Hence comparing off-diagonal elements, we get

$$\lambda_{ij} = \frac{k\lambda}{v},$$

where λ_{ij} is the number of times the pair of treatments i, j occur together in the blocks. Since λ_{ij}'s are all equal the design is Balanced Incomplete Block Design (BIBD) [2]. This result was proved in an alternative form by W. A. Thompson [3].

3. Concluding remarks. But these do not exclude the possibilities of the existence of balanced designs with different block sizes and the same number of replications. As an example consider the design whose incidence matrix is

$$N = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix},$$

$$r = (6, 6, 6, 6); \qquad k = (3, 3, 3, 3, 2, 2, 2, 2, 2, 2).$$

Here it can be verified that every elementary contrast is estimated with a variance equal to $3\sigma^2/7$, but the design is not a Balanced Incomplete Block Design.

It can also be seen that the example given above is obtained by adjoining two BIBD's with the same number of treatments. Such designs can be constructed from two BIBD's with the same number of treatments. Investigations on these lines are being carried out.

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THE SPACING OF OBSERVATIONS IN POLYNOMIAL REGRESSION

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1. Introduction and summary. De la Garza ([1], [2]) has considered the estimation of a polynomial of degree p from n observations in a given range of the

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independent variable x. This range may conveniently be taken to be from $+1$ to -1. He showed that for any arbitrary distribution of the points of observation there was a distribution of the n observations at only $p + 1$ points for which the variances (determined by the matrix $\mathbf{X}^T \mathbf{W} \mathbf{X}$) were the same. He then considered how these $p + 1$ points should be distributed so that the maximum variance of the fitted value in the range of interpolation should be as small as possible. In the present note general formulae will be obtained for the distribution of the points of observation and for the variances of the fitted values in the minimax variance case, and the variances will be compared with those for the uniform spacing case.[1]

2. Spacing for minimax variance. The fitted value is given by

$$u_p(x) = \sum_{j=0}^{p} L_j(x)\bar{y}_j, \tag{1}$$

where $L_j(x)$ is the Lagrangian coefficient corresponding to the point of observation x_j and $\bar{y}_j$ is the mean of the observed values at this point. The variance of the fitted value is $\operatorname{var} u_p(x) = \sum_{j=0}^{p} L_j^2(x) \operatorname{var} \bar{y}_j$.

At a point of observation

$$L_j(x_k) = \delta_{jk}$$

and

$$\operatorname{var} u_p(x_j) = \operatorname{var} \bar{y}_j .$$

The largest value of this variance will be as small as possible when the n observations are equally divided among the $p + 1$ points. When this is done

$$\operatorname{var} u_p(x_j) = (p + 1)\sigma^2/n \tag{2}$$

and

$$\operatorname{var} u_p(x) = \sum_{j=0}^{p} L_j^2(x)(p + 1)\sigma^2/n. \tag{2.1}$$

Since this is a polynomial of degree $2p$, the minimax variance conditions are obtained when the maxima of $\operatorname{var} u_p(x)$ are at the $p - 1$ internal points x_j, and the end points x_0 and x_p are $+1$ and -1; for then $\operatorname{var} u_p(x)$ never exceeds

$$(p + 1)\sigma^2/n$$

in the range $+1$ to -1. The minimax variance conditions are thus

$$L_j'(x_j) = 0, \qquad j = 1 \text{ to } p - 1. \tag{3}$$

[1] K. Smith, in an earlier discussion, has given details of curves up to the sixth degree (Biometrika 12 (1918), pp. 1–85).

Now, if

$$F(x) = \prod_{j=0}^{p} (x - x_j), \tag{4}$$

then

$$L_j(x) = \frac{F(x)}{(x - x_j)F'(x_j)},$$

and so

$$F''(x) = \{(x - x_j)L_j''(x) + L_j'(x)\}F'(x_j),$$

and (3) is equivalent to

$$F''(x_j) = 0, \qquad j = 1 \text{ to } p - 1. \tag{5}$$

The function $F(x)$ will be of the form $\alpha(x^2 - 1)\phi_{p-1}(x)$, where the polynomial $\phi_{p-1}(x)$ of degree $p - 1$ is determined by the $p - 1$ equations (5). The polynomial which satisfies these equations is readily shown to be the derivative $P_p'(x)$ of the Legendre polynomial. For if

$$F(x) = \alpha(x^2 - 1)P_p'(x), \tag{6}$$

then

$$F'(x) = \alpha \frac{d}{dx}\{(x^2 - 1)P_p'(x)\} = \alpha p(p + 1)P_p(x)$$

and

$$F''(x) = \alpha p(p + 1)P_p'(x),$$

and so $F''(x)$ vanishes at the internal points $F(x) = 0$.

The points of observation for minimax variance are then to be located at $+1$, -1, and the roots of $P_p'(x) = 0$.

Since the internal points of observation are points of maximum variance, the variance will be given by an equation of the form

$$\text{var } u_p(x) = \{1 + \beta(x^2 - 1)P_p'^2(x)\}(p + 1)\sigma^2/n. \tag{7}$$

The minima of the variance curve then occur at points for which

$$xP_p'(x) + (x^2 - 1)P_p''(x) = 0,$$

and this equation is equivalent to

$$xP_p'(x) = p(p + 1)P_p(x). \tag{8}$$

From (2.1),

$$\text{var } u_p(x) = \sum_{j=0}^{p} \left\{ \frac{\alpha(x^2 - 1)P_p'(x)}{(x - x_j)\alpha p(p + 1)P_p(x_j)} \right\}^2 (p + 1)\sigma^2/n, \tag{9}$$

and so, on comparing the coefficients of $x^2P_p'^2(x)$ in (7) and (9),

$$\beta = \sum_{j=0}^{p} \{p(p+1)P_p(x_j)\}^{-2}.$$

The Lobatto quadrature formula [3] with $f(x) \equiv 1$ gives

$$\int_{-1}^{1} dx = \sum_{j=0}^{p} 2\{p(p+1)P_p^2(x_j)\}^{-1} = 2.$$

Thus the explicit formula for the variance of the fitted value is

$$\text{(10)} \qquad \operatorname{var} u_p(x) = \left\{ 1 + \frac{x^2 - 1}{p(p+1)} P_p'^2(x) \right\} (p+1)\sigma^2/n.$$

In the region of extrapolation, when $|x|$ is large

$$P_p'(x) \doteqdot p\{(2p)!/2^p p!^2\}x^{p-1},$$

and so

$$\text{(11)} \qquad \operatorname{var} u_p(x) \doteqdot p\{(2p)!/2^p p!^2\}^2 x^{2p}\sigma^2/n.$$

3. Uniform spacing. When the observations are spaced at equal intervals the variance of the fitted value is

$$\operatorname{var} u_p(x) = \sum_{j=0}^{p} \left\{ T_j^2(x) \Big/ \sum_{i=1}^{n} T_j^2(x_i) \right\} \sigma^2,$$

where the $T_j(x)$ are the polynomials orthogonal over the n points of observation x_i. When n is large these polynomials will approximate to multiples of the Legendre polynomials $P_j(x)$ which are orthogonal over the continuous range $+1$ to -1. Thus

$$T_j(x) \sim k_j P_j(x)$$

and

$$\sum_i T_j^2(x_i)\Delta x_i \sim k_j^2 \int_{-1}^{1} P_j^2(x)\, dx = 2k_j^2/(2j+1).$$

The interval Δx_i between neighboring observations is $2/n$, and so

$$\sum T_j^2(x_i) \sim nk_j^2/(2j+1)$$

and

$$\text{(12)} \qquad \operatorname{var} u_p(x) \sim \sum_{j=0}^{p} (2j+1)P_j^2(x)\sigma^2/n.$$

The maxima and minima of variance are at points given by

$$\sum_{j=0}^{p} (2j+1)P_j(x)P_j'(x) = 0,$$

which from the recurrence relations for Legendre polynomials is

$$\sum_{j=0}^{p} P_j'(x)\{P_{j+1}'(x) - P_{j-1}'(x)\} = 0,$$

or

$$P_p'(x)P_{p+1}'(x) = 0.$$

The points of maximum variance are then the roots of $P_p'(x) = 0$ and the points of minimum variance the roots of $P_{p+1}'(x) = 0$. It is interesting to observe that the points of observation in the minimax variance method are points of maximum variance in the uniform spacing method. These points are also the points used in the Lobatto quadrature formula.

The Christoffel-Darboux identity [3] for the sum in equation (12) leads to the alternative form

$$\text{(12.1)} \qquad \operatorname{var} u_p(x) \sim \{P_p(x)P_{p+1}'(x) - P_p'(x)P_{p+1}(x)\}(p+1)\sigma^2/n.$$

By use of the recurrence relations for the Legendre polynomials this can be put in the form

$$\text{(12.2)} \qquad \operatorname{var} u_p(x) \sim \left\{(p+1)P_p^2(x) - \frac{x^2-1}{p+1}P_p'^2(x)\right\}(p+1)\sigma^2/n.$$

At the end-points $+1$ and -1, $P_p^2(x)$ is unity and

$$\operatorname{var} u_p(\pm 1) \sim (p+1)^2\sigma^2/n.$$

At the centre of the range the variance can be obtained by substituting the values of $P_p(0)$ and $P_p'(0)$ in (12.2). It is found that

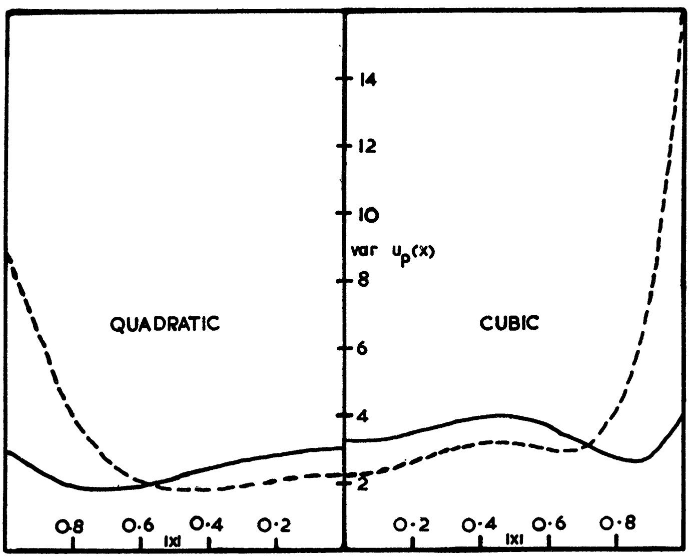

FIG. 1. The solid curve shows the variance of the fitted value for the minimax variance method and the dotted curve the variance for the uniform spacing method. The unit for the variance scale is σ^2/n.

(13) $$\operatorname{var} u_p(0) \sim \left\{\frac{(2q+1)(2q-1)\cdots 1}{2^q q!}\right\}^2 \sigma^2/n,$$

where q is $\frac{1}{2}p$ when p is even and $\frac{1}{2}(p-1)$ when p is odd. In the region of extrapolation, when $|x|$ is large (12.2) gives

$$\operatorname{var} u_p(x) \doteqdot (2p+1)\{(2p)!/2^p p!^2\}^2 x^{2p} \sigma^2/n.$$

The deviations from these formulae when n is not large have been discussed and tabulated [4].

4. Comparison of the two methods. In the central part of the range the uniform spacing method gives a smaller variance than the minimax variance method. An asymptotic expansion of (13) using Stirling's factorial approximation shows that the ratio of the variances is roughly $2/\pi$. This ratio increases steadily with $|x|$, and at the ends of the range the variance for the uniform spacing method exceeds that for the minimax variance method by a factor $p+1$, while in the region of extrapolation this factor approaches $2+p^{-1}$. The crossover points for the two variance curves occur at ± 0.58 for the quadratic and ± 0.72 for the cubic. Thus over most of the region of interpolation the advantage lies with the uniform spacing method, but at the extremes of the region of interpolation and in the region of extrapolation the advantage lies decidedly with the minimax variance method.

Fig. 1 shows the shape of the two variance curves in the region of interpolation for the second and third degree polynomials. Since the curves are symmetrical about the origin of x, only half of each curve is drawn.

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CONDITIONS THAT A STOCHASTIC PROCESS BE ERGODIC[1]

By Emanuel Parzen

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In his work on statistical inference on stochastic processes, Grenander has pointed out ([2], p. 257) that "the concept of metric transitivity seems to be

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