A CENTRAL LIMIT THEOREM FOR SUMS OF INTERCHANGEABLE RANDOM VARIABLES[1]

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1. Summary. A collection of random variables is defined to be interchangeable if every finite subcollection has a joint distribution which is a symmetric function of its arguments.

Double sequences of random variables X_{nk}, $k = 1, 2, \cdots, k_n$ $(\to \infty)$, $n = 1, 2, \cdots$, interchangeable (as opposed to independent) within rows, are considered. For each n, $X_{n1}, \cdots, X_{n,k_n}$ may (a) have a non-random sum, or (b) be embeddable in an infinite sequence of interchangeable random variables, or (c) neither. In case (a), a theorem is obtained providing conditions under which the partial sums have a limiting normal distribution. Applications to such well-known examples as ranks and percentiles are exhibited. Case (b) is treated elsewhere while case (c) remains open.

2. Terminology, notation and preliminaries. If X_n is a sequence of r.v.'s converging in probability (in measure) to a r.v. X, that is,

$$\lim P\{| X_n - X | > \epsilon\} = 0, \qquad \text{all } \epsilon > 0,$$

we abbreviate this by writing $X_n \xrightarrow{P} X$. This, in turn, implies $g(X_n) \xrightarrow{P} g(X)$ for any continuous function $g(x)$. If the corresponding c.d.f.'s $F_{X_n}(x) \to F_X(x)$ at all continuity points of the latter (in the sense of convergence of real numbers), we say X_n converges in law (or distribution) to X and write $X_n \xrightarrow{L} X$. We shall use frequently without ado the facts that if $X_n \underset{P}{\overset{L}{\to}} X$ and c_n is a sequence of positive constants such that $c_n \to c$, then $c_n X_n \underset{P}{\overset{L}{\to}} cX$[3].

The notation $P\{A \mid B\}$ will be used to designate the probability of an event A, given the occurrence of the event B, i.e., the conditional probability of A given B.

We shall be interested in and deal exclusively with r.v.'s whose joint c.d.f. is a symmetric function of its arguments. The same will then be true of the joint Fourier transform or characteristic function. This characteristic may also be expressed by stating that the joint distribution of $X_1, \cdots, X_k$ is invariant under permutations of the subscripts of the X's. Such random variables seem to have been introduced by de Finetti (cf. [4]). They have been termed "symmetrically dependent" by E. Sparre Anderson who also has studied some of their properties in a series of papers [1], [2]. By a quirk of terminology not in-

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frequent in mathematics, independent identically distributed r.v.'s are then subsumed under the category "symmetric dependence." On the grounds of brevity and connotation of the characteristic involved, we propose the sobriquet "interchangeable random variables" to denote any finite set of r.v.'s whose joint c.d.f. is symmetric.

In the case of an infinite sequence of r.v.'s, every finite subset of which has this property, Loève speaks of "exchangeable r.v.'s." However, the terminology of "interchangeability" of r.v.'s will be extended to include this case as well.

It is immediately evident that the r.v.'s, say $X_{i_1}, X_{i_2}, \cdots, X_{i_r}$, of any finite subcollection of a collection of interchangeable r.v.'s (i.r.v.'s) are themselves interchangeable and have a joint c.d.f. depending on r but not the permutation $(i_1, \cdots, i_r)$. In particular, the marginal c.d.f.'s $F_j(x) = F_{X_j}(x) = P\{X_j < x\}$ are identical for $j = 1, 2, \cdots, k$.

It is worth noting at the outset that it is, in general, not possible, to embed a given finite set of i.r.v.'s in an infinite set of i.r.v.'s (or even in a larger finite set). For example, if $P\{X_1 = 1, X_2 = 0\} = \frac{1}{2} = P\{X_1 = 0, X_2 = 1\}$ one cannot even adjoin a third r.v. so as to preserve interchangeability.

We commence with some elementary observations on the nature of i.r.v.'s. Two of these will be cast in the form of lemmas.

Suppose (as we shall throughout) that the i.r.v.'s under consideration have finite second order moments $EX_iX_j = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, dF_{X_i,X_j}(x, y), i, j = 1, 2, \cdots, k$. It is of course sufficient for this that when $m = 2$, $EX_1^m = \int_{-\infty}^{\infty} x^m\, dF_1(x) < \infty$. Take $\rho_{ii} = 1$, $i = 1, \cdots, k$ and define the (common) correlation coefficient between X_i and X_j by

$$\rho = \rho_{ij} \doteq \frac{\text{Cov}\,(X_i, X_j)}{\sqrt{\sigma_i^2 \sigma_j^2}} = \frac{EX_iX_j - (EX_i)(EX_j)}{\sqrt{E(X_i - EX_i)^2 E(X_j - EX_j)^2}}$$
$$= \frac{EX_1X_2 - (EX_1)^2}{E(X_1 - EX_1)^2}, \qquad i \neq j.$$

Then, the positive semi-definiteness of the correlation matrix

$$R = \{\rho_{ij}\} = \begin{pmatrix} 1\rho\cdots\rho \\ \rho 1\cdots\rho \\ \vdots \\ \cdot \\ \cdot \\ \rho\rho\cdots I \end{pmatrix}$$

constrains ρ to be at least $-[1/(k - 1)]$, where k is the number of i.r.v.'s. For if J is the $k \times k$ matrix consisting entirely of ones and I is the identity matrix of order k,

$$|R| = |\rho J + (1 - \rho)I| = [k\rho + (1 - \rho)][1 - \rho]^{k-1} \geqq 0.$$

Thus, $\rho \geqq -[1/(k - 1)]$. Consequently, the correlation between any pair of an infinite collection of interchangeable r.v.'s cannot be negative.

The following simple lemmas which we present without proof are useful. Let X and Y designate the vectors $(X_1, X_2, \cdots, X_k)$ and $(Y_1, Y_2, \cdots, Y_k)$.

LEMMA 1. *If* $X_1, X_2, \cdots, X_k$ *are interchangeable* r.v.'s *and* $Y = \psi(X)$ *is defined by* $Y_j = \Phi[X_j, g(X)], j = 1, 2, \cdots, k$, *where* Φ *and* g *are Borel measurable functions, the latter being symmetric in its* k *arguments, then* $Y_1, Y_2, \cdots, Y_k$ *are interchangeable.*

LEMMA 2. *If* $Y = (Y_1, Y_2, \cdots, Y_k)$ *is a random permutation of the interchangeable* r.v.'s $X_1, X_2, \cdots, X_k$, *then* Y *has the same distribution as* X.

3. Background and Framework. The term "Central Limit Theorem" is a loose designation for one of an agglomeration of theorems dealing with limiting normality of distributions of sums of random variables—in the classical treatment—independent random variables.

The early results of De Moivre and Laplace have been succeeded by ever more powerful theorems set in an increasingly general framework. Recent works [5], [6] commence with a double sequence of rowwise *independent* r.v.'s (i.e., the r.v.'s within each row are independent)

$$\begin{matrix} X_{11}, & X_{12}, & \cdots, & X_{1k_1} \\ X_{21}, & X_{22}, & \cdots, & X_{2k_2} \\ \vdots & \vdots & & \\ X_{n1}, & X_{n2}, & \cdots, & X_{nk_n} \\ \vdots & \vdots & \cdots & \vdots \end{matrix}$$

(where $k_n \to \infty$) and investigate the limiting distributions, i.e., c.d.f.'s of the row sums, say $S_n = \sum_{k=1}^{k_n} X_{nk}$. To render the problem more meaningful the r.v.'s are required to be "infinitesimal" (or asymptotically constant), i.e.,

$$\lim_{n\to\infty} \max_{1 \le i \le k_n} P\{|X_{ni}| > \epsilon\} = 0, \qquad \text{all } \epsilon > 0.$$

A famous theorem of Khintchine asserts that the class of limiting distribution of such sums S_n coincides with the class of infinitely divisible laws [5]. A necessary and sufficient condition that the limiting distribution (assuming one exists) of sums of row-wise independent infinitesimal r.v.'s be normal is well known, namely, $\max_{1<i<k_n} |X_{ni}| \xrightarrow{P} 0$. (This actually implies infinitesimality here). For purposes of comparison with Theorem 1 of the next section we state the following result of Raikov (cf. [5]):

If $Z_{nk}, k = 1, \cdots, k_n$ are infinitesimal rowwise independent r.v.'s with zero means and finite variances σ_{nk}^2 with $\sum_{k=1}^{k_n} \sigma_{nk}^2 = 1$, a necessary and sufficient condition that the c.d.f. of $\sum_{k=1}^{k_n} Z_{nk}$ converges to the normal c.d.f. with mean 0 and variance 1 is that $\sum_{k=1}^{k_n} Z_{nk}^2 \xrightarrow{P} 1$.

Attempts have been made to relax the requirement of independence with varying degrees of success. Perhaps a natural and useful generalization is to double sequences of interchangeable random variables.

In this direction, let $X_{ni}, i = 1, \cdots, k_n$ comprise a (finite) set of i.r.v.'s for every $n = 1, 2, \cdots,$.

If we stipulate that $\lim_{n\to\infty} P\{|X_{n1}| > \epsilon\} = 0$, all $\epsilon > 0$, the question of the nature of the class C^* of all limiting distributions of row sums may again be posed. Clearly, C^* includes all stable distributions but contains others as well. This follows from a result of von Mises [7] who showed that the distribution of the number S_{n,r_n} of unoccupied cells in a random casting of r_n objects into n cells approaches that of the Poisson when n, $r_n \to \infty$ in a manner such that the expected number of vacancies is constant. If the expected proportion of vacancies converges to a constant, then Irving Weiss [9] has shown that the limiting distribution, suitably normalized, is normal. But $S_{n,r_n} = \sum_{i=1}^{n} X_{ni}$ where the X_{ni} are i.r.v.'s assuming the values one or zero (according as the ith cell is empty or not). Therefore, the Poisson distribution and in fact all infinitely divisible distributions belong to C^*.

In this paper, we consider only the case of limiting normal distributions and treat the first of the following two situations:

(a) For each $n = 1, 2, \cdots$, the i.r.v.'s X_{ni}, $i = 1, 2, \cdots, k_n$ have a non-random sum.

(b) For each $n = 1, 2, \cdots$, the i.r.v.'s X_{ni}, $i = 1, \cdots, k_n$ are embeddable in an infinite sequence of i.r.v.'s.[2]

These cases are mutually exclusive since if $\sum_{i=1}^{k_n} X_{ni} = C_n$, the covariance of any pair of i.r.v.'s equals $-[1/(k_n - 1)]$ multiplied by the common variance. But then their correlation is negative, which is precluded (under case b) by a prior remark.

4. I.R.V.'s whose sum is non-random. For each $n = 1, 2, \cdots$, let X'_{nk}, $k = 1, 2, \cdots, k_n(\to \infty)$ be i.r.v.'s with finite variance $\sigma'^2_{nk} = \sigma'^2_{n1} = E(X'_{n1} - EX'_{n1})^2$ and satisfying the linear constraint

$$\text{(i}'\text{)} \qquad \sum_{i=1}^{k_n} X'_{ni} = C_n .$$

Naturally, under such a proviso we must investigate partial rather than complete row sums.

If we define

$$X_{ni} = \frac{1}{\sigma'_{ni}} \left(X'_{ni} - \frac{C_n}{k_n} \right),$$

the X_{ni} are, by Lemma 1, i.r.v.'s satisfying the relationships

$$\text{(i)} \qquad \sum_{i=1}^{k_n} X_{ni} = 0, \qquad n = 1, 2, \cdots,$$

and

(ii) $EX^2_{ni} = \sigma^2_{X_{ni}} = 1, i = 1, 2, \cdots, k_n$ and all $n = 1, 2, \cdots$,

[2] The theorems obtained for the case of infinite sequences of i.r.v.'s overlap results of Professors Blum and Rosenblatt of Indiana University and will appear in a joint publication.

We suppose, therefore, without loss of generality, that for each

$$n = 1, 2, \cdots, \{X_{nk}\}, k = 1, \cdots, k_n(\to \infty)$$

are rowwise i.r.v.'s satisfying (i) and (ii) and possessing the joint c.d.f. $F_n(x_1, x_2, \cdots, x_{k_n})$. We have then

THEOREM 1. *For each* $n = 1, 2, \cdots$, *let* $\{X_{ni}\}$ *be interchangeable random variables satisfying* (i) *and* (ii). *If*

$$\max_{1 \leq k \leq k_n} \frac{|X_{nk}|}{\sqrt{k_n}} \xrightarrow{P} 0, \tag{1}$$

$$\frac{1}{k_n} \sum_{k=1}^{k_n} X_{nk}^2 \xrightarrow{P} 1, \tag{2}$$

and $m_n < k_n$ *is a sequence of positive integers such that* $\lim_{n\to\infty} m_n/k_n = \alpha$, $0 < \alpha < 1$, *then*

$$\lim_{n\to\infty} P\left\{\frac{1}{\sqrt{m_n}} \sum_{k=1}^{m_n} X_{nk} < x\right\} = \frac{1}{\sqrt{2\pi(1-\alpha)}} \int_{-\infty}^{x} \exp\left(-\frac{y^2}{2(1-\alpha)}\right) dy.$$

PROOF. For any set of real numbers x_{ni}, if $\max_{1\leq i \leq k_n} |x_{ni}|/\sqrt{k_n} = o(1)$ and $\lim_{n\to\infty} (1/k_n) \sum_1^{k_n} x_{ni}^2 = 1$, then

$$\max_{1 \leq i \leq k_n} \frac{|x_{ni}|}{\sqrt{\sum_1^{k_n} x_{ni}^2}} = o(1).$$

It follows directly that if the x_{ni} are r.v.'s and the analogous conditions are true in "probability" the conclusion holds "in probability." That is, (1) and (2) imply

$$\max_{1 \leq i \leq k_n} \frac{|X_{ni}|}{\sqrt{\sum_{i=1}^{k_n} X_{ni}^2}} \xrightarrow{P} 0. \tag{5}$$

Next, let $Y_{n1}, \cdots, Y_{nk_n}$ be a randomly selected permutation of $X_{n1}, \cdots, X_{nk_n}$. Then even when it is stipulated that X_{ni} = fixed real number x_{ni}, $i = 1, 2, \cdots, k_n$, the quantity

$$U_n = \left(\sum_{i=1}^{k_n} X_{ni}^2\right)^{-1/2} \sum_{i=1}^{m_n} Y_{ni}$$

is a random variable.

Suppose that for some c.d.f. $G(u)$ and arbitrary $\epsilon > 0$, there exists $\delta_\epsilon > 0$ and integral $N_1(\epsilon)$ (all independent of $x_{n1}, \cdots, x_{nk_n}$) such that

$$\max_{1 \leq i \leq k_n} \frac{|x_{ni}|}{\sqrt{\sum_1^{k_n} x_{ni}^2}} < \delta_\epsilon$$

implies

$$|P\{U_n < u | X_{ni} = x_{ni}, i = 1, \cdots, k_n\} - G(u)| < \epsilon \tag{6}$$

for all $n > N_1(\epsilon)$ and continuity points u of $G(u)$. By (5), there exists $N_2(\epsilon)$ such that for all $n > N_2(\epsilon)$, say

$$\epsilon > P\left\{\max_{1 \leqq i \leqq k_n} \frac{|X_{ni}|}{\sqrt{\sum_1^{k_n} X_{ni}^2}} > \delta_\epsilon\right\} = P\{\bar{A}_n\}. \tag{7}$$

Then, from (6) and (7) for arbitrary $\epsilon > 0$ and $n > \max[N_1(\epsilon), N_2(\epsilon)]$ and continuity points u of $G(u)$,

$$\begin{aligned} &|P\{U_n < u\} - G(u)| \\ &= \left| \int_{R^{k_n}} [P\{U_n < u \mid X_{ni} = x_{ni}, i = 1, \cdots, k_n\} - G(u)]\, dF_n(x_1, \cdots, x_{k_n}) \right| \\ &\leqq \int_{A_n} |P\{U_n < u \mid X_{ni}, i = 1, \cdots, k_n\} - G(u)|\, dF_n + \int_{\bar{A}_n} dF_n \leqq 2\epsilon. \end{aligned} \tag{8}$$

For simplicity in writing, let Q be an r.v. with c.d.f. $G(u)$; then for $\lambda > 0$, $Q_\lambda = (1/\lambda)\, Q$ is an r.v. with distribution $G(\lambda u)$. Under the proviso (6), (8) shows that

$$U_n = \frac{\sum_{i=1}^{m_n} Y_{ni}}{\sqrt{\sum_1^{k_n} X_{ni}^2}} \xrightarrow{L} Q.$$

On the other hand, according to (2),

$$\sqrt{\frac{1}{k_n} \sum_{i=1}^{k_n} X_{ni}^2} \xrightarrow{P} 1.$$

Consequently, (see, e.g., [3]),

$$\frac{1}{\sqrt{m_n}} \sum_{i=1}^{m_n} Y_{ni} = \sqrt{\frac{k_n}{m_n}}\, U_n \sqrt{\frac{1}{k_n} \sum_{i=1}^{k_n} X_{ni}^2} \xrightarrow{L} \frac{1}{\sqrt{\alpha}} Q = Q_{\sqrt{\alpha}}.$$

But by Lemma 2, $\sum_{i=1}^{m_n} Y_{ni}$ and $\sum_{i=1}^{m_n} X_{ni}$ have the same distribution, and thus, under the proviso (6),

$$\frac{1}{\sqrt{m_n}} \sum_{i=1}^{m_n} X_{ni} \xrightarrow{L} Q_{\sqrt{\alpha}}. \tag{9}$$

It remains to verify (6) for $G(u)$ the c.d.f. of $Q = N_{o,\alpha(1-\alpha)}$, where N_{μ,σ^2} represents a normal random variable with mean μ and variance σ^2. To do so, it suffices to prove that

$$U_n = \left(\sum_{i=1}^{k_n} x_{ni}^2\right)^{-1/2} \sum_{i=1}^{m_n} Y_{ni} \xrightarrow{L} Q = N_{o,\alpha(1-\alpha)}, \tag{10}$$

providing $Y_{n1}, Y_{n2}, \cdots, Y_{nk_n}$ is a random permutation of the fixed real numbers

$x_{n1}, x_{n2}, \cdots, x_{nk_n}$, where

$$\max_{1 \leq k \leq k_n} \left(\sum_{i=1}^{k_n} x_{ni}^2 \right)^{-1/2} |x_{nk}| = o(1) \tag{11}$$

and $\sum_{i=1}^{k_n} x_{ni} = 0$. A theorem of Noether [8] states that the distribution of

$$L_n = \sum_{i=1}^{k_n} d_{ni} Y_{ni}$$

converges to the normal distribution (when normalized by its mean and standard deviation) if the d_{ni} are fixed real numbers such that

$$D_{n,s} = \frac{\frac{1}{k_n} \sum_{i=1}^{k_n} (d_{ni} - \bar{d}_n)^s}{\left[\frac{1}{k_n} \sum_{i=1}^{k_n} (d_{ni} - \bar{d}_n)^2 \right]^{s/2}} = O(1) \qquad \text{for } s = 3, 4, \cdots$$

and

$$A_{ns} = \frac{\sum_{i=1}^{k_n} (x_{ni} - \bar{x}_n)^s}{\left[\sum_{i=1}^{k_n} (x_{ni} - \bar{x}_{ni})^2 \right]^{s/2}} = o(1), \qquad \text{for } s = 3, 4, \cdots$$

with $\bar{d}_n = 1/k_n \sum_{i=1}^{k_n} d_{ni}$ and $\bar{x}_r = 1/k_n \sum_{i=1}^{k_n} x_{ni} = 0$. Let $d_{ni} = 1$ for $1 \leqq i \leqq m_n$ and 0 for $m_n + 1 \leqq i \leqq k_n$. Then $D_{n,s} = 0(1)$ for $s = 3, 4, \cdots$. Furthermore, from (11),

$$A_{ns} = \frac{\sum_{i=1}^{k_n} x_{ni}^s}{\left(\sum_{i=1}^{k_n} x_{ni}^2 \right)^{s/2}} \leqq \frac{\max_{1 \leq i \leq k_n} |x_{ni}|^{s-2}}{\left(\sum_{i=1}^{k_n} x_{ni}^2 \right)^{s-2/2}} = o(1) \quad \text{for } s = 3, 4, \cdots.$$

Thus Noether's theorem applies to $L_n = \sum_{i=1}^{m_n} Y_{ni}$ whose mean and variance we shall show to be given by $\mu_n = 0$ and

$$\sigma_n^2 = \left(\sum_{i=1}^{k_n} x_{ni}^2 \right) \frac{m_n(k_n - m_n)}{k_n(k_n - 1)}. \tag{12}$$

Then we shall have $L_n/\sigma_n \xrightarrow{L} N_{0,1}$ and the desired result

$$U_n = \frac{L_n}{\sigma_n} \sqrt{\frac{m_n(k_n - m_n)}{k_n(k_n - 1)}} \xrightarrow{L} N_{0,\alpha(1-\alpha)}.$$

We now conclude by evaluating μ_n and σ_n.

$$E(Y_{ni}) = \sum_{a=1}^{k_n} x_{na}/k_n = 0, \qquad E(Y_{ni}^2) = \sum_{a=1}^{n} x_{na}^2/k_n,$$

$$E(Y_{ni} Y_{nj}) = (\sum_{a \neq b} x_{na} x_{nb})/k_n(k_n - 1) = -\sum_{a=1}^{n} x_{na}^2/k_n(k_n - 1), \qquad i \neq j.$$

Hence

$$\mu_n = E\left(\sum_{i=1}^{m_n} Y_{ni}\right) = 0$$

and

$$\sigma_n^2 = E\left(\sum_{i=1}^{m_n} Y_{ni}\right)^2 = \left(\sum_{i=1}^{k_n} x_{ni}^2\right)\left(\frac{m_n}{k_n} - \frac{m_n(m_n - 1)}{k_n(k_n - 1)}\right),$$

which matches (12), concluding the proof.

COROLLARY 1. *For each positive integer* $n = 1, 2, \cdots$, *let* $\{X_{ni}\}$, $i = 1, 2, \cdots, k_n(\to \infty)$ *be i.r.v.'s satisfying* (i) *and* (ii). *If* $m_n < k_n$ *is a sequence of positive integers with* $\lim_{n\to\infty} m_n/k_n = \alpha$, $0 < \alpha < 1$ *and*

$$E[X_{n1}^4] = o(k_n), \qquad \mathrm{Cov}(X_{n1}^2, X_{n2}^2) = o(1), \tag{3}$$

then the conclusion of the theorem holds.

PROOF. For any $\eta > 0$,

$$P\left\{\max_{1\leq i\leq k_n} \frac{|X_{ni}|}{\sqrt{k_n}} > \eta\right\} = P\left\{\bigcup_1^{k_n}\left[\frac{|X_{ni}|}{\sqrt{k_n}} > \eta\right]\right\}$$

$$\leq k_n P\left\{\frac{|X_{n1}|}{\sqrt{k_n}} > \eta\right\} \leq \frac{k_n E|X_{n1}|^4}{k^2\eta^2} = o(1)$$

and

$$P\left\{\left|\frac{1}{k_n}\sum_1^{k_n}(X_{ni}^2 - 1)\right| > \eta\right\} \leq \frac{E\left[\sum_1^{k_n}(X_{ni}^2 - 1)\right]^2}{k_n^2\eta^2}$$

$$= \frac{k_n E(X_{n1}^2 - 1)^2 + k_n(k_n - 1)\,\mathrm{Cov}\,(X_{n1}^2, X_{n2}^2)}{k_n^2\eta^2} = o(1).$$

COROLLARY 2. *For each* $n = 1, 2, \cdots$, *let* $\{X'_{ni}\}$, $i = 1, 2, \cdots, k_n(\to \infty)$ *be i.r.v.'s with* $\sum_{i=1}^{k_n} X'_{ni} = C_n$ *and* $\sum_{i=1}^{k_n}(X'_{ni})^2 = D_n^2 > 0$. *If*

$$\max_{1\leq i\leq k_n} \frac{|X'_{ni} - C_n/k_n|}{\sqrt{1/k_n(D_n^2 - C_n^2/k_n)}} \xrightarrow{P} 0,$$

then the conclusion of the theorem holds for $(1/\sqrt{m_n})\sum_1^{m_n} X_{ni}$, *where*

$$X_{ni} = \frac{X'_{ni} - C_n/k_n}{[(1/k_n)(D_n^2 - C_n^2/k_n)]^{\frac{1}{2}}}.$$

PROOF. Condition (2) is certainly satisfied since $1/k_n\sum_1^{k_n} X_{ni}^2 = 1$.

COROLLARY 3. *For each* $n = 1, 2, \cdots$, *let* $\{X_{ni}\}$, $i = 1, \cdots, k_n(\to \infty)$ *be i.r.v.'s with* $EX_{n1} = 0$, $EX_{n1}^2 = 1$ *and* $\bar{X}_n = 1/k_n\sum_1^{k_n} X_{nk}$. *If the* $\{X_{ni}\}$ *satisfy* (1), (2), *and*

$$E(X_{n1}X_{n2}) = o(1), \tag{4}$$

then

$$\lim_{n\to\infty} P\left\{\frac{1}{\sqrt{m_n}}\sum_1^{m_n}(X_{ni}-\bar{X}_n)<x\right\}=\frac{1}{\sqrt{2\pi(1-\alpha)}}\int_{-\infty}^{x}\exp\left(-\frac{y^2}{2(1-\alpha)}\right)dy.$$

PROOF. Let

$$Y_{ni}=\sqrt{\frac{k_n}{k_n-1}}\frac{X_{ni}-\bar{X}_n}{\sqrt{1-EX_{n1}X_{n2}}}=a_n(X_{ni}-\bar{X}_n).$$

Then, applying Lemma 1 with $g(X)=\bar{X}$, it follows that the $\{Y_{ni}\}$ are i.r.v.'s. Further, $\sum_{i=1}^{k_n}Y_{ni}=0$ and $EY_{ni}^2=1$. Since

$$0\leqq\max_{1\leqq i\leqq k_n}\frac{|Y_{ni}|}{\sqrt{k_n}}\leqq\frac{2a_n}{\sqrt{k_n}}\max_{1\leqq i\leqq k_n}|X_{ni}|,$$

(1) and (4) imply

$$\max_{1\leqq i\leqq k_n}\frac{|Y_{ni}|}{\sqrt{k_n}}\xrightarrow{P}0.$$

Next, for every $\epsilon>0$,

$$P\left\{\frac{1}{k_n}\left|\sum_{i=1}^{k_n}X_{ni}\right|>\epsilon\right\}\leqq\frac{E\left(\sum_{i=1}^{k_n}X_{ni}\right)^2}{k_n^2\epsilon^2}=(k_n\epsilon^2)^{-1}+\frac{(k_n-1)}{k_n}\frac{EX_{n1}X_{n2}}{\epsilon^2}=o(1).$$

That is, $\bar{X}_n\xrightarrow{P}0$. Thus,

$$\frac{1}{k_n}\sum_{i=1}^{k_n}Y_{ni}^2=\frac{(k_n-1)^{-1}}{(1-EX_{n1}X_{n2})}\left[\sum_{j=1}^{k_n}X_{nj}^2-k_n\bar{X}_n^2\right]\xrightarrow{P}1.$$

A direct application of the theorem to the $\{Y_{ni}\}$ shows that

$$\sqrt{\frac{k_n}{(k_n-1)}\frac{1}{(1-EX_{n1}X_{n2})}}\frac{1}{\sqrt{m_n}}\sum_{i=1}^{m_n}(X_{ni}-\bar{X}_n)\xrightarrow{L}N_{0,1-\alpha},$$

which, in view of (4), implies that

$$\frac{1}{\sqrt{m_n}}\sum_{i=1}^{m_n}(X_{ni}-\bar{X}_n)\xrightarrow{L}N_{0,1-\alpha}.$$

COROLLARY 4. *For each* $n=1,2,\cdots$, *let* $\{X_{ni}\}$, $i=1,\cdots,k_n$, *be i.r.v.'s with* $EX_{n1}=0$, $EX_{n1}^2=1$. *If* m_n *is a sequence of positive integers such that* $\lim_n m_n/k_n=\alpha$, $0<\alpha<1$, *and the* $\{X_{ni}\}$ *satisfy*

$$\text{(4')}\qquad \operatorname{Cov}(X_{n1},X_{n2})=\frac{-1}{k_n-1}+o(k_n^{-1})$$

and either (3) *or* (1) *and* (2), *then*

$$\frac{1}{\sqrt{m_n}}\sum_{i=1}^{m_n}X_{ni}\xrightarrow{L}N_{0,1-\alpha}.$$

PROOF. Since, as shown in the proof of Corollary 1, (3) implies (1) and (2), it suffices to suppose that the latter obtain. But (4′) clearly implies (4) whence, according to Corollary 3,

$$\frac{1}{\sqrt{m_n}}\sum_{i=1}^{m_n} X_{ni} - \sqrt{m_n}\bar{X}_n \xrightarrow{L} N_{0,1-\alpha}.$$

However, for positive arbitrary positive,

$$\begin{aligned} P\{|\sqrt{m_n}\bar{X}_n| > \epsilon\} &\leqq \frac{m_n}{\epsilon^2 k_n^2} E\left(\sum_{i=1}^{k_n} X_{ni}\right)^2 \\ &= \frac{m_n}{\epsilon^2 k_n}\left[1 + (k_n - 1)\left\{\frac{-1}{k_n - 1} + o(k_n^{-1})\right\}\right] \\ &= \frac{m_n}{\epsilon^2 k_n} o(1) \end{aligned}$$

employing (4′). Thus, $\sqrt{m_n}\,\bar{X}_n \xrightarrow{P} 0$ and $1/\sqrt{m_n}\sum_{i=1}^{m_n} X_{ni} \xrightarrow{L} N_{0,1-\alpha}$.

In this instance, not only does $\bar{X}_n \xrightarrow{P} 0$, but even $1/\sqrt{k_n}\sum_{i=1}^{k_n} X_{ni} \xrightarrow{P} 0$, which is perhaps more than might be desired. Note that (4′) automatically prevails if the X_{ni} sum to C_n; in fact, $\mathrm{Cov}(X_{n1}, X_{n2}) = -[1/(k_n - 1)]$ in this case.

Define $Z_{ni} = X_{ni}/\sqrt{k_n}$. If (i) is replaced by (iii), $EX_{ni} = 0$, and (ii) still obtains, then $EZ_{ni} = 0$, $\sum_{i=1}^{k_n}\sigma_{Z_{ni}}^2 = 1$. Conditions (1) and (2) become

$$\max_{1\leqq i\leqq k_n} |Z_{ni}| \xrightarrow{P} 0 \tag{1′}$$

and

$$\sum_{i=1}^{k_n} Z_{ni}^2 \xrightarrow{P} 1. \tag{2′}$$

Then, in view of theorems cited in Section 3, the conditions (2′) implies (1′) (and correspondingly (2) implies (1)) for infinitesimal row-wise *independent* r.v.'s, satisfying (ii) and (iii).

Of course, condition (i) precludes independence. Nonetheless, it should be verified for interchangeable r.v.'s satisfying (i) and (ii) that conditions (1) and (2) do not overlap. This may be seen from the following examples:

EXAMPLE 1. Let $(X_{n1}, X_{n2}, \cdots, X_{n,2n})$ be a random permutation of

$$(\sqrt{n}, -\sqrt{n}, 0, 0, \cdots, 0).$$

Then $\sum_{i=1}^{2n} X_{ni} = 0$, $1/2n\sum_{i=1}^{2n} X_{ni}^2 = 1$, but $\max_{1\leqq i\leqq 2n}|X_{ni}|/\sqrt{2n} = 1/\sqrt{2}$.

EXAMPLE 2. Let $X = (X_{n1}, X_{n2}, \cdots, X_{n,2n}) = (0, 0, \cdots, 0)$ with probability $1 - p_n \to 1$, and otherwise let X be a random permutation of

$$\left(\frac{1}{\sqrt{p_n}}, -\frac{1}{\sqrt{p_n}}, \frac{1}{\sqrt{p_n}}, \cdots, -\frac{1}{\sqrt{p_n}}\right).$$

Then $\sum_{i=1}^{2n} X_{ni} = 0$, $E(X_{ni}^2) = 1$, and the X_{ni} are i.r.v.'s. Now

$$\max_{1 \leqq i \leqq 2n} \frac{|X_{ni}|}{\sqrt{2n}} = \frac{|X_{n1}|}{\sqrt{2n}} \xrightarrow{P} 0.$$

But $1/2n \sum_{i=1}^{2n} X_{ni}^2 = 0$ with probability $1 - p_n \to 1$ and hence converges to zero in probability.

5. Illustrations.

EXAMPLE 1. *Quantiles.* Let k, n be positive integers and $U_1, U_2, \cdots, U_{kn-1}$ independent r.v.'s each uniformly distributed on (0, 1). Take U_j^* to be the jth smallest of $(U_1, U_2, \cdots, U_{kn-1})$, $j = 1, 2, \cdots, kn - 1$. That is, $U_1^* \leqq U_2^* \leqq, \cdots, \leqq U_{kn-1}^*$ are the order statistics from a uniform or rectangular distribution. Designate the successive differences $U_i^* - U_{i-1}^*$ by V_i, $i = 1, 2, \cdots, kn$, where $U_0^* = 0$, $U_{kn}^* = 1$.

It is well known that $V_1, V_2, \cdots, V_{kn}$ are interchangeable random variables adding up to one. In fact, any $kn - 1$ of them have a joint density

$$\begin{aligned} f(v_1, v_2, \cdots, v_{i-1}, v_{i+1}, \cdots, v_{kn}) &= (kn-1)! \quad \text{for} \quad \textstyle\sum_{j \neq i} v_j \leqq 1, v_j \geqq 0, \\ &= 0, \qquad \text{otherwise.} \end{aligned}$$

A routine but tedious calculation or a non-routine exciting application of the Poisson stochastic process yields

$$E[V_1^r] = \binom{kn - 1 + r}{r}^{-1}, \qquad r = 1, 2, \cdots,$$

$$E[V_1^2 V_2^2] = \frac{4(kn-1)!}{(kn+3)!},$$

$$E[V_1 V_2] = \frac{(kn-1)!}{(kn+1)!},$$

$$E[V_1^2 V_2] = \frac{2(kn-1)!}{(kn+2)!}.$$

Further, $V_1, \cdots, V_{kn}$ are i.r.v.'s and likewise $X_{n1}, \cdots, X_{n,k_n}$, where $k_n = kn$ and

$$X_{ni} = \frac{kn\left(V_i - \dfrac{1}{nk}\right)}{\sqrt{(kn-1)(kn+1)^{-1}}}, \qquad i = 1, 2, \cdots, k_n.$$

Moreover, $\sum_{i=1}^{k_n} X_{ni} = 0$ and $\sigma_{X_{ni}}^2 = 1$, $i = 1, \cdots, kn$. The prior array of expected values furnishes the estimates:

$$EX_{n1}^4 = O(n^4)E\left[V_1 - \frac{1}{kn}\right]^4 = O(n^4)O\left(\frac{1}{n^4}\right) = O(1)$$

and

$$\begin{aligned}
&\operatorname{Cov}(X_{n1}^2, X_{n2}^2) \\
&= O(n^4)\operatorname{Cov}\left[\left(V_1 - \frac{1}{kn}\right)^2, \left(V_2 - \frac{1}{kn}\right)^2\right] \\
&= O(n^4)\operatorname{Cov}\left[V_1^2 - \frac{2}{kn}V_1, V_2^2 - \frac{2}{kn}V_2\right] \\
&= O(n^4)\left\{\left[E(V_1^2V_2^2) - \frac{4}{kn}E(V_1^2V_2) + \frac{4}{k^2n^2}E(V_1V_2)\right] - \left[E(V_1^2) - \frac{2}{kn}E(V_1)\right]^2\right\} \\
&= O(n^4)O(n^{-6}) = O(n^{-2}).
\end{aligned}$$

If, now, $m_n = n$, it follows from Corollary 1 to Theorem 1 that

$$\frac{1}{\sqrt{n}} \frac{kn(U_n - 1/k)}{\sqrt{(kn-1)(kn+1)^{-1}}} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_{ni}$$

has a limiting normal distribution with mean zero and variance $1 - 1/k$. The same statement then applies to $k\sqrt{n}(U_n - 1/k)$.

Thus, the sample quantile U_n of order $1/k$ in a sample of $kn - 1$ from a rectangular distribution is asymptotically normal with expected value $1/k$ and variance $(k - 1)/k^3n$.

Clearly, an analogous statement holds with $1/k$ replaced by any real number q in (0, 1). This conclusion extends to other distributions than the rectangular, e.g., if the c.d.f. $F(x)$ has a continuous non-zero derivative at the unique solution x_k of $F(x) - 1/k$. These facts are, of course, well known.

Note, in addition, that

$$\begin{aligned}
EX_{nl}X_{n2} &= O(n^2)E[(V_1 - 1/kn)(V_2 - 1/kn)] \\
&= O(n^2)\left[\frac{1}{kn(kn+1)} - \frac{2}{kn}\frac{1}{kn} + \frac{1}{(kn)^2}\right] \\
&= O(n^2)O(n^{-3}) = o(1).
\end{aligned}$$

Thus, if (for specificity) $k = 2$, an application of Corollary 3 yields the conclusion that

$$\frac{1}{\sqrt{n}}\left[\frac{2n(U_n - \frac{1}{2})}{\sqrt{(2n-1)(2n+1)^{-1}}} - n\bar{X}_n\right] = \sqrt{n}\left[2\sqrt{\frac{2n+1}{2n-1}}(\tilde{X}_n - \tfrac{1}{2}) - \bar{X}_n\right]$$

is normally distributed in the limit with mean zero and variance $\frac{1}{2}$ where $\tilde{X}_n$ denotes the sample median. This appears to be new but hardly of overwhelming interest. A comparable result may be demonstrated in the case of a random casting of r_n objects into n cells referred to in Section 3.

EXAMPLE 2. *Ranks.* Let $R_1, \cdots, R_{k_n}$ be a random permutation of the integers $(1, 2, \cdots, k_n)$. Define

$$X_{ni} = \frac{R_i - \frac{k_n + 1}{2}}{\sqrt{\frac{k_n^2 - 1}{12}}}.$$

Then, $(R_1, \cdots, R_{k_n})$ and $(X_{n1}, \cdots, X_{n,k_n})$ each comprise a set of i.r.v.'s. Moreover,

$$\sum_{i=1}^{k_n} X_{ni} = 0, \qquad \sum_{i=1}^{k_n} X_{ni}^2 = 1,$$

and

$$\max_{1 \le i \le k_n} \frac{|X_{ni}|}{\sqrt{k_n}} \frac{\sqrt{12}}{\sqrt{k_n}} \frac{\left| k_n - \frac{k_n + 1}{2} \right|}{\sqrt{k_n^2 - 1}} = \frac{1}{\sqrt{k_n}} \sqrt{\frac{3(k_n - 1)}{(k_n + 1)}} \to 0.$$

A direct application of Corollary 2 of Theorem 1 yields the limiting normality (mean 0, variance $1 - \alpha$) of

$$\sum_{i=1}^{m_n} \frac{R_i - \frac{k_n + 1}{2}}{\sqrt{\frac{k_n^2 - 1}{12}}},$$

where $\lim_{n \to \infty} m_n/k_n = \alpha, 0 < \alpha < 1$, a familiar result.

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