A MARKOVIAN FUNCTION OF A MARKOV CHAIN[1]

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1. Statement of the problem and the results obtained. Consider a Markov chain $X(n)$, $n = 0, 1, 2, \cdots$, with a finite number of states $1, \cdots, m$ and stationary transition probability matrix $P = (p_{ij})$

$$p_{ij} = P[X(n+1) = j \mid X(n) = i] \geqq 0, \qquad i, j = 1, \cdots, m, \tag{1}$$
$$\textstyle\sum_j p_{ij} = 1.$$

The probability structure of the chain is determined by P and the initial probability distribution vector $p = (p_i)$

$$p_i = P[X(0) = i] \geqq 0, \qquad i = 1, \cdots, m, \tag{2}$$
$$\textstyle\sum_i p_i = 1.$$

Suppose the experimenter does not observe the process $X(n)$ but rather a derived process $Y(n) = f(X(n))$ where f is a given function on $1, \cdots, m$. The states i of the original process $X(n)$ on which f equals some fixed constant are collapsed into a single state of the new process $Y(n)$. Call these collapsed sets of states S_i, $i = 1, \cdots, r$, $r \leqq m$. A natural question that arises is as to whether or not the new process is Markovian. It is clear that this is not generally the case.

Let us restrict ourselves to a process $X(n)$ with its initial probability distribution a left invariant vector of the matrix P, that is, $pP = p$. Further assume that all the components of p are positive (all transient states are thrown out). Let D be the diagonal matrix with its ith diagonal entry p_i. The process is said to be reversible if

$$DP = P'D$$

(P' is the transpose of P). The following result is obtained:

Theorem 1. *Let $X(n)$ be a stationary reversible process with $p_i > 0$ for all i. Then $Y(n)$ is Markovian if and only if for any fixed $\beta = 1, \cdots, r$*

$$\sum_{j \varepsilon S_\beta} p_{ij} = P[X(n+1) \,\varepsilon\, S_\beta \mid X(n) = i] = C_{S_\alpha, S_\beta} \tag{3}$$

has the same value for all i in any given collapsed set of states S_α, $\alpha = 1, \cdots, r$.[2]

A slightly different problem can be phrased in the following way. Let

$$w = (w_i),\ w_i > 0,\ i = 1, \cdots, m$$

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[2] J. L. Snell pointed out that the original proof, given for Markov processes $X(n)$ with a symmetric P, holds for the reversible processes.

be any initial probability distribution. Consider the Markov process $X(n)$ generated by initial distribution w and transition probability matrix P. Again consider $Y(n) = f(X(n))$ and require that $Y(n)$ be Markovian whatever the initial distribution w.

COROLLARY 1. *A sufficient condition that $Y(n)$ be Markovian whatever the initial distribution w of $X(n)$ is given by* (3). Nonetheless, condition (3) is not generally necessary if the collapsed process is to be Markovian even in the problem covered in Corollary 1.

THEOREM 2. *Let f be a function that collapses only one class of states S. $Y(n)$ is Markovian whatever the initial distribution w of $X(n)$ if and only if one of the following two conditions is satisfied*:

(4) (i) $$\sum_{l \varepsilon S} p_{kl}\, p_{lu} = p_{k,S}\, C_u$$

for all $u \not\varepsilon S$ and all k;

(5) (ii) $$p_{i,S} = 0 \quad \text{for all} \quad i \not\varepsilon S.$$

Here

$$p_{k,S} = \sum_{j \varepsilon S} p_{kj} = P[X(n+1) \varepsilon S \mid X(n) = k].$$

An example of a Markov chain satisfying (4) but not (3) is given in the body of the paper.

Condition (4) naturally suggests the condition given in Corollary 2.

COROLLARY 2. *A sufficient condition that $Y(n)$ be Markovian, whatever the initial distribution w of $X(n)$, is given by*

(4′) $$\sum_{l \varepsilon S_\alpha} p_{kl}\, p_{l,S_\beta} = p_{k,S_\alpha}\, C_{S_\alpha, S_\beta}$$

for all k, α, β.

Suppose we now go back and consider the class of stationary Markov chains $X(n)$ with $p_i > 0$, $i = 1, \cdots, m$, such that $Y(n) = f(X(n))$ is Markovian for any many-one transformation f.

THEOREM 3. *Let $X(n)$ be a stationary Markov chain with $p_i > 0$ $i = 1, \cdots, m$. $f(X(n))$ is Markovian for every many-one transformation f if and only if the transition probability matrix P of $X(n)$ is of the form*

(6) $$P = \alpha I + (1 - \alpha)U,$$

where U is a matrix with identical rows and α is a real number.

It is interesting to note that when one goes to the case of a decent continuous parameter Markov chain with a finite number of states, the analogue of (3) becomes almost necessary for $Y(t)$ to be Markovian, whatever the initial probability distribution w of $X(t)$.

THEOREM 4. *Let $X(t)$, $0 \leqq t < \infty$, be a Markov chain with a finite number of states $i = 1, \cdots, m$ and stationary transition probability function*

$$P(t) = (p_{ij}(t))$$
$$p_{ij}(t) = P[X(t + \tau) = j \mid X(\tau) = i]$$

continuous in t. Assume that

$$\lim_{t \downarrow 0} P(t) = I.$$

Clearly

$$P(t)P(s) = P(t + s), \qquad t, s > 0.$$

Let the initial probability distribution of $X(t)$ be w, $w_i > 0$, $i = 1, \cdots, m$. Then $Y(t) = f(X(t))$ is Markovian, whatever the initial distribution w of $X(t)$, if and only if for each $\beta = 1, \cdots, r$ separately either

$$(7) \quad \begin{array}{ll} \text{(i)} & p_{i,S_\beta}(t) \equiv 0 \quad \text{for all} \quad i \not\varepsilon S_\beta \quad \text{or} \\ \text{(ii)} & p_{i,S_\gamma}(t) = C_{S_\beta,S_\gamma}(t) \quad \text{for every} \quad i \varepsilon S_\beta \quad \text{and all} \quad \gamma = 1, \cdots, r. \end{array}$$

Part of the interest in the proofs of Theorems 1 and 4 lies in the fact that they show that if the collapsed processes in these cases satisfy the Chapman-Kolmogorov equations, they are Markovian.

Condition (3) can be reworded in the case of a Markov process $X(t)$, $0 \leqq t < \infty$, with stationary transition probabilities and values in an abstract space. Let Ω be a space of points x and $B(\Omega)$ a Borel field on Ω. Further let the sets (x) be elements of $B(\Omega)$. Consider a function

$$P(t; x, A), \qquad A \varepsilon B(\Omega)$$

satisfying

(i) $P(t; x, A)$ is a Baire function of x for fixed t, A;
(ii) $P(t; x, A)$ is a probability measure in $A \varepsilon B(\Omega)$ for fixed t, x;
(iii) $P(t; x, A)$ satisfies the Chapman-Kolmogorov equation

$$P(t + \tau; x, A) = \int_\Omega P(t; y, A) P(\tau; x, dy), \qquad t, \tau > 0.$$

Let $X(t)$ be a Markov chain with $P(t; x, A)$ as its transition probability function. Let f be a function from Ω onto another space of points Ω'. The function f induces a Borel field of sets $B(\Omega') = f(B(\Omega))$ on Ω'. This consists of sets of the form $fA = (y \varepsilon \Omega' \mid y = f(x), x \varepsilon A)$, $A \varepsilon B(\Omega)$. Now consider the inverse images of sets in $f(B(\Omega))$. The class of sets of this form we call $f^{-1}f(B(\Omega))$ and it is a subBorel field of $B(\Omega)$ consisting of sets of the form

$$\{z \varepsilon \Omega \mid z = f^{-1}f(x), x \varepsilon A\}, \qquad A \varepsilon B(\Omega).$$

The analogue of condition (3) is simply that

$$(8) \qquad P(t; x, A), \qquad A \varepsilon f^{-1}f(B(\Omega))$$

be a Baire function of x with respect to $f^{-1}f(B(\Omega))$ for fixed t, A.

COROLLARY 3. *$Y(t) = f(X(t))$ is a Markov process, whatever the initial probability distribution of $X(t)$, if condition* (8) *is satisfied.* Condition (8) is discussed in a paper of B. Rankin [4] as a sufficient condition for a collapsed Markovian process to be Markovian.

2. The stationary case. Let the assumptions of Theorem 1 be satisfied. The matrix of n-step transition probabilities of the process $Y(n)$ is of the form

$$Q^{(n)} = AP^nB = (q_{\alpha\beta}^{(n)}) = (P[X(t+n) \varepsilon S_\beta \mid X(t) \varepsilon S_\alpha]), \tag{9}$$

where A, B are $r \times m$ and $m \times r$ matrices respectively. The elements of B are of the form

$$b_{ij} = \begin{cases} 1 & \text{if} \quad i \varepsilon S_j, \\ 0 & \text{otherwise}; \end{cases}$$

while

$$A = (B'DB)^{-1}B'D, \tag{10}$$

where D is the diagonal matrix introduced above. If the new process is Markovian, the Chapman-Kolmogonov equation must be satisfied by the $Q^{(n)}$, that is,

$$Q^{(n)} = AP^nB = [Q^{(1)}]^n = (APB)^n, \qquad n = 2, 3, \cdots. \tag{11}$$

This condition can be reworded in an equivalent form

$$AP^nBAPB = AP^{n+1}B, \qquad n = 1, 2, 3, \cdots. \tag{12}$$

Note that

$$BAPB = PB \tag{13}$$

implies that (12) is satisfied. Condition (13) is just condition (3) expressed in matrix form when the assumptions of Theorem 1 are satisfied. We first verify that (3) implies that $Y(n)$ is Markovian. (To facilitate printing we sometimes write $\alpha(i)$ in place of α_i.) Clearly

$$\begin{aligned} P[Y(0) \varepsilon S_{\alpha(0)}, \cdots, Y(n) \varepsilon S_{\alpha(n)}] &= \sum_{j=0}^{n} \sum_{i_j \varepsilon S_{\alpha(j)}} p_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n} \\ &= \Big(\sum_{i \varepsilon S_{\alpha(0)}} p_i\Big) C_{S_{\alpha(0)}, S_{\alpha(1)}} \cdots C_{S_{\alpha(n-1)}, S_{\alpha(n)}} \end{aligned}$$

and it is easily seen that

$$C_{S_\alpha, S_\beta} = P[Y(n+1) \varepsilon S_\beta \mid Y(n) \varepsilon S_\alpha].$$

The sufficiency of condition (3) is thus verified. Note that the sufficiency argument given above holds for the case of any initial distribution w and without the condition of reversibility. We thus have Corollary 1.

Let us now consider the necessity of condition (3) when $X(n)$ is reversible. If $Y(n)$ is Markovian the Chapman-Kolmogorov equations are satisfied by the $Q^{(n)}$ and we must have

$$Q^{(2)} = [Q^{(1)}]^2$$

or

$$AP(I - BA)PB = 0.$$

But this implies that

$$B'DP(I - BA)PB = 0.$$

Because of reversibility, this can be written as

$$B'P'D(I - BA)PB = 0.$$

Now $D(I - BA)$ is positive definite so that

$$D(I - BA) = R'R$$

for some $m \times m$ matrix R. Thus

$$(RPB)'(RPB) = 0$$

and

$$RPB = 0.$$

But then

$$R'RPB = D(I - BA)PB = 0$$

and hence

$$(I - BA)PB = 0.$$

It is worth while noting that the problems we consider are related to issues of aggregation and consolidation in multisector models of mathematical economics (see [5]). There one has a stochastic matrix P and an invariant vector

$$p, pP = p.$$

One asks for the types of aggregation under which the aggregated invariant vector is an invariant vector of the aggregated matrix. The aggregated matrix $Q = APB$ where B is defined as before and $A = (B'D_vB)^{-1}B'D_v$. Here D_v is the diagonal matrix with its ith diagonal element v_i. The aggregation is determined by the sets of states S_i and the vector $v = (v_i)$. The aggregated vector is pB. The question is then for what aggregation schemes the relation

$$pBQ = pB(B'D_vB)^{-1}B'D_vPB = pB$$

is valid. Conditions (3) and (6) turn out to be crucial in some of the results obtained in [5].

3. Any initial distribution. Let the assumptions of Theorem 2 be satisfied. We first show that (4) is sufficient. It is enough to show that

$$\begin{aligned} P[X(n) = i, X(n+1) \varepsilon S, \cdots, X(n+h) \varepsilon S, X(n+h+1) = j] \\ = P[X(n) = i]P[X(n+1) \varepsilon S \mid X(n) = i] \\ \cdots P[X(n+h) \varepsilon S \mid X(n+h-1) \varepsilon S] \\ P[X(n+h+1) = j \mid X(n+h) \varepsilon S] \end{aligned}$$

for any $j \not\varepsilon S$ and any i, since then $Y(n)$ is clearly Markovian. Note that (4) implies that

$$\sum_{l \varepsilon S} p_{kl}\, p_{l,S} = p_{k,S}\, C_S \tag{14}$$

for all k. By making use of (4) and (14) the following relation is obtained

$$\begin{aligned} P[X(n+h+1) = j, X(n+h) \varepsilon S, \cdots, X(n+1) \varepsilon S \mid X(n) = i] \\ = \sum_{k=1}^{h} \sum_{i_k \varepsilon S} p_{i,i_1}\, p_{i_1,i_2} \cdots p_{i_{h-1},i_h}\, p_{i_h,j} \\ = p_{i,S}(C_S)^{h-1} C_j\,. \end{aligned}$$

But

$$C_j = P[X(n+1) = j \mid X(n) \varepsilon S], \qquad j \not\varepsilon S,$$

and

$$C_S = P[X(n+1) \varepsilon S \mid X(n) \varepsilon S].$$

An Argument paralleling the one given above indicates that (4′) implies that $Y(n)$ is Markovian so that we have Corollary 2. $Y(n)$ is obviously Markovian if (5) is satisfied.

Now consider the necessity of (4). Since $Y(n)$ is Markovian whatever the initial distribution w of $X(n)$, the transition probabilities of $Y(n)$ satisfy the Chapman-Kolmogorov equation. It may be that $p_{iS} = 0$ for all i. Then (4) is obviously satisfied. Suppose now that there is an i such that $p_{iS} \neq 0$. The Chapman-Kolmogorov equation then tells us that

$$p_{i,S} \frac{\sum_{l \varepsilon S} \sum_{k} w_k\, p_{kl}\, p_{lu}}{\sum_{k} w_k\, p_{kS}} = \sum_{l \varepsilon S} p_{il}\, p_{lu}$$

for all i, $u \not\varepsilon S$. If k is such that $p_{k,S} \neq 0$ then

$$p_{i,S} \sum_{l \varepsilon S} p_{kl}\, p_{lu} = p_{k,S} \sum_{l \varepsilon S} p_{il}\, p_{lu} \tag{15}$$

as is seen by letting $w_k \to 1$ and $w_l \to 0$, $l \neq k$. And if $p_{k,S} = 0$ (15) is obviously satisfied. Thus (15) holds for all k and all $i \not\varepsilon S$. If there is an $i \not\varepsilon S$ such that $p_{iS} \neq 0$ (15) is satisfied for all k and i. But this implies relation (4). There is still the possibility that $p_{i,S} = 0$ for all $i \not\varepsilon S$, namely condition (5).

In the context of Theorem 2 condition (3) implies that condition (4) is satisfied. However, the converse is not true. Consider the transition probability matrix

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} \end{bmatrix}.$$

Collapse the states 1, 2, 3 into a set S and leave the states 4, 5 alone. Note that (3) is not satisfied. But (4) is satisfied since

$$\frac{\sum_{l \varepsilon S} p_{kl}\, p_{lu}}{p_{k,S}} = \tfrac{1}{6}$$

for all $u \not\varepsilon S$ and all k.

4. Any function f. The answer obtained to the question posed in Theorem 3 is the same as the answer obtained in a similar problem posed by Bush, Mosteller and others [1]. The structure of interest in Bush and Mosteller's problem is not Markovian. Note that in our case we ask that $f(X(n))$ have the same structure (a Markovian structure) as $X(n)$ for any f and a specific initial probability vector, a left invariant vector p of P. Bush and Mosteller ask that $f(X(n))$ have the same structure as $X(n)$ for any f and any initial probability vector w.

Let us now prove Theorem 3. The condition imposed on the process will not be used in full strength. Just consider a consolidation in which two states j, k are consolidated into a set S and all other states are left the same. Let i, l be any indices distinct from j, k. Since the consolidated process is Markovian, its transition probabilities satisfy the Chapman-Kolmogorov equation and hence

$$p_{il}^{(2)} = \sum_{u=1}^{m} p_{iu}\, p_{ul} = \sum_{u \not\varepsilon S} p_{iu}\, p_{ul} + (p_{ij} + p_{ik}) \frac{p_j\, p_{jl} + p_k\, p_{kl}}{p_j + p_k}. \tag{16}$$

Equation (16) can be reduced to the following convenient form

$$(p_{ij}p_k - p_{ik}p_j)(p_{jl} - p_{kl}) = 0. \tag{17}$$

Further, (17) implies that

$$[(p_jp_{jj} + p_kp_{kj})p_k - (p_jp_{jk} + p_kp_{kk})p_j](p_{jl} - p_{kl}) = 0. \tag{18}$$

First consider the case in which for all i $p_{ij}p_k = p_{ik}p_j$ for all j, $k \neq i$. But then

$$p_{ij} = (1 - \lambda_i)p_j, \qquad i \neq j,$$

$$\lambda_i = \frac{p_{ii} - p_i}{1 - p_i},$$

so that P is of the form

$$P = \Lambda + (I - \Lambda)U,$$

where Λ is a diagonal matrix with diagonal elements λ_i and U is a matrix with identical rows $(p_1, \cdots, p_n)$. If

$$(p_jp_{jj} + p_kp_{kj})p_k = (p_jp_{jk} + p_kp_{kk})p_j \tag{19}$$

for some pair of indices j, k it follows that $\lambda_j = \lambda_k$. If (19) does not hold for the pair j, k, (18) implies that $p_{jl} = p_{kl}$ for all $l \neq j, k$. But then $\lambda_j = \lambda_k$. Thus it follows that in this case $\lambda_1 = \lambda_2 = \cdots = \lambda_n$.

Now on the contrary assume there is a row i for which $p_{ij}p_k = p_{ik}p_j$ does not hold for all $j, k \neq i$. Given any $j \neq i$ consider all k for which we can find a sequence $j_1, \cdots, j_\alpha$ such that

$$p_{ij}p_{j_1} = p_{ij_1}p_j, \qquad p_{ij_1}p_{j_2} = p_{ij_2}p_{j_1}, \cdots, p_{ij_\alpha}p_k = p_{ik}p_{j_\alpha}.$$

There is a maximal set of such indices k (including j of course). There are at least two such sets. The collection of all such maximal sets are disjoint. Given any j in one such maximal set and any j' in another we must have

$$p_{jl} = p_{j'l} \tag{20}$$

for all $l \neq j, j'$ and

$$p_{jj'} + p_{jj} - p_{j'j} - p_{j'j'} = 0. \tag{21}$$

For convenience let us assume $i = 1$. Keeping (20) and (21) in mind, it is clear that for any fixed $j \neq 1$ the p_{kj}'s must be equal for all $k \neq 1, j$. Call this common value u_j. Thus all rows except possibly for the first must be of the form

$$p_{kj} = \lambda\delta_{kj} + u_j.$$

There are now two possibilities. Either $p_{ij}p_k = p_{ik}p_j$ for all $i \neq 1$ and all

$$j, k \neq i$$

or this is not the case. If not we must have $p_{ij} = \lambda\delta_{ij} + u_j$ for all i. Since p is an invariant vector $u_j = (1 - \lambda)p_j$. On the other hand if $p_{ij}p_k - p_{ik}p_j = 0$ for all $i \neq 1$ and $j, k \neq i$ then $u_j = (1 - \lambda)p_j$. The elements of the first row are as yet unknown. But again making use of the fact that p is a stationary distribution we see that $p_{ij} = \lambda\delta_{1j} + (1 - \lambda)p_j$.

5. Finite state space and continuous time. The proof of the sufficiency of condition (7) in the case of Theorem 4 parallels the proof of Corollary 1.

We now show that (7) is necessary. A transition probability matrix-valued function $P(t)$ satisfying the regularity conditions posed in the assumptions in Theorem 4 is of the form (see [2])

$$P(t) = \exp(Gt),$$

where $G = (g_{ij})$ is such that

$$g_{ij} \geqq 0, \qquad i \neq j,$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{m} g_{ij} = -g_{ii}.$$

Let $w = (w_i)$, $w_i > 0$ be the initial distribution of $X(t)$. A necessary condition that the collapsed process be Markovian for an initial vector can be written down conveniently in matrix notation. As before, let

$$Q_w^{(t)} = (B'D_wB)^{-1}B'D_wP(t)B$$

denote the t-step transition probability matrix (from time zero to time t) for the collapsed process $Y(t)$ when the initial probability distribution vector of the original process $X(t)$ is w. If the collapsed process $Y(t)$ is Markovian $Q_w^{(t)}$ must satisfy the Chapman-Kolmogorov equation and thus

$$Q_w^{(t)} Q_{wP(t)}^{(\tau)} = Q_w^{(t+\tau)}, \qquad t, \tau > 0, \tag{22}$$

for all w, $w_i > 0$. It is clear that the w_i's only have to satisfy $w_i > 0$ and that the condition $\sum w_i = 1$ needn't be imposed. On differentiating relationship (22) with respect to τ at $\tau = 0$ we obtain

$$Q_w^{(t)}(B'D_{wP(t)}B)^{-1}B'D_{wP(t)}GB = (B'D_wB)^{-1}B'D_wP(t)GB. \tag{23}$$

Let us now differentiate (23) with respect to t at $t = 0$. We then have

$$B'D_wGB(B'D_wB)^{-1}B'D_wGB - (B'D_wB)^{-1}B'D_{wG}BB'D_wGB + B'D_{wG}GB = B'D_wG^2B.$$

This can be written more conveniently as

$$B'[D_wG - G_{wG}][B(B'D_wB)^{-1}(B'D_w) - I]GB = 0. \tag{24}$$

Let

$$w_{S_\alpha} = \sum_{i \varepsilon S_\alpha} w_i,$$

$$g_{i,S_\alpha} = \sum_{j \varepsilon S_\alpha} g_{ij}.$$

Condition (24) can be written down elementwise as

$$\begin{aligned}\sum_{i \varepsilon S_\alpha} \sum_\gamma w_i\, g_{i,S_\alpha}\, w_{S_\gamma}^{-1} \sum_{i \varepsilon S_\gamma} w_i\, g_{i,S_\beta} - \sum_{i \varepsilon S_\alpha} \sum_k w_i\, g_{ik}\, g_{k,S_\beta} \\ - \sum_i w_i\, g_{i,S_\alpha}\, w_{S_\alpha}^{-1} \sum_{i \varepsilon S_\alpha} w_i\, g_{i,S_\beta} + \sum_i w_i \sum_{k \varepsilon S_\alpha} g_{ik}\, g_{k,S_\beta} = 0.\end{aligned} \tag{25}$$

If we set $w_i = u_i h$, $i \varepsilon S_\alpha$, in (25) and then let $h \downarrow 0$, the following relation is obtained since the first two terms drop out

$$- \sum_{i \not\varepsilon S_\alpha} w_i\, g_{i,S_\alpha}\, u_{S_\alpha}^{-1} \sum_{i \varepsilon S_\alpha} u_i\, g_{i,S_\beta} + \sum_{i \not\varepsilon S_\alpha} w_i \sum_{k \varepsilon S_\alpha} g_{i,k}\, g_{k,S_\beta} = 0.$$

But this is valid if and only if

$$g_{i,S_\alpha}\, u_{S_\alpha}^{-1} \sum_{i \varepsilon S_\alpha} u_i\, g_{i,S_\beta} = \sum_{k \varepsilon S_\alpha} g_{i,k}\, g_{k,S_\beta}$$

for all $i \not\varepsilon S_\alpha$. Further, since this holds for all u_i,

$$g_{i,S_\alpha}\, g_{j,S_\beta} = \sum_{k \varepsilon S_\alpha} g_{i,k}\, g_{k,S_\beta} \tag{26}$$

for all $i \not\varepsilon S_\alpha$ and all $j \varepsilon S_\alpha$. There are only two alternatives that arise. If

$$g_{i,S_\alpha} = 0$$

for all $i \not\varepsilon S_\alpha$ relationship (26) is obviously satisfied (we then say that S_α satisfies (i)). Otherwise $g_{i,s_\alpha} \neq 0$ for some $i \not\varepsilon S_\alpha$ in which case g_{j,s_β} for each β is a constant for all $j \varepsilon S_\alpha$, that is,

$$g_{j,s_\beta} = K_{s_\alpha,s_\beta} \tag{27}$$

for all $j \varepsilon S_\alpha$, $\beta = 1, \cdots, r$ (we then say that S_α satisfies (ii)). The matrix G is said to satisfy (7) if for each α separately S_α satisfies either (i) or (ii). Note that if G satisfies (7) the nth power of G, $G^n = (g_{ij}^{(n)})$, satisfies (7) in a consistent manner, that is, S_α satisfies (i) for G^n if and only if S_α satisfies (i) for G. Since

$$P(t) = \exp(Gt) = \sum_{k=0}^{\infty} G^k t^k / k!$$

$P(t)$ satisfies (7). It should be noted that our proof has shown that the condition that the Chapman-Kolmogorov equation be satisfied by the collapsed process is enough to imply that the new process be Markovian. P. Levy [3] has shown that this is generally not the case.

6. Abstract state space. Consider a Markov process $X(t)$ with initial probability distribution

$$P[X(0) \varepsilon A] = P(A), \qquad A \varepsilon B(\Omega)$$

and transition probability function

$$P(t; x, A)$$

satisfying the assumptions of Corollary 3. Then $Y(t) = f(X(t))$ is a Markovian process with initial distribution

$$P[Y(0) \varepsilon A'] = P[X(0) \varepsilon f^{-1}(A')] = Q(A')$$

$A' \varepsilon f(B(\Omega))$, and transition probability function

$$\begin{aligned} Q(t; y, A') &= P[Y(t+\tau) \varepsilon A' \mid Y(\tau) = y] \\ &= P[X(t+\tau) \varepsilon f^{-1}(A') \mid X(\tau) \varepsilon f^{-1}(y)] \\ &= P(t; x, f^{-1}(A')), \; y \varepsilon \Omega', \; A' \varepsilon f(B(\Omega)), \end{aligned}$$

where x is such that $y = f(x)$. This follows immediately from condition (8).

It is interesting to note that one can generate new Markovian processes from old ones by setting up f so that it is consistent with the symmetries of the transition probability mechanism of the old process. Consider $X(t)$ Brownian motion on the line. Here the transition probability density is

$$P(t; x, y) = (2\pi t)^{-1/2} \exp\left(-\frac{1}{2t}(x-y)^2\right), \qquad t > 0.$$

If we set

$$f(x) = x - a[x/a], \qquad a > 0,$$

where $[x]$ is the greatest integer less than or equal to x, the new Markovian process $Y(t) = f(X(t))$ is Brownian motion on the circle. If

$$f(x) = z$$

on all points of the form $2ka \pm z$, $0 \leqq z < a$, $k = 0, \pm 1, \cdots$, $Y(t)$ is Brownian motion on a line segment of length a with reflecting barriers at the endpoints.

As a further example consider starting out with two-dimensional Brownian motion $(X_1(t), X_2(t))$, that is, the transition probability density is

$$p(t; (x_1, x_2), (y_1, y_2)) = (2\pi t)^{-1} \exp\left(-\frac{1}{2t}\left[\left(x_1 - y_1\right)^2 + \left(x_2 - y_2\right)^2\right]\right), \quad t > 0.$$

If

$$f(x_1, x_2) = (u_1, u_2)$$

for all points (x_1, x_2) of the form $(u_1 + ja, u_2 + ka)$ $0 \leqq u_1, u_2 < a$, $j, k = 0, \pm 1, \cdots$ $(Y_1(t), Y_2(t))$ is Brownian motion on a torus. If

$$f(x_1, x_2) = (u_1, u_2)$$

for all points of the form $(u_1 + ja, (2k + j)\, a \pm u_2)$ $0 \leqq u_1, u_2 < a$, $j, k = 0, \pm 1, \cdots$ $(Y_1(t), Y_2(t))$ is Brownian motion on a Moebius strip with reflecting barriers on the edges of the strip.

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