SOME EXACT RESULTS FOR THE FINITE DAM

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1. Summary. In the discrete finite dam model due to Moran, the storage process $\{Z_t\}$ is known to be a Markov chain. Stationary distributions of Z_t are obtained for the cases where the release is a unit amount of water per unit time, and the input is of (i) geometric, (ii) negative binomial and (iii) Poisson type.

The paper concludes with a discussion of the problem of emptiness in the finite dam and considers the probability that, starting with an arbitrary storage, the dam becomes empty before it overflows.

2. Introduction. This paper is concerned with a storage system whose probability model is due to Moran [9]. The storage Z_t of a dam of finite capacity K is defined for discrete time t ($t = 0, 1, 2, \cdots$) as the dam content just after an instantaneous release at t, and just before an input X_t flows into it over the time-interval $(t, t + 1)$. The model is subject to the conditions that

(i) the inputs X_t during the intervals $(t, t + 1)$ are independently and identically distributed;

(ii) there is an overflow $\text{Max}(Z_t + X_t - K, 0)$ during the interval $(t, t + 1)$, a quantity $\text{Min}(K, Z_t + X_t)$ being left in the dam just before the release occurs; and

(iii) the amount of water released at time $t + 1$ is $\text{Min}(M, Z_t + X_t)$ where M is a constant ($<K$).

A fuller description of the model and further references on the subject are given by Gani [3]. It is seen the stochastic processes $\{Z_t\}$ and $\{Z_t + X_t\}$ are both Markov chains, and the problem of obtaining their stationary distributions, given the probability distribution of the input, is of some interest. Moran ([9], [10]) and Gani and Moran [4] have obtained a few approximate solutions to this problem by numerical methods, and some important observations on the solution in the general case have been made by Moran [11], but the only exact solution known so far is the one due to Moran [10] for the case of the geometric input. The problem is considerably simplified when $K = \infty$, i.e. when the dam is of infinite capacity; it is then seen (Gani and Prabhu, [5]) that the transition-matrix of the Markov chain $\{Z_t + X_t\}$ also occurs in the theory of queues in connection with the length of a queue at epochs just before service. For this case Bailey [1] has obtained, by the method of probability generating functions (p.g.f.), the stationary distributions arising from a given distribution of X_t. A dam of finite capacity K can be considered as the analogue of a queueing system in which there is accommodation for only K customers to wait, those in excess of K being compelled to leave the queue altogether (as may happen, for

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instance, in an airport of limited capacity); we proceed to obtain for such a dam the stationary distribution of the storage Z_t.

3. Stationary distribution of the storage. We shall be concerned with the case where M, the amount of water released at time t, is unity. Let $\{g_j\}$ be the probability distribution of X_t, so that

$$\Pr\{X_t = j\} = g_j, \qquad (j = 0, 1, 2, \cdots). \tag{1}$$

We assume that $g_j > 0$ for all j. Also, let

$$G(z) = \sum_{j=0}^{\infty} g_j z^j, \qquad |z| < 1 \tag{2}$$

be the p.g.f. of $\{g_j\}$, and

$$\rho = G'(1) = \sum_{j=0}^{\infty} j g_j \tag{3}$$

the mean input. The transition-matrix of the Markov chain $\{Z_t\}$ is $P \equiv \{P_{ij}\}$, where

$$P = \begin{array}{c|cccccc} \nearrow & 0 & 1 & \cdots & K-2 & K-1 \\ \hline 0 & g_0 + g_1 & g_2 & \cdots & g_{K-1} & h_K \\ 1 & g_0 & g_1 & \cdots & g_{K-2} & h_{K-1} \\ 2 & 0 & g_0 & \cdots & g_{K-3} & h_{K-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ K-1 & 0 & 0 & \cdots & g_0 & h_1 \end{array} \tag{4}$$

where $h_i = \sum_{j=i}^{\infty} g_j$, $(i = 1, 2, \cdots, K)$. Clearly, the chain is irreducible and contains a finite number K of states, so that the stationary probability distribution $\{u_i\}$, $(i = 0, 1, \cdots, K-1)$ exists, where the u_i are the unique solutions of the equations

$$u_j = \sum_{i=0}^{K-1} u_i p_{ij}, \qquad (j = 0, 1, \cdots, K-1) \tag{5}$$

together with $\sum_0^{K-1} u_i = 1$. We first prove the following theorem due to Moran [11].

THEOREM.

(i) *If* $\{u_i^{(K)}\}$, $(i = 0, 1, \cdots, K-1)$ *is the stationary probability distribution of storage in a dam of capacity* K, *then the ratios*

$$v_i = \frac{u_i^{(K)}}{u_0^{(K)}}, \qquad (i = 1, 2, \cdots, K-1) \tag{6}$$

are independent of K, *and*

(ii) *the v_i's can be found as the coefficients of z^i in $V(z)$, where*

$$v(z) = \frac{g_0(1 - z)}{G(z) - z} \tag{7}$$

The first part of the theorem is easily proved; in fact, writing out the equations (5) in full we obtain

$$\begin{aligned} u_0 &= (g_0 + g_1)u_0 + g_0 u_1 \\ u_1 &= g_2 u_0 \quad + g_1 u_1 + g_0 u_2 \\ &\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \\ u_{K-2} &= g_{K-1}u_0 + g_{K-2}u_1 + \cdots + g_0 u_{K-1} \\ u_{K-1} &= h_K u_0 \quad + h_{K-1}u_1 + \cdots + h_1 u_{K-1} \end{aligned}$$

Solving these equations successively for the ratios $v_i = u_i / u_0$ we obtain

$$\begin{aligned} v_1 &= \frac{1 - g_0 - g_1}{g_0} \\ v_2 &= \frac{1 - g_1}{g_0} v_1 + \frac{g_2}{g_0} \end{aligned} \tag{8}$$

and in general, the v_i's $(i = 1, 2, \cdots, K - 1)$ are seen to be independent of K. Now consider the function $V(z)$ defined by (7). We shall first prove that $V(z)$ can be expanded as a power series which is convergent for suitable values of $|z|$. Let us first consider the case $\rho \leqq 1$. Writing

$$G(z) - z = (1 - z)\left\{1 - \frac{1 - G(z)}{1 - z}\right\}$$

and following Kendall ([6], p. 159) we obtain

$$\frac{1 - G(z)}{1 - z} = \sum_{n=0}^{\infty} z^n \sum_{n+1}^{\infty} g_i$$

so that, for $|z| < 1$,

$$\left|\frac{1 - G(z)}{1 - z}\right| < \sum_{n=0}^{\infty} \sum_{n+1}^{\infty} g_i = \sum_{i=1}^{\infty} i g_i = \rho \leqq 1$$

Hence $|G(z) - z| \neq 0$, and we have the power series expansion

$$V(z) = g_0\left\{1 - \frac{1 - G(z)}{1 - z}\right\}^{-1} = v_0 + v_1 z + v_2 z^2 + \cdots$$

convergent for $|z| < 1$.

Next, let $\rho > 1$. In this case there exists a positive λ such that the power series expansion

$$\frac{1}{G(z) - z} = c_0 + c_1 z + c_2 z^2 + \cdots$$

is valid for $|z| < \lambda$ (Knopp, [8], p. 182). Hence it follows that $V(z)$ also possesses a power series expansion convergent for $|z| < \lambda$.

Thus whether or not $\rho \leqq 1$, $V(z)$ has a power series expansion

$$V(z) = \frac{g_0(1 - z)}{G(z) - z} = v_0 + v_1 z + v_2 z^2 + \cdots \tag{9}$$

The coefficients v_i are determined from the relation

$$g_0(1 - z) = \{G(z) - z\} \sum_{i=0}^{\infty} v_i z^i$$

and hence it is seen that $v_0 = 1$, and $v_1, v_2, \cdots, v_{K-1}$ satisfy the relations (8). Thus they are, in fact, the quantities defined in (6).

If $\rho < 1$, the stationary probability distribution exists in the case of the infinite chain ($K = \infty$), and its g.f. is proportional to $V(z)$. However, the above results hold, as we have shown, even when $\rho \geqq 1$. It is now obvious that the general method of obtaining the stationary probability distribution $\{u_i\}$ for the discrete dam of finite capacity K consists of (i) finding $V(z)$, (ii) expanding $V(z)$ to obtain the v_i's, and (iii) normalising $v_0, v_1, \cdots, v_{K-1}$ to obtain a probability distribution. We proceed to do this in some particular cases.

3.1. Geometric input. Consider, for instance, an input distribution of the geometric type,

$$g_j = \Pr\{X_t = j\} = ab^j, \qquad (j = 0, 1, \cdots,) \tag{10}$$

where $0 < a < 1$ and $b = 1 - a$. The p.g.f. of X_t is then

$$G(z) = \frac{a}{1 - bz} \tag{11}$$

and the function $V(z)$ is given by

$$\begin{aligned} V(z) &= \frac{a(1 - z)}{a(1 - bz)^{-1} - z} = \frac{1 - bz}{1 - \rho z} \\ &= (1 - bz) \sum_{i=0}^{\infty} \rho^i z^i \left(|z| < \min\left(\frac{1}{\rho}, 1\right)\right), \end{aligned} \tag{12}$$

where $\rho = b/a$ is the mean input. Hence we obtain

$$v_0 = 1, \qquad v_i = \rho^i - b\rho^{i-1} = b\rho^i, \qquad (i = 1, 2, \cdots, K - 1)$$

and

$$\sum_0^{K-1} v_i = 1 + b \sum_{i=1}^{K-1} \rho^i = a \frac{1 - \rho^{K+1}}{1 - \rho}.$$

The stationary distribution in this case is therefore given by $\{u_i\}$, where

$$u_0 = \frac{(1 - \rho)}{a(1 - \rho^{K+1})}, \qquad u_i = \frac{\rho^{i+1}(1 - \rho)}{1 - \rho^{K+1}}, \qquad (i = 1, 2, \cdots, K - 1). \tag{13}$$

Thus the storage of a dam of finite capacity K into which flows an input of the geometric type has a stationary distribution of the geometric type, which is truncated at $Z = K - 1$ and has a modified initial term. This result is implied in Moran's solution (referred to in Section 2) for the general case $M > 1$, although it is not explicitly mentioned by him; for $M = 1$ his solution is given by the formulae $u_0 = \pi_0 + \pi_1$, $u_i = \pi_{i+1}$ $(i = 1, 2, \cdots, K - 1)$, where

$$\pi_{K-r}/\pi_K = {}^rS_1\, a - {}^{r-1}S_2\, a^2 + {}^{r-2}S_3\, a^3 \cdots$$

$$(14) \qquad {}^nS_p = \binom{n-1}{p-1} b^{-n} - \binom{n-2}{p-1} b^{1-n}. \qquad (r = 1, 2, \cdots, K)$$

From this we obtain (13) after some simple reduction.

3.2. Negative binomial input. Consider next the more general case of the negative binomial input,

$$(15) \qquad g_j = \Pr\{X_t = j\} = n_j \binom{n+j-1}{j} a^n b^j, \qquad (j = 0, 1, \cdots)$$

where $0 < a < 1$, $b = 1 - a$, and n is a positive integer; the p.g.f. of X_t is then

$$G(z) = \frac{a^n}{(1 - bz)^n}$$

and the mean input is $\rho = nb/a$. We have then

$$(16) \qquad V(z) = \frac{a^n(1 - z)}{a^n(1 - bz)^{-n} - z} = \frac{a^n(1 - z)(1 - bz)^n}{a^n - z(1 - bz)^n}$$

Obviously $z = 1$ is a zero of the denominator of the expression on the right hand side of (16); in addition to this it has n other zeros $z_1, z_2, \cdots, z_n$. We consider here the case where $z_1, z_2, \cdots, z_n$ are all distinct and different from unity; however, the general case can be treated along similar lines. When $(1, z_1, z_2, \cdots, z_n)$ are all different we can break up $V(z)$ into partial fractions of the form

$$(17) \qquad V(z) = d_0 + \sum_{p=1}^{n} \frac{d_p}{1 - z/z_p}$$

where obviously $d_0 = a^n$ and the d_p's are given by

$$d_p = \lim_{z \to z_p} \left(1 - \frac{z}{z_p}\right) V(z)$$

$$(18) \qquad = \lim_{z \to z_p} \frac{a^n(1 - z)(1 - bz)^n \left(1 - \frac{z}{z_p}\right)}{a^n - z(1 - bz)^n} = \frac{a^n(1 - 1/z^p)}{\rho a z_p/(1 - bz_p) - 1}$$

$$(p = 1, 2, \cdots, n).$$

Now let λ be the least among the quantities $1, |z_1|, |z_2|, \cdots, |z_n|$; then for $|z| < \lambda$ we can express each term under the summation sign in (17) as a power series. Thus

$$V(z) = d_0 + \sum_{p=1}^{n} d_p \sum_{i=0}^{\infty} \left(\frac{z}{z_p}\right)^i = d_0 + \sum_{i=0}^{\infty} z^i \sum_{p=1}^{n} d_p \left(\frac{1}{z_p}\right)^i,$$

whence we obtain

$$(19)\qquad \begin{aligned} v_0 &= d_0 + \sum_{p=1}^{n} d_p = \lim_{z\to 0} V(z) = 1 \\ v_i &= \sum_{p=1}^{n} \frac{d_p}{(z_p)^i}, \qquad (i = 1, 2, \cdots, K-1), \end{aligned}$$

so that

$$\sum_{i=0}^{K-1} v_i = d_0 + \sum_{p=1}^{n} d_p \sum_{i=0}^{K-1} \left(\frac{1}{z_p}\right)^i = d_0 + \sum_{p=1}^{n} d_p \frac{1-(1/z_p)^K}{1-(1/z_p)}.$$

It follows that the stationary probabilities u_i are given by

$$(20)\qquad \begin{aligned} u_0 &= \left\{d_0 + \sum_{p=1}^{n} d_p \frac{1-(1/z_p)^K}{1-(1/z_p)}\right\}^{-1} \\ u_i &= u_0 \sum_{p=1}^{n} d_p \left(\frac{1}{z_p}\right)^i, \qquad (i = 1, 2, \cdots K-1). \end{aligned}$$

From (20) we see that the stationary distribution of the dam storage is the weighted sum of n geometric distributions, each of which is truncated at $Z = K - 1$, and has a modified initial term.

3.3. Poisson input. Finally we consider the case where the input has the Poisson distribution with mean ρ,

$$(21)\qquad g_j = \Pr\{X_t = j\} = e^{-\rho}\frac{\rho^j}{j!}, \qquad (j = 0, 1, \cdots).$$

The rigorous procedure here consists of writing down $V(z)$ and obtaining the coefficients v_i by complex variable methods. We shall, however, argue heuristically and consider (21) as the limiting case of the negative binomial (15) as $n \to \infty$, $a \to 1$ and $\rho = nb / a$ is held fixed. In fact, putting $a = 1 / (1 + \rho/n)$, $b = \rho/(n + \rho)$, it is seen that the p.g.f. of (15) reduces to

$$(1 + \rho/n)^{-n}\left\{1 - \frac{1}{n}\rho\left(1 + \frac{1}{n}\rho\right)^{-1} z\right\}^{-n} \to e^{-\rho+\rho z}$$

which is the p.g.f. of (21). Also, $d_0 \to e^{-\rho}$, and

$$d_p = \frac{a_n(1 - 1/z_p)}{\rho a z_p/(1 - b z_p) - 1} \to \frac{e^{-\rho}(1 - 1/z_p)}{\rho z_p - 1}$$

where $z_1, z_2, \cdots$ are the roots (other than unity) of the equation

$$e^{-\rho+\rho z} = z \tag{22}$$

(which are infinite in number). Hence the stationary probabilities of the dam storage are given by

$$\begin{aligned} u_0 &= \left\{e^{-\rho} + \sum_{p=1}^{\infty} \frac{e^{-\rho}(1 - 1/z_p)}{\rho z_p - 1} \frac{1 - (1/z_p)^K}{1 - (1/z_p)}\right\}^{-1} \\ u_i &= u_0 \sum_{p=1}^{\infty} \frac{e^{-\rho}(1 - 1/z_p)}{\rho z_p - 1} \left(\frac{1}{z_p}\right)^i, \qquad (i = 1, 2, \cdots, K - 1). \end{aligned} \tag{23}$$

4. The problem of emptiness in the finite dam. The analogy between the dam process and the random walk has already been pointed out by several authors (see the discussion in [3]). In fact, putting $U_t = X_t - 1$, we see that the storage Z_t in a dam of capacity K satisfies the relations

$$Z_{t+1} = \begin{cases} Z_t + U_t & \text{if } 0 < Z_t + U_t < K - 1 \\ 0 & \text{if } \quad Z_t + U_t \leqq 0 \\ K - 1 & \text{if } \quad Z_t + U_t \geqq K - 1 \end{cases} \tag{24}$$

which, however, define a random walk with impenetrable barriers at $Z = 0$ and $Z = K - 1$. If $K = \infty$, there is only the first barrier and the problem of 'duration of the game' (i.e. the distribution of time required for the dam to become empty for the first time) has been discussed by Kendall [7] for the case where the input is of the Gamma type and the release is continuous. For finite K this problem is much more difficult; however, for this case we propose to discuss the probability of absorption at $Z = 0$ (i.e. the conditional probability V_i that, starting with a storage $Z_0 = i$, the dam becomes empty ($Z_t = 0$) before it overflows). This is a familiar problem in random walk theory, and has been discussed, for instance, by Feller ([2], pp. 300–303); it is seen that the probabilities V_i $(i = 1, 2, \cdots, K - 2)$ satisfy the relations

$$\begin{aligned} V_1 &= \sum_{j=1}^{K-2} g_j V_j + g_0 \\ V_i &= \sum_{j=i-1}^{K-2} g_{j-i+1} V_j, \qquad (i = 2, 3, \cdots, K - 2). \end{aligned}$$

These equations simplify to some extent if we note that the states 0 and $K - 1$ are absorbing, so that $V_0 = 1$, $V_{K-1} = 0$; for we can then write

$$V_i = \sum_{j=i-1}^{K-1} g_{j-i+1} V_j, \qquad (i = 1, 2, \cdots, K - 2). \tag{25}$$

Clearly, the coefficients on the right hand side of these equations correspond to the rows of the transition-matrix (4). It will now be found easiest to start at the bottom right hand corner and work up to the left: thus

$$g_0 V_{K-3} + g_1 V_{K-2} + h_2 \cdot 0 = V_{K-2}$$

so that

$$V_{K-3} = \frac{1 - g_1}{g_0} V_{K-2},$$

and similarly

$$V_{K-4} = \frac{(1 - g_1) V_{K-3} - g_2 V_{K-2}}{g_0},$$

etc. This shows that the ratios of the quantities

$$w_i = V_{K-1-i} \tag{26}$$

are again independent of K ($w_0 = 0$, $w_{K-1} = 1$); rewriting the equations (25) in terms of these quantities we obtain

$$w_i = \sum_{j=0}^{i} g_j w_{i-j+1}, \qquad (i = 1, 2, \cdots, K - 2). \tag{27}$$

Consider the system of equations (27) for $i = 1, 2, \cdots$ ad. inf., and put

$$W(z) = \sum_{i=1}^{\infty} \frac{w_i}{w_1} z^{i-1} \tag{28}$$

we have

$$\begin{aligned} zW(z) &= \sum_{i=1}^{\infty} \frac{z^i}{w_1} \sum_{j=0}^{i} g_j w_{i-j+1} \\ &= \sum_{j=1}^{\infty} g_j \sum_{i=j}^{\infty} \frac{w_{i-j+1}}{w_1} z^i + g_0 \sum_{i=1}^{\infty} \frac{w_{i+1}}{w_1} z^i \\ &= \sum_{j=1}^{\infty} g_j \sum_{i=1}^{\infty} \frac{w_i}{w_1} z^{1+j-1} + g_0 \sum_{i=2}^{\infty} \frac{w_i}{w_1} z^{i-1} \\ &= G(z)W(z) - g_0, \end{aligned}$$

whence we obtain the relation

$$W(z) = \frac{g_0}{G(z) - z}. \tag{29}$$

Folowing the same lines of argument as for $V(z)$, we can prove that $W(z)$ can be expanded as a power series convergent for suitable values of $|z|$. Let $W(z) = \sum_{i=0}^{\infty} \omega_{i+1} z^i$; then since $\omega_{K-1} = w_{K-1} / w_1 = 1/w_1$, we must have

$$w_i = \frac{\omega_i}{\omega_{K-1}}, \qquad (i = 1, 2, \cdots, K - 2) \tag{30}$$

which are, therefore, the required solutions to the equations (27). The absorption probabilities V_i can then be obtained from (26).

Let us now consider the particular case where the input is geometric with probabilities $g_j = ab^j$, $(j = 0, 1, 2, \cdots)$, and $G(z) = a(1 - bz)^{-1}$; then (29) gives

$$W(z) = \frac{a}{a(1-bz)^{-1} - z} = \frac{(1-bz)}{(1-z)(1-\rho z)}$$

$$(31) \qquad = \begin{cases} \dfrac{a}{1-\rho}\left\{\dfrac{1}{1-z} - \dfrac{\rho^2}{1-\rho z}\right\} & \text{if } \rho \neq 1 \\[2ex] \dfrac{1-bz}{(1-z)^2} & \text{if } \rho = 1. \end{cases}$$

Hence it follows that

$$(32) \qquad \omega_i = \begin{cases} \dfrac{a(1-\rho^{i+1})}{1-\rho} & \text{if } \rho \neq 1 \\[2ex] a(i+1) & \text{if } \rho = 1 \end{cases} \qquad (i \neq 1)$$

and

$$(33) \qquad w_i = \begin{cases} \dfrac{1-\rho^{i+1}}{1-\rho^K} & \text{if } \rho \neq 1 \\[2ex] \dfrac{i+1}{K} & \text{if } \rho = 1 \end{cases} \qquad (i = 1, 2, \cdots, K-2).$$

Thus the absorption probabilities V_i in the case of the geometric input are given by

$$(34) \qquad V_i = \begin{cases} \dfrac{1-\rho^{K-i}}{1-\rho^K} & \text{if } \rho \neq 1 \\[2ex] 1 - \dfrac{i}{K} & \text{if } \rho = 1 \end{cases} \qquad (i = 1, 2, \cdots, K-2).$$

A similar procedure could be used, when the input is of a more general type, to obtain the exact expressions for the probabilities V_i. However, in many cases, it may suffice to know the bounds within which V_i lie, and these bounds are given by Feller ([2], inequalities 8.11 and 8.12 on p. 303). In fact, noting that $E(U_t) = E(X_t - 1) = \rho - 1$, where ρ is the mean input, we have that

$$(35) \qquad \begin{aligned} \frac{z_0^{K-1} - z_0^i}{z_0^{K-1} - 1} &\leqq V_i \leqq 1 && \text{if } \rho < 1 \\ \frac{z_0^i - z_0^{K-1}}{1 - z_0^{K-1}} &\leqq V_i \leqq z_0^i && \text{if } \rho > 1 \\ 1 - \frac{i}{K-1} &\leqq V_i \leqq 1 && \text{if } \rho = 1 \end{aligned}$$

where z_0 is the unique positive root (other than unity) of the equation $\sum_j z^j \Pr\{U_t = j\} = 1$, i.e. $\sum_{j=0}^{\infty} g_j z^j = z$, and $z_0 \gtrless 1$ according as $\rho \lessgtr 1$.

5. Concluding remarks and acknowledgements. When the input X_t has a continuous probability distribution, it is seen that the stationary distribution function of $Z_t + X_t$ satisfies an integral equation, which has been solved by the author in a recent paper (Prabhu, [12]) for the special case when the input distribution is of the Gamma type. A more realistic problem on which some work is in progress at the moment is the one dealing with the finite dam process in continuous time; however, our solutions for discrete time may be taken as useful approximations to this continuous case.

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