There is no loss of generality in taking the loss function equal to zero where the action is preferred as differences of the loss functions are the only relevant quantities involved. The risk function is again given by (3.1). This time p is assumed to be strictly Pólya Type 3.

The assertion that will be proven under certain hypothesis of smoothness is that the game $G = (\{F\}, \{\phi\}, \rho)$ has a value and both players have minimax strategies. The statistician has a monotone minimax strategy, and nature has a minimax strategy which concentrates on at most 3 points. To establish this assertion we impose three conditions:

Condition A. If Ω is open at d and a is interior to the convex hull of the spectrum of μ, then as $\omega \to d$

$$L_2(\omega) \int_{-\infty}^{a} P(x, \omega) \, d\mu(x) \to 0.$$

Condition B. If Ω is open at c and b is interior to the convex hull of the spectrum of μ, then as $\omega \to c$

$$L_2(\omega) \int_{b}^{\infty} P(x, \omega) \, d\mu(x) \to 0.$$

These conditions require that if either endpoint is open then as ω tends to this endpoint the mass of probability shifts away from the opposite end of the axis in such a way that the probability at the opposite end of the axis must tend to zero at a faster rate than the loss L blows up. These conditions are similar to those imposed in the one-sided problem.

Condition C. Let $l(c) = \lim_{\omega \to c} L_2(\omega)$ and $l(d) = \lim_{\omega \to d} L_2(\omega)$ (the existence of the limits is postulated).

(i) $l = \min(l(c), l(d)) > \max_{\omega_1 \leqq \omega \leqq \omega_2} L_1(\omega)$

(ii) If Ω is open at c, then $l(c) > \max_{a \leqq \omega \leqq b} L_2(\omega)$ for any closed interval contained in Ω.

(iii) If Ω is open at d, then $l(d) > \max_{a \leqq \omega \leqq b} L_2(\omega)$ for any closed interval contained in Ω.

Condition C has essentially the effect of eliminating the possibility that nature will desire to concentrate at the ends of the parameter space in choosing an optimal strategy. This condition is fulfilled, for instance, when the losses tend to ∞ at both ends.

Suppose Ω is not a closed interval (the case treated in Theorem 7). Then the result is not immediate because the space $\{F\}$ is not compact. In fact it is no longer true unless for example the three conditions A, B, and C are imposed on the problem; that is, some conditions are necessary. The method of proof consists in considering the sequence of games $G^n = (\{F^n\}, \{\phi\}, \rho)$ where $\Omega^n = [\omega_n', \omega_n'']$ and $\omega_n' \to c$, $\omega_n'' \to d$. That is, we consider a sequence of games defined over closed intervals contained in Ω which in the limit approach Ω. If one end of Ω is closed, say d, then $\omega_n'' \equiv d$ for all n. Each game G^n has a value, and for each game G^n the statistician has a monotone minimax strategy ϕ^n and nature has a strategy (distribution $\bar{F}^n$ which concentrates at at most three points by Theorem 7. The problem is to show that subsequences can be selected from $\{\phi^n\}$ and $\{\bar{F}^n\}$ which converge to strategies yielding the desired properties in the original game G.

Let ν_n be the value of the game G^n. The sequence of ν_n's is bounded away from zero. Indeed, consider the closed interval $[\omega_1', \omega_1'']$. Choose three points ω^1, ω^2, and ω^3 such that $\omega_1' < \omega^1 < \omega_1$, $\omega_1 < \omega^2 < \omega_2$, and $\omega_2 < \omega^3 < \omega_1''$. (We assume of course that $\omega_1' < \omega_1$ and $\omega_2 < \omega_1''$. The game G^n has no meaning otherwise.) Let F' be the strategy for nature which plays ω^1, ω^2, and ω^3 with equal probability. Clearly $\rho(F', \phi) \geqq \alpha > 0$ for all ϕ when $\alpha = \min(L_2(\omega^1), L_1(\omega^2), L_2(\omega^3))$. Hence $\nu_n \geqq \alpha > 0$ for all n.

Let $T_n = \{\omega : \omega \epsilon [\omega_n', \omega_n''], \rho(\omega, \phi^n) = \nu_n\}$. T_n contains points in both preference regions; i.e., T_n contains ω in the interval (ω_1, ω_2) and T_n also meets at least one of the intervals (c, ω_1), (ω_2, d). Suppose not. Suppose $T_n \subset (\omega_1, \omega_2)$. $\bar{F}^n$ must concentrate at points of T_n. Let $\phi_0(x) \equiv 0$ for all x. Then $\rho(\bar{F}^n, \phi_0) = 0$ which contradicts the fact that $\nu_n > 0$. An analogous argument using $\phi_1(x) \equiv 1$ eliminates the possibility that T_n contain no points of (ω_1, ω_2).

The monotone minimax strategies ϕ^n are characterized by two points x_n, y_n ($x_n < y_n$). In order to show that two subsequences of $\{\phi^n\}$ and $\{\bar{F}^n\}$ can be selected which converge to minimax strategies in the original game G we need the fact that the sets T_n are bounded away from the open ends or end of Ω. This is established by showing that the x_n's or y_n's or both are bounded away from c' and d', the ends of the spectrum of μ. c' may be $-\infty$ and d' may be $+\infty$.

Suppose Ω is open at c; Ω can be open or closed at d. We assert that c' cannot be a limit of the sequence $\{x_n\}$. Suppose there were a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with the limit c'.

Case 1. c' is a limit point of the sequence $\{y_{n_i}\}$.

There exists a subsequence $\{\phi^{n_j}\}$ of strategies such that $\{x_{n_j}\}$ and $\{y_{n_j}\}$ each have the limit c'. For $\omega < \omega_1$ and

$$\omega > \omega_2 \; \rho(\phi^{n_j}, \omega) = L_2(\omega) \int_{x_{n_j}}^{y_{n_j}} P(x, \omega) \, d\mu(x), \qquad \text{Therefore } \rho(\phi^{n_j}, \omega) \to 0$$

as $n_j \to \infty$ for $\omega < \omega_1$ and $\omega > \omega_2$. Since $\nu_n \geqq \alpha > 0$ for all n, this means that T_n is totally contained in the interval (ω_1, ω_2), a contradiction.

Case 2. c' is not a limit point of $\{y_{n_i}\}$.

It is assumed there exists a $y_{c'}$ such that $y_{n_i} \geqq y_{c'}$ for all n_i.

LEMMA A. *Under the conditions that* $x_{n_i} \rightarrow c'$, *there exists a limit point of* $\{\nu_{n_i}\} \geqq l(c)$.

PROOF. By condition B given $\eta > 0$ there exists $M(\eta)$ such that $\int_{-\infty}^{y_c} P(x, \omega) \, d\mu(x) \geqq (1 - \eta)$ for $\omega < M(\eta)$. Given $\epsilon > 0$ there exists $\omega_\epsilon < M(\eta)$ such that $L_2(\omega_\epsilon) > l(c) - \epsilon$ (or if $l(c) = \infty$ for arbitrarily large K there exists $\omega_K < M(\eta)$ such that $L_2(\omega_K) > K$). For ω_ϵ (or ω_K) there exists $N(\omega_\epsilon, \epsilon')$ such that

$$\int_{-\infty}^{y_c} P(x, \omega_\epsilon) \, d\mu(x) - \int_{x_{n_i}}^{y_c} P(x, \omega_\epsilon) \, d\mu(x) < \epsilon'$$

for $n_i \geqq N(\omega_\epsilon, \epsilon')$. Therefore, when $l(c) < \infty$

$$\rho(\phi^{n_i}, \omega_\epsilon) \geqq L_2(\omega_\epsilon) \int_{x_{n_i}}^{y_c} P(x, \omega_\epsilon) \, d\mu(x) \geqq (l(c) - \epsilon)(1 - \eta - \epsilon')$$

for $n_i \geqq N(\omega_\epsilon, \epsilon')$. But ϵ, η, ϵ' are arbitrary constants so the assertion follows. When $l(c) = \infty$ $\rho(\phi^{n_i}, \omega_K) \geqq K(1 - \eta - \epsilon')$, and the assertion still holds.

LEMMA B. *Under the same conditions as in Lemma* A, $\nu_{n_i} \leqq l(c) - \beta$ *for all* n_i *where* $\beta > 0$.

PROOF. Let ϕ' be a monotone strategy for which the characterizing points x' and y' are interior to the spectrum of μ.

$$\rho(\omega, \phi') = L_1(\omega) \left[\int_{-\infty}^{x'} P(x, \omega) \, d\mu(x) + \int_{y'}^{\infty} P(x, \omega) \, d\mu(x) \right] + L_2(\omega) \int_{x'}^{y'} P(x, \omega) \, d\mu(x).$$

Let $a = l(c) - \max L_1(\omega)$. $a > 0$ and

$$L_1(\omega) \left[\int_{-\infty}^{x'} P(x, \omega) \, d\mu(x) + \int_{y'}^{\infty} P(x, \omega) \, d\mu(x) \right] \leqq l(c) - a.$$

If Ω is also open at d, then by virtue of conditions A and B for every $\epsilon > 0$ there exist two constants $H(\epsilon)$ and $K(\epsilon)$ such that $L_2(\omega) \int_{x'}^{y'} P(x, \omega) \, d\mu(x) < \epsilon$ for $\omega < H(\epsilon)$ and $\omega > K(\epsilon)$. For $H(\epsilon) \leqq \omega \leqq K(\epsilon)$ the second factor in $\rho(\omega, \phi')$ is $\leqq l(c) - b$ for some $b > 0$ by condition C. Thus $\rho(\omega, \phi') \leqq \max(l - a, \epsilon, l - b) < l$ for all ω, and $\nu_{n_i} \leqq \max(l - a, \epsilon, l - b)$ for all n_i. If Ω is closed at d, by condition B there exists $H(\epsilon)$ such that $L_2(\omega) \int_{x'}^{y'} P(x, \omega) \, d\mu(x) < \epsilon$ for $\omega < H(\epsilon)$. For $H(\epsilon) \leqq \omega \leqq d$ this factor is $\leqq l - b$ for some $b > 0$ by condition C. Again $\nu_{n_i} \leqq l - \beta$ where $\beta = l - \max(l - a, \epsilon, l - b) > 0$.

But Lemmas A and B are contradictory assertions. Hence the original assumption that c' was a limit point of $\{x_n\}$ is untenable.

An analogous argument shows that if Ω is open at d the sequence $\{y_n\}$ cannot have d' as a limit point.

The required theorem follows almost immediately.

LEMMA C. *If* Ω *is open at* c *and* d, *there exist two constants* C_1 *and* C_2 *such that*

$c < C_1 < C_2 < d$ and $T_n \subseteq [C_1, C_2]$ for all n. If Ω is open at $c(d)$ and closed at $d(c)$, there exists a constant $C_1(C_2)$ such that $c < C_1(C_2 < d)$ and $T_n \subseteq [C_1, d]$ $([c, C_2])$ for all n.

PROOF. For ω near c and d $\rho(\omega, \phi^n) = L_2(\omega) \int_{x_n}^{y_n} P(x, \omega)\, d\mu(x)$. By the previous discussion if Ω is open at both ends there exist constants $x_{c'}$ and $y_{d'}$ interior to the spectrum of μ such that $x_{c'} \leqq x_n$ and $y_n \leqq x_{d'}$ for all n. $\rho(\omega, \phi^n) \leqq L_2(\omega)$ $\int_{x_{c'}}^{y_{d'}} P(x, \omega)\, d\mu(x)$. By conditions A and B there exist constants $C_1(\eta)$ and $C_2(\eta)$ such that

$$L_2(\omega) \int_{x_{c'}}^{y_{d'}} P(x, \omega)\, d\mu(x) < \eta$$

for $c < \omega < C_1(\eta)$ and $C_2(\eta) < \omega < d$. Choose $\eta = \alpha/2$ where α is the bound of $\nu_n(\nu_n \geqq \alpha > 0$ for all $n)$. Thus T_n cannot have points below $C_1 = C_1(\alpha/2)$ or above $C_2 = C_2(\alpha/2)$. A similar argument works when Ω is open at just one end.

For each game $G^n = (\{F^n\}, \{\phi\}, \rho)$ in which Ω is open at both ends there is a triplet $(\tilde{F}^n, \phi^n, \nu_n)$ where $\tilde{F}^n$ is the minimax strategy for nature which concentrates at three points, ϕ^n is the monotone minimax strategy for the statistician characterized by two points, and ν_n is the value of the game. $\tilde{F}^n$ concentrates at exactly three points (and not at at most three) for it has been shown that ϕ^n defines a split selection region for action 1, and by virtue of Theorem 5 and the fact that ϕ^n is Bayes against $\tilde{F}^n$ this would be impossible unless $\tilde{F}^n$ concentrates at three points. A subsequence $\{\phi^{n_i}\}$ can be selected which converges to a monotone strategy ϕ^* for the statistician. ϕ^* also defines a split selection region for action 1 since the x_{n_i}'s and y_{n_i}'s are bounded away from the ends of the spectrum of μ. Since the T_n's are contained in a closed interval contained in Ω, a subsubsequence $\{\tilde{F}^{n_j}\}$ can be selected which converges to at most a three-point distribution, F^*, for nature. Finally a subsubsequence $\{\nu_{n_k}\}$ can be chosen which converges to a value ν. $\rho(F, \phi^{n_k}) \leqq \nu_{n_k}$ so by the Lebesgue convergence theorem $\rho(F, \phi^*) \leqq \nu$, for every F. Similarly $\rho(F^{n_k}, \phi) \geqq \nu_{n_k}$ so $\rho(F^*, \phi) \geqq \nu$. Thus ν is the value of the game, and F^* and ϕ^* are minimax strategies for nature and the statistician respectively. F^* concentrates at exactly three points since ϕ^* defines a split selection region for action 1.

When Ω is closed at one end, analogous arguments prove the existence of minimax strategies ϕ^* and F^* where F^* concentrates at at most three points.

Summing up the foregoing results, we have

THEOREM 8. *If conditions* A, B, *and* C *are satisfied and no other restriction on the parameter space Ω is made, then the game with payoff kernel $\rho(F, \phi)$ has a value, where optimal strategies exist with the same properties as is given in Theorem* 7.

Sec. 3. Computation of minimax strategies. The previous discussion has been an existence discussion, and no mention was made of how the statistician's monotone minimax strategy can be found or constructed. The remainder of this section is devoted to giving two general methods for constructing the minimax strategy—one for the one-sided problem and one for the two-sided problem.

In the one-sided problem

$$\rho(\omega, \phi) = \begin{cases} L_1(\omega) \int_{-\infty}^{x_0} P(x, \omega)\, d\mu(x), & \omega \geqq \omega_0 \\ L_2(\omega) \int_{x_0}^{\infty} P(x, \omega)\, d\mu(x), & \omega \leqq \omega_0 \end{cases}$$

where x_0 is the point characterizing the monotone strategy ϕ. As x_0 decreases $L_1(\omega) \int_{-\infty}^{x_0} P(x, \omega)\, d\mu(x)$ decreases for each $\omega \geqq \omega_0$ and $L_2(\omega) \int_{x_0}^{\infty} P(x, \omega)\, d\mu(x)$ increases for each $\omega \leqq \omega_0$. The method is now obvious. Choose an arbitrary x_0. If

$$\max_{\omega \leqq \omega_0} \int_{x_0}^{\infty} P(x, \omega)\, d\mu(x) < \max_{\omega \geqq \omega_0} \int_{-\infty}^{x_0} P(x, \omega)\, d\mu(x),$$

decrease x_0 until the maximums are equal. If the reverse is true, increase x_0. That x_0 which implies equal maximums above and below ω_0 defines the monotone minimax strategy. There is no danger of the maximums not existing since by conditions 1 and 2, $\rho(\omega, \phi) \to 0$ as $\omega \to c, d$.

In the two-sided problem

$$\rho(\omega, \phi) = \begin{cases} L_1(\omega) \left[\int_{-\infty}^{x_0} P(x, \omega)\, d\mu(x) + \int_{y_0}^{\infty} P(x, \omega)\, d\mu(x) \right] & \omega_1 \leqq \omega \leqq \omega_2 \\ L_2(\omega) \int_{x_0}^{y_0} P(x, \omega)\, d\mu(x) & \omega \leqq \omega_1,\ \omega \geqq \omega_2 \end{cases}$$

where x_0, y_0 are the points characterizing the monotone strategy ϕ. Assume Ω is open at both ends. Choose an arbitrary x_0. Determine y_0 (as a function of x_0) so that

$$\max_{\omega \leqq \omega_1} L_2(\omega) \int_{x_0}^{y_0} P(x, \omega)\, d\mu(x) = \max_{\omega_2 \leqq \omega} L_2(\omega) \int_{x_0}^{y_0} P(x, \omega)\, d\mu(x).$$

This cannot be done for all x_0. As $\omega \to d$, $L_2(\omega) \int_{x_0}^{d'} P(x, \omega)\, d\mu(x) \to l(d)$ and $l(d) > \max_{a \leqq \omega \leqq b} L_2(\omega)$ for any closed interval $[a, b] \subset \Omega$ so that for y_0 sufficiently large $\max_{\omega_2 \leqq \omega} > \max_{\omega \leqq \omega_1}$. Both maximums equal 0 for $y_0 = x_0$ so unless $\max_{\omega_2 \leqq \omega} \geqq \max_{\omega \leqq \omega_1}$, for all $y_0 > x_0$ there will be equality at some point. $\max_{\omega_2 \leqq \omega} > \max_{\omega \leqq \omega_1}$ for all $y_0 > x_0$ when x_0 is chosen too close to d'. In this case decrease x_0 until it is possible to determine y_0 so that the maximums are equal. There will be some point x_m such that for all $x_0 \leqq x_m$ a y_0 can be found. Now vary x_0 in the appropriate direction until the outer maximums are equal to

$$\max_{\omega_1 \leqq \omega \leqq \omega_2} \left[\int_{-\infty}^{x_0} P(x, \omega)\, d\mu(x) + \int_{y_0(x_0)}^{\infty} P(x, \omega)\, d\mu(x) \right].$$

The x_0 and corresponding y_0 which give three equal maximums determine the monotone minimax strategy for the statistician.

A further useful fact is that the monotone minimax strategy for the statistician is unique. This can be demonstrated with the aid of Lemma 4.

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