CONSENSUS OF SUBJECTIVE PROBABILITIES: THE PARI-MUTUEL METHOD

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A certain probability space is contemplated by a group of m individuals, each of whom endows it with his own subjective probability distribution. Suppose, now, that we wish to form a distribution which represents, in some sense, a consensus of those individual distributions. Various possibilities suggest themselves: the average, the convolution—but wait. There actually exists a popular institution which, theoretically at least, does perform just such an aggregation of personal probabilities. We refer to the pari-mutuel method of betting on horse races. In this system the final "track's odds" on a given horse are proportional to the amount bet on the horse. We shall here investigate the type of consensus given by this mechanism, which turns out to be quite different from any of the obvious aggregation schemes that might occur to one.

In formulating the pari-mutuel model we assume the m individuals involved are bettors, labeled $B_1, \cdots, B_m$, concerned with a race involving n horses, labeled $H_1, \cdots, H_n$. We assume further that each B_i, after careful study of the form sheets, the condition of the track, and other relevant material, has arrived at an estimate of the relative merits of each of the H_j's which he expresses in quantitative terms. Specifically, we are given an $m \times n$ *subjective probability matrix* $P = (p_{ij})$ where p_{ij} is the probability, in the opinion of B_i, that H_j will win the race.

Having determined his subjective probability distribution, B_i will now bet the amount b_i, a fixed positive number called B_i's *budget*, in a way which maximizes his *subjective expectation*. This means, of course, that B_i will not necessarily bet the whole amount b_i on that H_j for which p_{ij} is largest. In general, B_i will "bet the odds," that is, he will wait until the final track odds, or more conveniently, track probabilities, are announced. If these are $\pi_1, \cdots, \pi_n$, he will examine the ratios p_{ij}/π_j and distribute b_i among those H_j for which this ratio is a maximum. We shall refer to this course of action as B_i's *strategy*.

A technical difficulty is immediately apparent. We have already stated that the final track probabilities $\pi_1, \cdots, \pi_n$ are proportional to the amounts bet on $H_1, \cdots, H_n$, respectively (this is true whether or not the track retains a percentage). Thus, in practice, the π_j's are not known until each B_i has made his bet. On the other hand, B_i must know $\pi_1, \cdots, \pi_n$ before he can determine his bet. There is, therefore, a serious question as to whether there exist final track probabilities and individuals' bets compatible both with the bettors' strategies and the pari-mutuel principle. It is the purpose of this note to show that such probabilities and bets do exist and that the probabilities are in fact unique,

Received June 16, 1958.

thus giving a well-defined notion of consensus. Of course, the "influence" of B_i on the consensus will depend on his budget b_i, the case of equal influence being, by definition, that of equal budgets.

It will be convenient to choose the unit of money so that $\sum_{i=1}^{m} b_i = 1$. We shall also assume that each column of the matrix P contains at least one positive entry. If this were not so then, say, $p_{ij} = 0$ for all i and none of the B_i's would bet on H_j under any circumstances. We could then eliminate H_j from consideration entirely.

We shall now arithmetize the conditions which must be satisfied under the pari-mutuel system. Let β_{ij} be the amount which B_i bets on H_j. These must satisfy the *budget relation.*

$$\sum_{j=1}^{n} \beta_{ij} = b_i . \tag{1}$$

Next, the *pari-mutuel* condition requires that

$$\sum_{i=1}^{m} \beta_{ij} = \pi_j , \tag{2}$$

which is simply the statement that the final track probability π_j is proportional to the total amount bet on H_j. Equality holds here because of the normalization of the monetary unit. (We are using Greek letters to represent unknowns, Latin letters for the given constants of the problem.)

Finally, we must express the fact that each B_i is maximizing his expectation. The reader will easily verify that the condition is the following:

$$\text{if } \mu_i = \max_s \frac{p_{is}}{\pi_s} \text{ and } \beta_{ij} > 0, \text{ then } \mu_i = \frac{p_{ij}}{\pi_j}, \tag{3}$$

which states that B_i bets only on those H_j's for which his expectation is a maximum.

Nonnegative numbers π_j and β_{ij} which satisfy (1), (2) and (3) are called *equilibrium probabilities* and *bets*. Their existence can be proved by means of fixed-point theorems. We prefer, however, to prove existence in an elementary manner using a variational method which seems to be of interest in itself. We define a function ϕ and show that the variables which maximize it correspond to a solution of (1), (2) and (3).

The function ϕ has mn arguments ξ_{ij} and is defined by the rule:

$$\phi(\xi_{11}, \cdots, \xi_{mn}) = \sum_{i=1}^{m} b_i \log \sum_{j=1}^{n} p_{ij}\,\xi_{ij} ,$$

the variables ξ_{ij} being restricted to the domain D defined by:

$$\xi_{ij} \geqq 0, \qquad \text{for all } i, j, \tag{4}$$

$$\sum_{i=1}^{m} \xi_{ij} = 1, \qquad \text{for all } j. \tag{5}$$

We shall come back and discuss the meaning of the function ϕ after we have shown its relation to the pari-mutuel problem.

If we include minus infinity in the range of ϕ, then ϕ is continuous on the compact set D, hence attains a maximum at some point $(\bar{\xi}_{11}, \cdots, \bar{\xi}_{mn})$ of D. At this maximum the term $\sum_{j=1}^{n} p_{ij}\bar{\xi}_{ij}$ is positive for every i (otherwise ϕ would be minus infinity, which is clearly not its maximum value). The partial derivatives of ϕ at the maximum are given by

$$\frac{\partial \phi}{\partial \bar{\xi}_{ij}} = \frac{b_i\, p_{ij}}{\sum_s p_{is}\, \bar{\xi}_{is}}.$$

We now assert:

Existence Theorem. *A set of equilibrium probabilities π_j and bets β_{ij} are given by*

$$\pi_j = \max_i \frac{\partial \phi}{\partial \bar{\xi}_{ij}} = \max_i \frac{b_i\, p_{ij}}{\sum_s p_{is}\, \bar{\xi}_{is}} \tag{6}$$

$$\beta_{ij} \doteq \bar{\xi}_{ij}\, \pi_j . \tag{7}$$

Proof. We must show that the numbers π_j, β_{ij} satisfy (1), (2) and (3). The *pari-mutuel* condition (2) follows at once upon summing (7) on i and using condition (5) on the $\bar{\xi}_{ij}$.

The verification of (1) and (3) depends on the fact that

$$\text{if } \bar{\xi}_{ij} > 0, \text{ then } \pi_j = \frac{\partial \phi}{\partial \bar{\xi}_{ij}}. \tag{8}$$

To see this, suppose (8) is false and for some i, j we have $\bar{\xi}_{ij} > 0$ and $\pi_j > \partial\phi/\partial\bar{\xi}_{ij}$. By definition of π_j we have $\pi_j = \partial\phi/\partial\bar{\xi}_{kj} > \partial\phi/\partial\bar{\xi}_{ij}$ for some index k. Which means that by slightly decreasing $\bar{\xi}_{ij}$ and increasing $\bar{\xi}_{kj}$ by the same amount (which would not violate (4) or (5)), we could increase the value of ϕ, which is impossible since we are already at a maximum. Thus (8) is established.

We next verify condition (1). From (7) and (8) we have

$$\beta_{ij} = \bar{\xi}_{ij}\, \pi_j = \bar{\xi}_{ij} \frac{b_i\, p_{ij}}{\sum_s p_{is}\, \bar{\xi}_{is}}.$$

Summing the above on j,

$$\sum_j \beta_{ij} = b_i \frac{\sum_j p_{ij}\, \bar{\xi}_{ij}}{\sum_s p_{is}\, \bar{\xi}_{is}} = b_i .$$

Finally, we must prove (3). Since we assumed that none of the columns of the matrix P is identically zero, we know that each π_j is positive. Thus from (6) we have

$$\frac{p_{ij}}{\pi_j} \leqq \frac{\sum_s p_{is}\, \bar{\xi}_{is}}{b_i} \tag{9}$$

and μ_i, as defined in (3), is $(1/b_i) \sum_s p_{is}\bar{\xi}_{is}$. We see, then, from (7) and (8) that if β_{ij} is positive, $\bar{\xi}_{ij}$ is positive and hence $\mu_i = p_{ij}/\pi_j$, thus showing that (3) holds. (The fact that the π_j's sum to 1 is, of course, a consequence of (1) and (2) and the normalization of the b_i's.) Q.E.D.

The function ϕ can be interpreted as follows. From (7) we have $\sum_j p_{ij}\bar{\xi}_{ij} = \sum_j \beta_{ij}(p_{ij}/\pi_j)$, which is exactly the subjective expectation of B_i when he bets β_{ij} on H_j with track probabilities $\pi_1, \cdots, \pi_n$. Thus, at equilibrium the bettors, as a group, maximize a weighted sum of logarithms of subjective expectations, the weights being the bettors' budgets. As noted previously, equilibrium probabilities turn out to be unique, although equilibrium bets need not be unique. Furthermore, not every collection of β_{ij}'s, obtained by having each B_i act according to his strategy at equilibrium probabilities, need be equilibrium bets.

As a final result we show

UNIQUENESS THEOREM. *Equilibrium probabilities are unique.*

PROOF. Let $\pi_1, \cdots, \pi_n$ and $\bar{\pi}_1, \cdots, \bar{\pi}_n$ be equilibrium probabilities, let β_{ij}, $\bar{\beta}_{ij}$ be corresponding bets, and μ_i, $\bar{\mu}_i$ as defined in (3). Then for all i, j, k we have:

$$\beta_{ij}\mu_i\pi_j = \beta_{ij}p_{ij} \leqq \beta_{ij}\bar{\mu}_i\bar{\pi}_j$$

$$\bar{\beta}_{ik}\bar{\mu}_i\bar{\pi}_k = \bar{\beta}_{ik}p_{ik} \leqq \bar{\beta}_{ik}\mu_i\pi_k$$

whence, since μ_i, $\bar{\mu}_i$, π_j, π_k are positive, $\beta_{ij}\bar{\beta}_{ik}(\bar{\pi}_k/\pi_k) \leqq \beta_{ij}\bar{\beta}_{ik}\bar{\pi}_j/\pi_j$. Summing on j, k we get: $b_i \sum_k \bar{\beta}_{ik}(\bar{\pi}_k/\pi_k) \leqq b_i \sum_j \beta_{ij}\bar{\pi}_j/\pi_j$; dividing by b_i and summing on i: $\sum_k (\bar{\pi}_k\bar{\pi}_k/\pi_k) \leqq \sum_j \bar{\pi}_j = 1$.

Let $x_k = \bar{\pi}_k/\sqrt{\pi_k}$, $y_k = \sqrt{\pi_k}$. From the Cauchy-Schwartz inequality we have:

$$\left(\sum x_k y_k\right)^2 = \left(\sum \bar{\pi}_k\right)^2 = 1 \leqq \left(\sum x_k^2\right)\left(\sum y_k^2\right) = \left(\sum \frac{\bar{\pi}_k\bar{\pi}_k}{\pi_k}\right)\left(\sum \pi_k\right) \leqq 1.$$

Thus the vectors $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n)$ are dependent and $\bar{\pi}_k = \mu\pi_k$. But $\sum \bar{\pi}_k = \sum \pi_k = 1$, hence $\mu = 1$ and $\pi_k = \bar{\pi}_k$, proving uniqueness.

The referee has suggested the following instructive example which indicates the somewhat "pathological" nature of the pari-mutuel consensus. In the case of two bettors with equal budgets if the first bettor's subjective probability distribution on two horses is $(\frac{1}{2}, \frac{1}{2})$, then the equilibrium probabilities will be $(\frac{1}{2}, \frac{1}{2})$ regardless of the subjective probabilities of the second bettor, as the reader will easily verify.