BAYES ACCEPTANCE SAMPLING PROCEDURES FOR LARGE LOTS[1]

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1. Introduction and statement of the main results. A lot consisting of N items may be characterized by N non-negative random variables X_i, $i = 1, 2, \cdots, N$, where the value of X_i indicates the quality of the ith item. In a typical case X_i might take on the values zero and one according to whether the ith item is non-defective or defective. Alternatively, X_i might be defined to be the number of defects in the ith item so that the possible values of X_i would be $0, 1, 2, \cdots$. In still another formulation X_i might be a continuous random variable related to the deviation from standard of some characteristic of the item. We shall assume that the random variables X_i, $i = 1, 2, \cdots, N$, are independent and identically distributed with common distribution function $F(x \mid \lambda)$ depending on a single parameter λ.

The fixed size sampling inspection scheme to be considered consists of the random selection of n items from the lot and the observation of the values of the corresponding X_i's. Thus, the sample may be described by the random variables $X_1, X_2, \cdots, X_n$. The two possible actions to be taken on the basis of the sample are acceptance or rejection of the uninspected remainder of the lot. The consequences of these alternative actions are appraised by the following cost model where we let $S_k = \sum_{i=1}^{k} X_i$ for any $k = 1, 2, \cdots, N$:

(1.1)

Action	Cost
Acceptance	$a_1(S_N - S_n) + a_2(N - n) + s_1S_n + s_2n$
Rejection	$r_1(S_N - S_n) + r_2(N - n) + s_1S_n + s_2n.$

Thus, for $i = n + 1, n + 2, \cdots, N$, the contributions to the total cost due to the acceptance or rejection of the ith item without inspection are given by $a_1X_i + a_2$ and $r_1X_i + r_2$ respectively. For $i = 1, 2, \cdots, n$ the cost of inspection (and possibly replacement) of the ith item is given by $s_1X_i + s_2$. If, for example, an item is classified as defective or non-defective by X_i, then S_N and S_n are the number of defective items in the lot and in the sample respectively. Suppose that the cost of accepting an item is a_1 if the item is defective and zero if the item is non-defective, and that the cost of rejecting the uninspected remainder of the lot is proportional to the number of items remaining in the lot. Then $a_2 = r_1 = 0$. If, in addition, all items found to be defective in the sample are replaced with good items, each at a cost of s_1 units, and s_2 represents the cost of the time and labor required to inspect each item in the sample, then (1.1)

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becomes

(1.1a)

Action	Cost
Acceptance	$a_1(S_N - S_n) + s_1S_n + s_2n$
Rejection	$r_2(N - n) + s_1S_n + s_2n.$

The cost model (1.1) includes a wide variety of sampling inspection and acceptance sampling problems corresponding to various choices of the cost parameters and the family of distribution functions $F(x \mid \lambda)$. Similar formulations of this problem have been given by several authors, notably [1], [2], and [3].

The authors of [1] and [2] have attempted to characterize optimal sample sizes in terms of a minimax criterion. When the lot size is large this approach seems to lead to sample sizes which are appropriate when the true state of nature (value of λ) has a high *a priori* probability of being very close to the "indifference state" where either acceptance or rejection leads to the same expected cost. Such an *a priori* assumption about the true state of nature will not generally be reasonable, which suggests that the minimax criterion is not suitable for this problem.

The purpose of this paper is to find explicit asymptotic characterizations for large N of the decision procedures and sample sizes which are optimal in the Bayes sense for various classes of *a priori* probability distributions defined over the values of the parameter λ. This problem is considered for certain families of distribution functions $F(x \mid \lambda)$ of the exponential type having the property that $E(X \mid \lambda) = \lambda$. This parametrization of distribution functions of the exponential type is convenient because (1) for this case the range of possible values of X coincides with the range of λ, and (2) any available *a priori* information will usually be most easily expressed in terms of the expected quality of an item, i.e., the value of the parameter λ.

The two principal reasons for investigating the Bayes solutions to this problem are as follows: (1) In most practical situations the statistician will possess some subjective *a priori* information concerning the probable values of the parameter λ and such information may often be reasonably summarized and made objective by the choice of a suitable *a priori* distribution; (2) for statistical decision problems of the type under consideration, the class of Bayes decision procedures coincides with the admissible class so that all procedures discussed will have the optimal property of admissibility (see, e.g., [4]).

Each family of distribution functions $F(x \mid \lambda)$ to be considered will be defined in terms of a given measure μ on the Borel sets of the positive real half-line as follows: Let

$$(1.2)\qquad \begin{aligned} a &= \inf\left\{x: \int_x^{x+\epsilon} d\mu > 0, \quad \text{all} \quad \epsilon > 0\right\}, \\ b &= \sup\left\{x: \int_{x-\epsilon}^{x} d\mu > 0, \quad \text{all} \quad \epsilon > 0\right\}. \end{aligned}$$

Let I_μ be the interval $[a, b]$ if $b < \infty$ and $[a, \infty)$ if $b = \infty$. We assume that μ satisfies the conditions

(1.3) A) I_μ is non-empty,

B) There exists a function $\omega(\lambda)$ such that for each $\lambda \varepsilon I_\mu$

$$\frac{\int_{[a,\infty)} xe^{\omega(\lambda)x}\, d\mu(x)}{\int_{[a,\infty)} e^{\omega(\lambda)x}\, d\mu(x)} = \lambda.$$

We now define $F(x \mid \lambda)$ by

$$F(x \mid \lambda) = \begin{cases} 0, & x \leqq a, \\ \dfrac{\int_{[a,x)} e^{\omega(\lambda)t}\, d\mu(t)}{\int_{[a,\infty)} e^{\omega(\lambda)t}\, d\mu(t)}, & a < x \leqq b, \\ 1, & x > b. \end{cases} \tag{1.4}$$

The following theorems concerning such families are proved in Section 2.

THEOREM 2.1: *The function* $\omega(\lambda)$ *given by* (1.3B) *is unique and* $d\omega(\lambda)/d\lambda$ *exists and is positive for* $\lambda \varepsilon I_\mu$.

THEOREM 2.2: *If* $F(x \mid \lambda)$ *is defined by* (1.4), *then all moments of* $F(x \mid \lambda)$ *exist and all derivatives of* $\omega(\lambda)$ *exist and are finite for* $\lambda \varepsilon I_\mu$.

THEOREM 2.3: *The distribution function* $F(x \mid \lambda)$ *defined by* (1.4) *may be represented for* $a < x \leqq b$ *and for* $\lambda \varepsilon I_\mu$ *by*

$$F(x \mid \lambda) = K(\gamma) \int_{[a,x)} \exp\left\{\omega(\lambda)t - \int_\gamma^\lambda u\omega'(u)\, du\right\} d\mu(t), \tag{1.5}$$

if and only if assumption (1.3B) *is satisfied, where* $\gamma \varepsilon I_\mu$ *and* $K(\gamma)$ *is a normalizing factor depending on the choice of* γ *and determined so that* $F(b+ \mid \lambda) = 1$.

We may unambiguously define $F(x \mid a)$ and, if b is finite, $F(x \mid b+)$ by

$$F(x \mid a) = \lim_{\lambda \searrow a} F(x \mid \lambda), \tag{1.6}$$

and

$$F(x \mid b+) = \lim_{\lambda \nearrow b} F(x \mid \lambda). \tag{1.7}$$

It is easily verified that the n-fold convolution of $F(x \mid \lambda)$ may be written

$$F^{(n)}(x \mid \lambda) = (K(\gamma))^n \int_{[na,x)} \exp\left\{\omega(\lambda)t - n\int_\gamma^\lambda u\omega'(u)\, du\right\} d\mu^{(n)}(t), \tag{1.8}$$

where $u^{(n)}$ is the n-fold convolution of μ. We define the interval $I_\mu^{(n)} = [na, nb]$ if $b < \infty$ and $I_\mu^{(n)} = [na, \infty)$ if $b = \infty$. Now since $X_1, X_2, \cdots, X_n$ are assumed to be independent with common distribution function $F(x \mid \lambda)$, we see that the sum S_n is distributed according to (1.8). Furthermore, by applying the factorization criterion for sufficiency to the joint distribution of $X_1, X_2, \cdots, X_n$ (see, e.g., [5]) it is easily seen that S_n is a sufficient statistic for the problem under consideration so that we may confine our attention to decision procedures depending only on the value of S_n.

Some particular examples of families of distribution functions $F(x \mid \lambda)$ which are of practical interest are as follows:

Example 1: If μ is the counting measure on the integers zero and one, and $\omega(\lambda) = \ln(\lambda/1 - \lambda)$ for $0 < \lambda < 1$, then $F(x \mid \lambda)$ is the distribution function of a Bernoulli random variable taking on the value one with probability λ and zero with probability $1 - \lambda$.

Example 2: If ν is the counting measure on the non-negative integers, $d\mu(x)/d\nu(x) = 1/x!$, and $\omega(\lambda) = \ln \lambda$ for $0 < \lambda < \infty$, then $F(x \mid \lambda)$ is the distribution of a Poisson random variable with expected value λ.

Example 3: If ν is Lebesgue measure on the positive half-line, $d\mu(x)/d\nu(x) = \eta^\eta x^{\eta-1}/\Gamma(\eta)$ for known $\eta > 0$, and $\omega(\lambda) = -\eta/\lambda$, then for each η, $F(x \mid \lambda)$ is a gamma distribution with $E(X \mid \lambda) = \lambda$ and $\operatorname{Var}(X \mid \lambda) = \lambda^2/\eta$.

In order to discuss the properties of the Bayes sample size it will be necessary to consider a further specialization of the class of distribution functions $F(x \mid \lambda)$. To this end we let

$$\omega(\lambda) = \frac{1}{k} \ln \frac{\lambda}{k\alpha + \beta\lambda}, \tag{1.9}$$

where k is a positive number and α and β are numbers such that either (i) $\alpha > 0$ and $\beta \geqq 0$, or (ii) $\alpha > 0$ and $\beta = -\alpha/b^*$ where b^* is a positive integer. Let $\mu(x)$ be a measure such that

$$\frac{d\mu(x)}{d\nu(x)} = \begin{cases} 1, & x = 0, \\ \dfrac{\alpha(\alpha+\beta)\cdots\left(\alpha + \left(\frac{x}{k} - 1\right)\beta\right)}{\left(\frac{x}{k}\right)!}, & x = k, 2k, \cdots, \end{cases} \tag{1.10}$$

where ν is counting measure on $0, k, 2k, \cdots$. For case (i) $I_\mu = [0, \infty)$ and for case (ii) $I_\mu = [0, b]$, where $b = kb^*$. These definitions permit us to define the class $\mathfrak{F}_1$ of distribution functions $F(x \mid \lambda)$ as follows:

(1.11) $\mathfrak{F}_1$ = The class distribution functions $F(x \mid \lambda)$ of the form (1.4) for which the corresponding $\omega(\lambda)$ and $\mu(x)$ are determined by (1.9) and (1.10) respectively.

The class of distribution functions $\mathfrak{F}_1$ clearly contains the Bernoulli ($\alpha = k = 1$, $\beta = -1$) and Poisson ($\alpha = k = 1, \beta = 0$) examples discussed earlier That the distribution functions in the class $\mathfrak{F}_1$ are well defined follows from

THEOREM 2.4: *If the function* $\omega(\lambda)$ *and the measure* $\mu(x)$ *are defined by* (1.9) *and* (1.10), *then condition* (1.3B) *is satisfied.*

The form of the n-fold convolution $F^{(n)}(x \mid \lambda)$ of a distribution function $F(x \mid \lambda)$ in the class $\mathfrak{F}_1$ is needed in the derivation of an asymptotic expansion for the Bayes risk. The following theorem gives a formula for $F^{(n)}(x \mid \lambda)$.

THEOREM 2.5: *If* $F(x \mid \lambda) \in \mathfrak{F}_1$, *then for all integer values of* m, $F^{(n)}(x \mid \lambda)$ *is given by*

$$(1.12) \quad F^{(n)}(km \mid \lambda) = \begin{cases} 0, & m \leqq 0, \\ 1 - mr^{(n)}(km) \int_0^\lambda \frac{t^{m-1}}{(k\alpha + \beta t)^m} \cdot \exp\left\{-n \int_0^t \frac{\alpha\, du}{(k\alpha + \beta u)}\right\} dt, & m = 1, 2, \cdots, \end{cases}$$

where

$$(1.13) \quad r^{(n)}(x) = \begin{cases} 1, & x = 0, \\ \dfrac{n\alpha(n\alpha + \beta) \cdots \left(n\alpha + \left(\frac{x}{k} - 1\right)\beta\right)}{\left(\frac{x}{k}\right)!}, & x = k, 2k, \cdots. \end{cases}$$

In addition to the class $\mathfrak{F}_1$ of discrete distributions we will consider the class of continuous gamma type families defined by (1.4) with

$$(1.14) \quad \omega(\lambda) = -\eta/\lambda,$$

and

$$\frac{du(x)}{d\nu(x)} = \begin{cases} 0, & x < 0, \\ \dfrac{\eta^\eta x^{\eta-1}}{\Gamma(\eta)}, & x \geqq 0, \end{cases}$$

where $\nu(x)$ is Lebesgue measure on the positive half-line. The class $\mathfrak{F}_2$ is defined by

(1.16) $\quad \mathfrak{F}_2 =$ the family of distribution functions $F(x \mid \lambda)$ of the form (1.4) with $\omega(\lambda)$ and $\mu(x)$ given by (1.14) and (1.15).

This definition leads to

THEOREM 2.6: *For any distribution function* $F(x \mid \lambda) \in \mathfrak{F}_2$, *(i) condition* (1.3B) *is satisfied, and (ii)*

$$(1.17) \quad F^{(n)}(x \mid \lambda) = \frac{(\eta x)^{n\eta}}{\Gamma(n\eta)} \int_\lambda^\infty u^{-n\eta-1} \exp\left\{-\frac{x\eta}{u}\right\} du.$$

Returning now to the underlying decision problem, for any fixed n let $\delta(s_n)$ be a decision rule which is to be interpreted as the probability of acceptance of the uninspected remainder of the lot when s_n is the observed value of the sufficient statistic S_n. Regarding λ as a random variable Λ by virtue of the assumed existence of an *a priori* probability distribution we observe that, given the value λ of Λ, $S_N - S_n$ and S_n are conditionally independently distributed according to $F^{(N-n)}(x \mid \lambda)$ and $F^{(n)}(x \mid \lambda)$ respectively so that

$$\begin{aligned} E\{S_N - S_n \mid S_n\} &= E\{E\{S_N - S_n \mid \Lambda, S_n\} S_n\} \\ &= E\{(N-n)\Lambda \mid S_n\} \\ &= (N-n)E\{\Lambda \mid S_n\}. \end{aligned} \tag{1.18}$$

Hence, referring to (1.1) the risk incurred by using the rule δ may be written as

$$\begin{aligned} R(\delta, n, N) &= E\{\delta(S_n)[a_1(S_N - S_n) + a_2(N-n)]\} \\ &\quad + E\{[1 - \delta(S_n)][r_1(S_N - S_n) + r_2(N-n)]\} \\ &\quad + E\{s_1 S_n + s_2 n\} \\ &= E\{\delta(S_n)[a_1 - r_1)(S_N - S_n) + (a_2 - r_2)(N-n)]\} \\ &\quad + (s_1 n + r_1(N-n))E(\Lambda) + s_2 n + r_2(N-n) \\ &= (N-n)E\{\delta(S_n)[(a_1 - r_1)E(\Lambda \mid S_n) + a_2 - r_2\} \\ &\quad + [s_1 n + r_1(N-n)]E(\Lambda) + s_2 n + r_2(N-n). \end{aligned} \tag{1.19}$$

From this it is clear that the essentially unique Bayes decision rule is given by

$$\delta^*(s_n) = \begin{cases} 1, & E\{\Lambda \mid S_n = s_n\} \leqq c, \\ 0, & \text{otherwise}, \end{cases} \tag{1.20}$$

where $c = r_2 - a_2/a_1 - r_1$, provided $a < c < b$. To avoid trivial cases where acceptance or rejection is determined without sampling we shall assume that $a < c < b$, which implies either (1) $r_1 < a_1$ and $r_2 > a_2$, or (2) $r_1 > a_1$ and $r_2 < a_2$. Referring to (1.1) we see that for any given cost situation where (1) holds we may find a corresponding second situation where (2) holds which becomes identical with the first when the two actions are interchanged. Hence in the sequel we shall assume without loss of generality that (2) holds.

Unfortunately, for many *a priori* distributions and many families $F(x \mid \lambda)$ of interest the quantity $E\{\Lambda \mid S_n = s_n\}$ cannot be expressed explicitly. The following results which are proved in Section 3 give more explicit characterizations of δ^* for the case where n is large. These results are also needed for the determination of the Bayes sample size.

Let $G(\lambda)$ be the *a priori* distribution function of the parameter λ, i.e., $G(\lambda) = P\{\Lambda < \lambda\}$. We assume that $G(\lambda)$ assigns probability one to the closed interval $[a, b]$. We assume further that $E(\Lambda)$ is finite and that $G(\lambda)$ does not assign probability one to any single point. Define the function $\varphi_n(t)$ for $t \,\varepsilon\, I_\mu^{(n)}$ by

$$\varphi_n(t) = \frac{\int_0^\infty \lambda \exp\left\{t\omega(\lambda) - n\int_\gamma^\lambda u\omega'(u)\,du\right\} dG(\lambda)}{\int_0^\infty \exp\left\{t\omega(\lambda) - n\int_\gamma^\lambda u\omega'(u)\,du\right\} dG(\lambda)}. \tag{1.21}$$

It is easily verified that $\varphi_n(t)$ coincides with $E\{\Lambda \mid S_n = t\}$ for almost all values of t which are possible values of S_n. From this definition we obtain

THEOREM 3.1: *The function* $\varphi_n(t)$ *given by* (1.21) *is finite and strictly increasing for* $t \varepsilon I_\mu^{(n)}$.

We now observe that exactly one of the following must hold:

$$\text{(1.22)} \qquad \begin{aligned} &\text{i) } \varphi_n(nb) < c,\\ &\text{ii) } \varphi_n(na) > c,\\ &\text{iii) } \varphi_n(t_n) = c \text{ for a unique } t_n \,\varepsilon\, I_\mu^{(n)}. \end{aligned}$$

Hence the Bayes decision rule $\delta^*(S_n)$ given by (1.20) is equivalently expressed by

$$\text{(1.23)} \qquad \delta^*(S_n) = \begin{cases} 1, & \text{if } S_n \leqq t(n),\\ 0, & \text{otherwise,} \end{cases}$$

where

$$\text{(1.24)} \qquad t(n) = \begin{cases} na - 1, & \varphi_n(nb) < c,\\ nb, & \varphi_n(na) > c,\\ t_n, & \varphi_n(t_n) = c,\ t_n \,\varepsilon\, I_\mu^{(n)}. \end{cases}$$

An asymptotic characterization of the function $t(n)$ for *a priori* distributions $G(\lambda)$ placing positive weight on both sides of c is given by

THEOREM 3.2. *If* $\lambda_0 = \sup\{\lambda : \lambda \leqq c; G(\lambda+) - G(\lambda - \epsilon) > 0$ *all* $\epsilon > 0\}$ *and* $\lambda_1 = \inf\{\lambda : \lambda \geqq c; G(\lambda + \epsilon) - G(\lambda) > 0$, *all* $\epsilon > 0\}$, *then*

$$\text{(1.25)} \qquad \lambda_0 \leqq \liminf_{n\to\infty} \frac{t(n)}{n} \leqq \limsup_{n\to\infty} \frac{t(n)}{n} \leqq \lambda_1 .$$

Thus for the particular case where $G(\lambda)$ assigns positive weight to every interval about c we have $\lambda_0 = \lambda_1$ and

$$\text{(1.26)} \qquad t(n) = cn + o(n).$$

In order to obtain a more precise asymptotic characterization of $t(n)$ we define two classes of *a priori* distributions as follows:

(1.27) $\mathcal{G}_1 =$ the class of all $G(\lambda)$ which are twice continuously differentiable in some open interval about c with $G'(c) > 0$;

(1.28) $\mathcal{G}_2 =$ the class of all $G(\lambda)$ for which there exist numbers l_G and u_G, $l_G < c < u_G$, which are assigned positive weight by $G(\lambda)$ and are such that $G(u_G) - G(c+) = 0$ and $G(c) - G(l_G+) = 0$.

The class of *a priori* distributions assigning probability one to a finite set of points is of course a subset of $\mathcal{G}_2$. We now have

THEOREM 3.3: *If* $G(\lambda) \,\varepsilon\, \mathcal{G}_1$, *then*

$$\text{(1.29)} \qquad t(n) = cn + \frac{\omega''(c)}{(\omega'(c))^2} - \frac{G''(c)}{G'(c)\omega'(c)} + o(1).$$

THEOREM 3.4: *If* $G(\lambda) \varepsilon \mathcal{G}_2$, $\zeta_1 = G(l_G+) - G(l_G)$, *and* $\zeta_2 = G(u_G+) - G(u_G)$, *then*

$$t(n) = n \frac{\int_{l_G}^{u_G} u\omega'(u)\, du}{\omega(u_G) - \omega(l_G)} + \frac{\ln \frac{\zeta_1(c - l_G)}{\zeta_2(u_G - c)}}{\omega(u_G) - \omega(l_G)} + o(1). \tag{1.30}$$

Although the classes $\mathcal{G}_1$ and $\mathcal{G}_2$ are not exhaustive we have characterized $t(n)$ and hence $\delta^*(s_n)$ sufficiently for most practical purposes as long as n is reasonably large. We now turn our attention to the problem of determining the Bayes sample size $n^* = n^*(N)$ which minimizes the risk $R(\delta^*, n, N)$, and seek an asymptotic characterization of the Bayes sample size $n^*(N)$ for large N.

The parameter k appearing in (1.9) is essentially a scale parameter in the sense that if X is a random variable distributed according to (1.9) with $k = 1$ and if λ is replaced by λ^*/k then kX has a distribution of the same form as (1.9) with arbitrary k and with λ^* playing the role of λ. A similar remark applys to the parameter η appearing in (1.15). Hence the cases where k and η are arbitrary may be obtained from the cases where $k = 1$ and $\eta = 1$ by multiplying the appropriate cost coefficients by k or η and making suitable changes of variables in the *a priori* distribution functions. For the sake of simplicity the remaining results are stated for the cases $k = 1$ and $\eta = 1$ only.

The asymptotic behavior of $n^*(N)$ is characterized by the following theorems, which are proved in Section 4.

THEOREM 4.1: *If* $F(x \mid \lambda) \varepsilon \mathfrak{F}_1$ (*with* $k = 1$) and $G(\lambda) \varepsilon \mathcal{G}_1$, *then the Bayes risk for fixed n and N is given by*

$$\begin{aligned} R(\delta^*, n, N) &= n((s_1 - r_1)E(\Lambda) + (s_2 - r_2) + (r_1 - a_1) \int_0^c (\lambda - c)\, dG(\lambda)) \\ &\quad + N(r_1 E(\Lambda) + r_2 + (a_1 - r_1) \int_0^c (\lambda - c)\, dG(\lambda)) \\ &\quad + (N - n) \frac{(r_2 - a_2)(\alpha + \beta c)G'(c)}{2\alpha n} + (N - n) o\left(\frac{1}{n}\right) \end{aligned} \tag{1.31}$$

and the Bayes sample size is

$$n^*(N) = \begin{cases} N, & A_G \leqq 0, \\ N^{1/2} \left(\frac{(r_2 - a_2)(\alpha + \beta c)G'(c)}{2\alpha A_G} \right)^{1/2} + o(N^{1/2}), & A_G > 0, \end{cases} \tag{1.32}$$

where

$$A_G = (s_1 - r_1)E(\Lambda) + s_2 - r_2 + (r_1 - a_1) \int_0^c (\lambda - c)\, dG(\lambda). \tag{1.33}$$

THEOREM 4.2: *If* $F(x \mid \lambda) \varepsilon \mathfrak{F}_1$ (*with* $k = 1$), $G(\lambda) \varepsilon \mathcal{G}_2$ *and* A_G *is defined by* (1.33), *then the Bayes sample size is given by*

$$(1.34) \qquad n^*(N) = \begin{cases} N, & A_G \leqq 0, \\ K \ln N - \frac{K}{2} \ln \ln N + O(1), & A_G > 0. \end{cases}$$

(The definition of K is lengthy and is contained in the proof in Section 4.)

THEOREM 4.3: *If* $F(x \mid \lambda) \,\varepsilon\, \mathfrak{F}_2$ *(with* $\eta = 1$*),* $G(\lambda) \,\varepsilon\, \mathcal{G}_1$ *and* A_G *is defined by* (1.33), *then the Bayes sample size is given by*

$$(1.35) \qquad n^*(N) = \begin{cases} N, & A_G \leqq 0, \\ N^{1/2} \left(\frac{(r_2 - a_2) c G'(c)}{2A_G} \right)^{1/2} + o(N^{1/2}), & A_G > 0. \end{cases}$$

THEOREM 4.4: *If* $F(x \mid \lambda) \,\varepsilon\, \mathfrak{F}_2$ *(with* $\eta = 1$*),* $G(\lambda) \,\varepsilon\, \mathcal{G}_2$ *and* A_G *is defined by* (1.33), *then the Bayes sample size is given by*

$$(1.36) \qquad n^*(N) = \begin{cases} N, & A_G \leqq 0, \\ K \ln N - \frac{K}{2} \ln \ln N + O(1), & A_G > 0. \end{cases}$$

In all cases where $A_G \leqq 0$ the proper procedure is to screen the lot completely (i.e., take $n^*(N) = N$). Theorems 4.1 and 4.3 show that if an *a priori* probability density for the parameter λ exists in the vicinity of the critical point c and if this density is smooth and positive at c, then the optimal sample size when $A_G > 0$ is approximately proportional to the square root of the lot size N when N is large. For the cases covered by Theorems 4.2 and 4.4 where the *a priori* probability that λ lies within a certain neighborhood of c is zero, the optimal sample size when $A_G > 0$ is approximately proportional to the logarithm of N when N is large. It is clear from these results that the optimal rate of increase for the sample size depends critically on the fine structure of the *a priori* information about λ in the vicinity of c. This is especially remarkable in view of the fact that c is actually the "indifference" value of λ in the sense that if $\lambda = c$ then either acceptance or rejection of the lot leads to the same expected cost.

Referring to (1.3) and (1.32) of Theorem 4.1 we may write

$$(1.37) \qquad R(\delta^*, n, N) = A_G n + B_G N + C_G(N - n) \left(\frac{1}{n} + o\left(\frac{1}{n}\right) \right),$$

and if $A_G > 0$

$$(1.38) \qquad n^*(N) = \left(\frac{C_G}{A_G} \right)^{1/2} N^{1/2} + o(N^{1/2}),$$

where A_G is given by (1.33), and B_G and C_G are coefficients depending on the costs and the *a priori* distribution G. It is easily verified that the term $B_G N$ represents the "subminimal" risk which would result if the value λ of Λ were known exactly without sampling and the decision to accept or reject determined accordingly. From (1.37) and (1.38) we obtain

$$R(\delta^*, n^*(N), N) = B_G N + 2(A_G C_G)^{1/2} N^{1/2} + o(N^{1/2}), \tag{1.39}$$

which shows that the amount by which the Bayes risk exceeds the subminimal risk due to the uncertainty concerning the value of Λ is of smaller order in N than the subminimal risk itself. Expression (1.39) is still valid if the sample size is determined by taking only the first term in the asymptotic expansion for $n^*(N)$ so that not much is lost by making this approximation if N is large. Similar remarks may be made for the cases covered by Theorems 4.2, 4.3 and 4.4. For the cases covered by Theorems 4.2 and 4.4 the term added to the subminimal risk in the expressions for $R(\delta^*, n^*(N), N)$ is of order $\ln N$.

2. Theorems concerning the class of distribution functions $F(x \mid \lambda)$.

THEOREM 2.1: *The function $\omega(\lambda)$ given by (1.9) is unique and $d\omega(\lambda)/d\lambda$ exists and is positive for $\lambda \varepsilon I_\mu$.*

PROOF: By assumption (1.3) there exist numbers ω_0, ω_1 such that the ratio

$$\rho(\omega) = \frac{\int_{[a,\infty)} xe^{\omega x}\, d\mu(x)}{\int_{[a,\infty)} e^{\omega x}\, d\mu(x)} \tag{2.1}$$

is finite for $\omega_0 < \omega < \omega_1$. Furthermore, $\rho(\omega)$ is differentiable with respect to ω and

$$\rho'(\omega) = \int_{[a,\infty)} (x - \rho(\omega))^2 \frac{e^{\omega x}}{\int_{[a,\infty)} e^{\omega x}\, d\mu(x)}\, d\mu(x) > 0, \tag{2.2}$$

for $\omega_0 < \omega < \omega_1$. Now for $\lambda \varepsilon I_\mu$ we have, by (1.3B), $\rho(\omega(\lambda)) = \lambda$ which implies that $\omega(\lambda)$ is unique and that $d\omega(\lambda)/d\lambda$ exists and is given by

$$\frac{d\omega(\lambda)}{d\lambda} = \frac{1}{\rho'(\omega(\lambda))} > 0. \tag{2.3}$$

THEOREM 2.2: *If $F(x \mid \lambda)$ is defined by (1.4), then all moments of $F(x \mid \lambda)$ exist, and all derivatives of $\omega(\lambda)$ exist and are finite for $\lambda \varepsilon I_\mu$.*

PROOF: The function $\omega(\lambda)$ is continuous and strictly increasing for $\lambda \varepsilon I_\mu$ by Theorem 2.1. Therefore, the moment generating function given by

$$m_\lambda(t) = \frac{\int_{[a,\infty)} e^{(t+\omega(\lambda))x}\, d\mu(x)}{\int_{[a,\infty)} e^{\omega(\lambda)x}\, d\mu(x)} \tag{2.4}$$

exists for each $\lambda \varepsilon I_\mu$ for all t in some open interval about zero since both the numerator and the denominator of the ratio (1.3B) must be finite for any $\lambda \varepsilon I_\mu$. That is, for any fixed $\lambda \varepsilon I_\mu$, we may choose $t \neq 0$ small enough in magnitude so that there exists a $\lambda^* \varepsilon I_\mu$ for which $|t| + \omega(\lambda) < \omega(\lambda^*)$ so that the integral in the numerator must converge.

Repeated formal differentiations of $\rho(\omega)$ yield sums of ratios involving products of integrals of the form $\int_{[a,\infty)} x^k e^{\omega x} \, d\mu(x)$ in the numerators and powers of $\int_{[a,\infty)} e^{\omega x} \, d\mu(x)$ in the denominators. These integrals are finite and those in the denominators do not vanish for $\omega = \omega(\lambda)$, $\lambda \varepsilon I_\mu$, so all derivatives of $\rho(\omega)$ exist for such values of ω. As before, for $\lambda \varepsilon I_\mu$,

$$\frac{d\omega(\lambda)}{d\lambda} = \frac{1}{\rho'(\omega(\lambda))} \tag{2.5}$$

and repeated application of the rule for differentiation of implicit functions shows that $\omega(\lambda)$ possesses derivatives of all orders for $\lambda \varepsilon I_\mu$.

THEOREM 2.3: *The distribution function $F(x \mid \lambda)$ defined by* (1.4) *may be represented for $a < x \leqq b$ and for $\lambda \varepsilon I_\mu$* by

$$F(x \mid \lambda) = K(\gamma) \int_{[a,x)} \exp\left\{\omega(\lambda) - \int_\gamma^\lambda u\omega'(u) \, du\right\} d\mu(t), \tag{2.6}$$

if and only if assumption (1.3B) *is satisfied, where $\gamma \varepsilon I_\mu$ and $K(\gamma)$ is a normalizing factor depending on the choice of γ determined so that $F(b+ \mid \lambda) = 1$.*

PROOF: We observe that

$$\frac{d}{d\lambda} \int_{[a,\infty)} e^{\omega(\lambda)x} \, d\mu(x) = \omega'(\lambda) \int_{[a,\infty)} x e^{\omega(\lambda)x} \, d\mu(x) \tag{2.7}$$

so that dividing both sides by $\int_{[a,\infty)} e^{\omega(\lambda)x} \, d\mu(x)$ and referring to assumption (1.3B) we have for $\lambda \varepsilon I_\mu$,

$$\frac{d}{d\lambda} \ln \int_{[a,\infty)} e^{\omega(\lambda)x} \, d\mu(x) = \lambda\omega'(\lambda). \tag{2.8}$$

Hence

$$\begin{aligned} \int_{[a,\infty)} e^{\omega(\lambda)x} \, d\mu(x) &= \exp\left\{\int \lambda\omega'(\lambda) \, d\lambda + c\right\} \\ &= K(\gamma) \exp\left\{\int_\gamma^\lambda u\omega'(u) \, du\right\} \end{aligned} \tag{2.9}$$

for $\gamma \varepsilon I_\mu$. The fact that assumption (1.3B) is satisfied whenever (2.6) is valid follows immediately by differentiating the expression $F(b+ \mid \lambda) = 1$ with respect to λ.

We now verify that the classes $\mathfrak{F}_1$ and $\mathfrak{F}_2$ defined by (1.11) and (1.16) satisfy the above assumptions, and determine closed expressions for the n-fold convolution of distribution functions in these classes.

THEOREM 2.4: *If the function $\omega(\lambda)$ and the measure $\mu(x)$ are defined by* (1.9) *and* (1.10), *then condition* (1.3B) *is satisfied.*

PROOF: Let the function $r(x)$, $x = 0, k, 2k, \cdots$ be defined by the generating function

$$\sum_{x=0}^{b^*} r(kx)t^x = \begin{cases} e^{\alpha t}, & \beta = 0, \\ (1 - \beta t)^{-\alpha/\beta}, & \beta \neq 0. \end{cases} \tag{2.10}$$

It is easily verified by successive differentiation of (2.10) that $r(x) = d\mu(x)/d\nu(x)$ as given by 1.10. Substituting $\lambda/(k\alpha + \beta\lambda)$ for t in (2.10), we obtain

$$\sum_{x=0}^{b^*} r(kx)\left(\frac{\lambda}{k\alpha + \beta\lambda}\right)^x = \begin{cases} e^{\lambda/k}, & \beta = 0, \\ \left(1 + \frac{\beta\lambda}{k\alpha}\right)^{\alpha/\beta}, & \beta \neq 0. \end{cases} \tag{2.11}$$

Noting that $\omega(\lambda) = (1/k) \ln (\lambda/(k\alpha + \beta\lambda))$ by (1.9) we may write (2.11) as

$$\int_{[0,\infty)} e^{x\omega(\lambda)}\, d\mu(x) = \begin{cases} e^{\lambda/k}, & \beta = 0, \\ \left(1 + \frac{\beta\lambda}{k\alpha}\right)^{\alpha/\beta}, & \beta \neq 0. \end{cases} \tag{2.12}$$

Differentiating (2.12) with respect to λ and dividing both members by $\omega'(\lambda) = (\alpha/\lambda(k\alpha + \beta\lambda))$, we obtain

$$\int_{[0,\infty)} xe^{x\omega(\lambda)}\, d\mu(x) = \begin{cases} \lambda e^{-\lambda/k}, & \beta = 0, \\ \lambda\left(1 + \frac{\beta\lambda}{k\alpha}\right)^{\alpha/\beta}, & \beta \neq 0. \end{cases} \tag{2.13}$$

The proof is completed by noting that the ratio of (2.13) to (2.12) is always λ so that (1.3B) is satisfied.

Theorems 2.3 and 2.4 imply that for any $F(x \mid \lambda) \varepsilon \mathfrak{F}_1$

$$F(kx \mid \lambda) = \sum_{t=0}^{x-1} r(kt) \exp\left\{kt\omega(\lambda) - \int_0^\lambda u\omega'(u)\, du\right\}, \tag{2.14}$$

for integer values of x, where $\omega(\lambda)$ is given by (1.9), $r(x) = d\mu(x)/ d\nu(x)$ is given by (1.10), and the value of γ appearing in (2.6) is taken to be zero. The fact that we must have $K(0) = 1$ if $F(b+ \mid \lambda)$ is to equal one follows from the observation that $\omega(\lambda) \to -\infty$ as $\lambda \to 0$ together with the assumption that $r(0) = 1$.

The following theorem provides an integral formula for the n-fold convolution $F^{(n)}$ of $F(x \mid \lambda)$ when F is in $\mathfrak{F}_1$.

THEOREM 2.5: *If $F(x \mid \lambda) \varepsilon \mathfrak{F}_1$, then for integer values of m, $F^{(n)}$ is given by*

$$F^{(n)}(km \mid \lambda) = \begin{cases} 0, & m \leqq 0, \\ 1 - mr^{(n)}(km) \int_0^\lambda \frac{t^{m-1}}{(k\alpha + \beta t)^m} \cdot \exp\left\{-n \int_0^t \frac{\alpha\, du}{k\alpha + \beta u}\right\} dt, & m = 1, 2, \cdots, \end{cases} \tag{2.15}$$

where

$$(2.16)\qquad r^{(n)}(x) = \begin{cases} 1, & x = 0, \\ \dfrac{n\alpha(n\alpha+\beta)\cdots\left(n\alpha+\left(\frac{x}{k}-1\right)\beta\right)}{\left(\frac{x}{k}\right)!}, & x = k, 2k, \cdots. \end{cases}$$

PROOF: By (1.7)

$$(2.17)\qquad F^{(n)}(\infty \mid \lambda) = \int_{[0,\infty)} \exp\left\{\omega(\lambda)x - n\int_0^\lambda u\omega'(u)\,du\right\} d\mu^{(n)}(x) = 1$$

where $\mu^{(n)}$ is the n-fold convolution of μ. Hence, letting $r^{(n)}(x) = d\mu^{(n)}(x)/d\nu(x)$ and $t = \lambda/(k\alpha + \beta\lambda)$ and recalling that $\omega(\lambda) = (1/k)\ln(\lambda/(k\alpha + \beta t))$ for this case, we have

$$(2.18)\qquad \begin{aligned} \sum_{x=0}^{nb^*} r^{(n)}(kx)t^x &= \exp\left\{n\int_0^{k\alpha t/1-\beta t} \frac{\alpha\,du}{k\alpha+\beta u}\right\} \\ &= \begin{cases} e^{n\alpha t}, & \beta = 0, \\ (1-\beta t)^{-n\alpha/\beta}, & \beta \neq 0. \end{cases} \end{aligned}$$

Successive differentiation of (2.17) with respect to t yields (2.16). Referring to (1.8), we may write

$$(2.19)\qquad F^{(n)}(km \mid \lambda) = \sum_{x=0}^{m-1} r^{(n)}(kx)\left(\frac{\lambda}{k\alpha+\beta\lambda}\right)^x \exp\left\{-n\int_0^\lambda \frac{\alpha\,du}{k\alpha+\beta u}\right\},$$

for integer values of m. Assuming that (2.15) holds for some integer m, we have

$$(2.20)\qquad \begin{aligned} &F^{(n)}(k(m+1) \mid \lambda) \\ &= F^{(n)}(km \mid \lambda) + r^{(n)}(km)\left(\frac{\lambda}{k\alpha+\beta\lambda}\right)^m \exp\left\{-n\int_0^\lambda \frac{\alpha\,du}{k\alpha+\beta u}\right\} \\ &= 1 - mr^{(n)}(km)\int_0^\lambda \frac{t^{m-1}}{(k\alpha+\beta t)^m}\exp\left\{-n\int_0^t \frac{\alpha\,du}{k\alpha+\beta u}\right\}dt \\ &\quad + r^{(n)}(km)\left(\frac{\lambda}{k\alpha+\beta\lambda}\right)^m \exp\left\{-n\int_0^\lambda \frac{\alpha\,du}{k\alpha+\beta u}\right\}. \end{aligned}$$

Integrating the second term on the right by parts yields

$$(2.21)\qquad \begin{aligned} &mr^{(n)}(km)\int_0^\lambda t^{m-1}\left[(k\alpha+\beta t)^{-m}\exp\left\{-n\int_0^t \frac{\alpha\,du}{k\alpha+\beta u}\right\}\right]dt \\ &= r^{(n)}(km)(n\alpha+m\beta)\int_0^\lambda \frac{t^m}{(k\alpha+\beta t)^{m+1}}\exp\left\{-n\int_0^t \frac{\alpha\,du}{k\alpha+\beta u}\right\}dt \\ &\quad + r^{(n)}(km)\left(\frac{\lambda}{k\alpha+\beta\lambda}\right)^m \exp\left\{-n\int_0^\lambda \frac{\alpha\,du}{k\alpha+\beta u}\right\}. \end{aligned}$$

Substituting this expression in (2.20) and recalling (2.16), we see that (2.15) is verified for $F^{(n)}(k(m+1))$. It is easily shown directly that (2.15) holds for $m = 1$ so that the desired result follows by induction.

We now consider the family $\mathfrak{F}_2$ of distribution functions obtained when $\omega(\lambda)$ and $\mu(x)$ are defined by (1.14) and (1.15).

THEOREM 2.6: *If $F(x \mid \lambda) \varepsilon \mathfrak{F}_2$, then (i) condition (1.3B) is satisfied, and (ii)*

$$F^{(n)}(x \mid \lambda) = \frac{(\eta x)^{n\eta}}{\Gamma(n\eta)} \int_\lambda^\infty u^{-n\eta-1} e^{-x\eta/u} \, du. \tag{2.21}$$

PROOF: Any distribution in $\mathfrak{F}_2$ is a gamma distribution with parameters determined so that the first moment is λ, thereby satisfying (1.3B). The convolution of such gamma distributions is well known (cf. [6]) to be given by

$$F^{(n)}(x \mid \lambda) = \int_0^x \frac{\eta^{n\eta} t^{n\eta-1}}{\Gamma(n\eta)\lambda^{n\eta}} e^{-t\eta/\lambda} \, dt. \tag{2.22}$$

By making the change of variable $u = x\lambda/t$ we obtain (2.21).

3. The Bayes decision rule.

THEOREM 3.1: *The function $\varphi_n(t)$ given by (1.21) is finite and strictly increasing for $t \varepsilon I_\mu^{(n)}$.*

PROOF: The finiteness of $E(\Lambda) = \int_0^\infty \lambda \, dG(\lambda)$ insures the finiteness of $\varphi_n(t)$ for all $t \varepsilon I_\mu^{(n)}$ with the possible exception of a set of $\mu^{(n)}$-measure zero since

$$\varphi_n(t) = E\{\Lambda \mid S_n = t\}, \tag{3.1}$$

for all $t \varepsilon I_\mu^{(n)} - A$, where A is the exceptional null set. Hence, for any fixed $t \varepsilon I_\mu^{(n)}$, we may choose t_1, $t_2 \varepsilon I_\mu^{(n)}$ such that $t_1 \leqq t \leqq t_2$ and $\varphi_n(t_1)$, $\varphi_n(t_2)$ are finite. Then $t\omega(\lambda) \leqq \max (t_1\omega(\lambda), t_2\omega(\lambda))$ for all $\lambda \varepsilon I_\mu$ and the finiteness of $\varphi_n(t)$ follows from the finiteness of the integrals in the expressions for $\varphi_n(t_1)$ and $\varphi_n(t_2)$. Now choose t and $\delta > 0$ so that $[t, t + \delta] \subset I_\mu^{(n)}$ and let

$$H_t(z) = \frac{\int_0^z \exp\left\{t\omega(\lambda) - n \int_\gamma^\lambda u\omega'(u) \, du\right\} dG(\lambda)}{\int_0^\infty \exp\left\{t\omega(\lambda) - n \int_\gamma^\lambda u\omega'(u) \, du\right\} dG(\lambda)}. \tag{3.2}$$

Then $H_t(z)$ may be interpreted as the distribution function of some random variable Z and we may write

$$\varphi_n(t + \delta) - \varphi_n(t) = \frac{E\{Z \exp\{\delta\omega(Z)\}\} - E\{Z\}E\{\exp\{\delta\omega(Z)\}\}}{E\{\exp\{\delta\omega(Z)\}\}}. \tag{3.3}$$

It is intuitively clear and follows rigorously from the inequality on page 43 of [7] that the right hand side of (3.3) is strictly positive for all $\delta > 0$ whenever $G(\lambda)$ and hence $H_t(z)$ are non-degenerate. Thus $\varphi_n(t)$ is strictly increasing and the proof is completed.

We now recall that the Bayes decision rule (1.20) is equialvent to

$$\delta^*(S_n) = \begin{cases} 1, & S_n \leqq t(n), \\ 0, & S_n > t(n), \end{cases} \tag{3.4}$$

where $t(n)$ is defined by (1.24).

THEOREM 3.2: *If* $\lambda_0 = \sup \{\lambda : \lambda \leqq c; G(\lambda+) - G(\lambda - \epsilon) > 0$, *all* $\epsilon > 0\}$ *and* $\lambda_1 = \inf \{\lambda : \lambda \geqq c; G(\lambda + \epsilon) - G(\lambda) > 0$, *all* $\epsilon > 0\}$, *and if* $t(n)$ *is defined by* (1.24), *then*

$$\lambda_0 \leqq \liminf_{n\to\infty} \frac{t(n)}{n} \leqq \limsup_{n\to\infty} \frac{t(n)}{n} \leqq \lambda_1 . \tag{3.5}$$

PROOF: Let I be an indicator function defined on the product of the space of all sequences of numbers $\{s_n\}$ and the real line by

$$I(\{s_n\}, \lambda) = \begin{cases} 1, & \text{if } s_n/n \to \lambda \quad \text{as} \quad n \to \infty, \\ 0, & \text{otherwise.} \end{cases} \tag{3.6}$$

Given $\Lambda = \lambda$, S_n is a sum of conditionally independent identically distributed random variables with mean λ. Hence, by the strong law of large numbers

$$E\{I(\{S_n\}, \lambda) \mid \Lambda = \lambda\} = 1, \qquad \text{a.s.,} \tag{3.7}$$

and this can be shown to be equivalent to

$$E\{I(\{S_n\}, \Lambda) \mid \Lambda = \lambda\} = 1, \qquad \text{a.s.} \tag{3.8}$$

Thus

$$P\{S_n/n \to \Lambda\} = E\{E\{I(\{S_n\}, \Lambda\} \mid \Lambda\}\} = 1. \tag{3.9}$$

Furthermore, by a martingale convergence theorem (cf., p. 398 of [8]),

$$P\{E(\Lambda \mid S_n) \to \Lambda\} = 1. \tag{3.10}$$

Therefore for all x which are continuity points of $G(x)$,

$$P\{E(\Lambda \mid S_n) < x\} \to G(x), \tag{3.11}$$

and

$$P\{S_n/n < x\} \to G(x), \tag{3.12}$$

as $n \to \infty$. Suppose that $\limsup_{n\to\infty} (t(n)/n) > \lambda_1$. Then there is a $\delta > 0$ such that $\lambda_1 + \delta$ is a continuity point of $G(x)$, and $t(n)/n > \lambda_1 + \delta$ for arbitrarily large values of n. Hence, for these values of n

$$P\{E(\Lambda \mid S_n) \leqq c\} = P\{S_n/n \leqq t(n)/n\} \geqq P\{S_n/n \leqq \lambda_1 + \delta\}. \tag{3.13}$$

However, as $n \to \infty$

$$P\{S_n/n < \lambda_1 + \delta\} \to G(\lambda_1 + \delta) > G(\lambda_1+) \tag{3.14}$$

and for sufficiently large n

$$P\{E(\Lambda \mid S_n) \leqq c\} \leqq G(\lambda_1+) + \epsilon \tag{3.15}$$

for arbitrary $\epsilon > 0$. If we choose $\epsilon > 0$ such that $\epsilon < G(\lambda_1 + \delta) - G(\lambda_1+)$ we are led to a contradiction and hence, $\limsup_{n\to\infty} t(n)/n \leqq \lambda_1$. A similar argument establishes the other inequality of the theorem.

THEOREM 3.3: *If* $G(\lambda) \varepsilon \mathcal{G}_1$, *then*

$$t(n) = cn + \frac{\omega''(c)}{(\omega'(c))^2} - \frac{G''(c)}{G'(c)\omega'(c)} + o(1). \tag{3.16}$$

The proof of Theorem 3.3 will require a sequence of preliminary lemmas, the first two of which will also be used in the later derivation of an asymptotic expansion for the Bayes risk.

Since $P\{E(\Lambda \mid S_n) \to \Lambda\} = 1$, there exists a $\delta > 0$ such that (i) $c + \delta$ is a continuity point of $G(x)$, (ii) $G(c + \delta) < 1$, and (iii) as $n \to \infty$

$$P\{E(\Lambda \mid S_n) > c + \delta\} \to 1 - G(c + \delta) > 0. \tag{3.17}$$

But $E(\Lambda \mid S_n) \leqq \varphi_n(nb)$, a.s., hence $\varphi_n(nb) > c$ for all sufficiently large n. Similarly, $\varphi_n(na) < c$ for all sufficiently large n. Hence, referring to (1.22) we see that for all sufficiently large n, $t(n)$, defined by (1.24), is the solution to

$$I(n) = \int_0^\infty (\lambda - c) \exp\left\{t(n)\omega(\lambda) - n\int_\gamma^\lambda u\omega'(u)\,du\right\} dG(\lambda) = 0. \tag{3.18}$$

The result of Theorem 3.2 suggests that if $G(\lambda) \varepsilon \mathcal{G}_1$, then we should write $t(n) = cn + \psi(n)$ so that

$$I(n) = \int_0^\infty (\lambda - c) \exp\{nh(\lambda) + \psi(n)\omega(\lambda)\}\, dG(\lambda) \tag{3.19}$$

where

$$h(\lambda) = c\omega(\lambda) - \int_\gamma^\lambda u\omega'(u)\,du. \tag{3.20}$$

This form of $I(n)$ will be convenient for the application of results from the theory of the asymptotic expansion of integrals.

LEMMA 3.1: *Let $g(t)$ be any function integrable with respect to a distribution function $H(t)$, and let $\varphi(n, t)$ be a function, $\{t_n\}$ a sequence and c a number such that for all sufficiently small $\epsilon > 0$ there exists a $\delta > 0$ such that for all sufficiently large n, $\varphi(n, t) < \varphi(n, t_n) - \delta$ whenever $\mid t - c \mid \geqq \epsilon$. Then for any fixed m*

$$\int_{-\infty}^{c-\epsilon} g(t) \exp\{n\varphi(n, t)\}\, dH(t) = o(\exp\{n\varphi(n, t_n)\}n^{-m}), \tag{3.21}$$

and

$$\int_{c+\epsilon}^\infty g(t) \exp\{n\varphi(n, t)\}\, dH(t) = o(\exp\{n\varphi(n, t_n)\}n^{-m}). \tag{3.22}$$

PROOF: For all sufficiently large n,

$$\left| \int_{-\infty}^{c-\epsilon} g(t) \exp\{n\varphi(n, t)\}\, dH(t) \right| \leq \exp\{n\varphi(n, t_n) - n\delta\} \int_{-\infty}^{c-\epsilon} |g(t)|\, dH(t), \tag{3.23}$$

and (3.21) follows immediately. Expression (3.22) follows by the same argument.

LEMMA 3.2: *If $g(t)$ is any function which is four times continuously differentiable in some interval containing c, and if $g'(c) = 0$, $g''(c) < 0$, then*

$$\begin{aligned}\int_{c-\epsilon}^{c+\epsilon} (t-c)^r \exp\{ng(t)\}\, dt = \exp\{ng(c)\} \Bigg\{ & n^{-\frac{r+1}{2}} \left(-\frac{2}{g''(c)}\right)^{\frac{r+1}{2}} \\ & \cdot \left(\frac{1+(-1)^r}{2}\right) \Gamma\left(\frac{r+1}{2}\right) + n^{-\frac{r+2}{2}} \left(\frac{g'''(c)}{3!}\right)\left(-\frac{2}{g''(c)}\right)^{\frac{r+4}{2}} \\ & \cdot \Gamma\left(\frac{r+4}{2}\right)\left(\frac{1+(-1)^{r+1}}{2}\right) + n^{-\frac{r+3}{2}} \left[\left(\frac{1+(-1)^r}{2}\right)\left(-\frac{2}{g''(c)}\right)^{\frac{r+5}{2}} \right. \\ & \left. \cdot \left\{\left(\frac{g''''(c)}{4!}\right)\Gamma\left(\frac{r+5}{2}\right) - \frac{\left(\frac{g'''(c)}{3!}\right)^2 \Gamma\left(\frac{r+7}{2}\right)}{g''(c)}\right\}\right] + o\left(n^{-\frac{r+3}{2}}\right)\Bigg\}.\end{aligned} \tag{3.24}$$

This is a standard result from the theory of asymptotic expansions and will not be proved here. A proof is outlined, for example, in [9].

The next lemma establishes the boundedness of $\psi(n)$.

LEMMA 3.3: *If $G(\lambda) \varepsilon \mathcal{G}_1$ and $t(n) = cn + \psi(n)$, then $\psi(n) = O(1)$.*

PROOF: The method of proof is to derive an asymptotic expansion for $I(n)$ as defined by (3.19) and show that the assumption that $|\psi(n)| \to \infty$ as $n \to \infty$ leads to a contradiction.

The expression $h(\lambda) + (\psi(n)/n)\omega(\lambda)$ is maximized when $\lambda = c + (\psi(n)/n)$ and, noting that $\psi(n)/n \to 0$ as $n \to \infty$ by Theorem 3.2, it is easily verified that $h(\lambda) + (\psi(n)/n)\omega(\lambda)$ has the properties of the function $\varphi(n, t)$ of Lemma 3.1 with $t = \lambda$ and $t_n = c + (\psi(n)/n)$. Hence choosing $\epsilon > 0$ such that $dG(\lambda) = G'(\lambda)\, d\lambda$ for λ in $(c - \epsilon, c + \epsilon)$ we have from (3.19) by Lemma 3.1

$$\begin{aligned} I(n) = & \int_{c-\epsilon}^{c+\epsilon} (\lambda - c) \exp\{nh(\lambda) + \psi(n)\omega(\lambda)\} G'(\lambda)\, d\lambda \\ & + o\left(\exp\left\{nh\left(c + \frac{\psi(n)}{n}\right) + \psi(n)\omega\left(c + \frac{\psi(n)}{n}\right)\right\} n^{-m}\right), \end{aligned} \tag{3.25}$$

for any $m \geqq 0$. By the definition of $\mathcal{G}_1$, $G'(\lambda) = G'(c) + O(\lambda - c)$ for $c - \epsilon < \lambda < c + \epsilon$. Hence

$$I(n) = G'(c) \int_{c-\epsilon}^{c+\epsilon} (\lambda - c) \exp \{nh(\lambda) + \psi(n)\omega(\lambda)\}\, d\lambda$$
$$(3.26) \qquad + O\left(\int_{c-\epsilon}^{c+\epsilon} (\lambda - c)^2 \exp \{nh(\lambda) + \psi(n)\omega(\lambda)\}\, d\lambda\right)$$
$$+ o\left(\exp\left\{nh\left(c + \frac{\psi(n)}{n}\right) + \psi(n)\omega\left(c + \frac{\psi(n)}{n}\right)\right\} n^{-m}\right).$$

Letting $\tau_n = \psi(n)/n$ and $t = \lambda - c - \tau_n$, and expanding the exponent in the integrands about $t = 0$ we obtain

$$I(n) = \exp \{nh(c + \tau_n) + \psi(n)\omega(c + \tau_n)\} \Big[G'(c) \int_{-\epsilon-\tau_n}^{\epsilon-\tau_n} (t + \tau_n)$$
$$\cdot \exp \{t^2[nh''(c + \tau_n) + \psi(n)\omega''(c + \tau_n)] + \xi(n, t)\}\, dt\Big]$$
$$(3.27) \qquad + O\left(\int_{-\epsilon-\tau_n}^{\epsilon-\tau_n} (t + \tau_n)^2 \exp \{t^2[nh''(c + \tau_n) + \psi(n)\omega''(c + \tau_n)]\right.$$
$$\left. + \xi(n, t)\}\, dt\right) + o(\exp \{nh(c + \tau_n) + \psi(n)\omega(c + \tau_n)\} n^{-m})$$

where $|\xi(n, t)| \leqq knt^3$ for some $k > 0$, all n, and all t in $(-\epsilon - \tau_n, \epsilon - \tau_n)$.

Since $\tau_n \to 0$, changing the range of integration from $(-\epsilon - \tau_n, \epsilon - \tau_n)$ to $(-\epsilon^*, \epsilon^*)$ for $0 < \epsilon^* < \epsilon$ adds only terms of negligibly small order by Lemma 3.1. Hence, $I(n)$ may be expressed in terms of integrals of the form

$$(3.28) \qquad \int_{-\epsilon^*}^{\epsilon^*} t^r \exp \{t^2[nh''(c + \tau_n) + \psi(n)\omega''(c + \tau_n)] + \xi(n, t)\}\, dt$$

for $r = 0, 1, 2$. Applying Lemma 3.2 to these integrals, regarding

$$[nh''(c + \tau_n) + \psi(n)\omega''(c + \tau_n)]$$

as the parameter which becomes large, and noting that the first terms in the expansions remain unchanged if either the upper or lower bound for $\xi(n, t)$ is used, we obtain

$$I(n) = \exp\{nh(c + \tau_n) + \psi(n)\omega(c + \tau_n)\}$$
$$(3.29) \qquad \cdot \left[G'(c)\, \frac{\psi(n)}{n^{3/2}} \left(\frac{-\pi}{h''(c + \tau_n) + \tau_n\, \omega''(c + \tau_n)}\right)^{1/2}\right.$$
$$\left. + o\left(\frac{\psi(n)}{n^{3/2}}\right) + O\left(\frac{1}{n^{3/2}}\right)\right].$$

However, if $|\psi(n_k)| \to \infty$ for any subsequence $n_k \to \infty$, (3.29) implies that $I(n) \neq 0$ for arbitrarily large values of n. This, however, is a contradiction, hence, $\psi(n) = O(1)$.

These lemmas now permit us to complete the proof of the theorem.

PROOF OF THEOREM 3.3: Expanding

$$f(n, \lambda) = G'(\lambda) \exp \psi(n)\omega(\lambda) \tag{3.30}$$

about $\lambda = c$ in (3.25) and regarding $h(\lambda)$ as the function $\varphi(n, t)$ appearing in Lemma 3.1 with $t = \lambda$ and $t_n \equiv c$, we have

$$\begin{aligned} I(n) = \int_{c-\epsilon}^{c+\epsilon} (\lambda - c) \exp \{nh(\lambda) + \psi(n)\omega(c)\}[G'(c) \\ + (\lambda - c)(G''(c) + G'(c)\omega(c)\psi(n)) \\ + R(n, \lambda)]\, d\lambda + o(\exp \{nh(c)\}n^{-m}) \end{aligned} \tag{3.31}$$

for any $m \geqq 0$, where

$$R(n, \lambda) = (\lambda - c) \left[\frac{\partial}{\partial\lambda} f(n, c + \theta(n, \lambda)(\lambda - c)) - \frac{\partial}{\partial\lambda} f(n, c)\right] \tag{3.32}$$

for some $0 < \theta(n, \lambda) < 1$. Now let

$$T(n) = \int_{c-\epsilon}^{c+\epsilon} (\lambda - c) \exp \{nh(\lambda)\}R(n, \lambda)\, d\lambda. \tag{3.33}$$

For any arbitrarily small $\delta > 0$ we can find an $\eta(\delta) > 0$ such that $| R(n, \lambda) | < \delta(\lambda - c)$ for λ in $(c - \eta(\delta), c + \eta(\delta))$, since

$$\partial/\partial\lambda f(n, \lambda) = [G''(\lambda) + G'(\lambda)\psi(n)\omega'(\lambda)] \exp \{\psi(n)\omega(\lambda)\}$$

regarded as a function of λ is continuous at c uniformly in n. Applying Lemma 3.2 we have

$$\begin{aligned} | T(n) | &= \left| \int_{c-\eta(\delta)}^{c+\eta(\delta)} R(n,\lambda)(\lambda - c) \exp \{nh(\lambda)\}\, d\lambda \right| + o(\exp \{nh(c)\}n^{-m}) \\ &< \delta \int_{c-\eta(\delta)}^{c+\eta(\delta)} (\lambda - c)^2 \exp \{nh(\lambda)\}\, d\lambda + o(\exp \{nh(c)\}n^{-m}) \\ &= \delta O\left(\frac{\exp \{nh(c)\}}{n^{3/2}}\right). \end{aligned} \tag{3.34}$$

But δ may be taken arbitrarily small, hence

$$\lim \sup_{n\to\infty} (| T(n) |/\exp \{nh(c)\}n^{-3/2})$$

is less than an arbitrarily small quantity, so that $T(n) = o\,(\exp \{nh(c)\}n^{-3/2})$. Using this fact and applying Lemma 3.2 to the terms of (3.31) involving $(\lambda - c)$ and $(\lambda - c)^2$ we have,

$$I(n) = n^{-3/2} \exp \{nh(c) + \psi(n)\omega(c)\} \Big[G'(c) \left(\frac{h'''(c)}{3!} \right)$$

$$(3.35) \qquad \cdot \left(-\frac{2}{h''(c)} \right)^{5/2} \Gamma(5/2) + (G''(c) + \psi(n)\omega'(c)G'(c))$$

$$\cdot \left(-\frac{2}{h''(c)} \right)^{3/2} \Gamma(3/2) + o(1) \Big] = 0$$

which yields

$$(3.36) \quad \psi(n) = \frac{h'''(c)}{2h''(c)\omega'(c)} - \frac{G''(c)}{G'(c)\omega'(c)} + o(1) = \frac{\omega''(c)}{(\omega(c))^2} - \frac{G''(c)}{G'(c)\omega'(c)} + o(1).$$

This establishes the expression (3.16) for $t(n)$ if $G(\lambda) \varepsilon \mathcal{G}_1$. We now turn to consideration of $G(\lambda) \varepsilon \mathcal{G}_2$.

THEOREM 3.4: *If* $G(\lambda) \varepsilon \mathcal{G}_2$, $\zeta_1 = G(l_G+) - G(l_G)$, *and* $\zeta_2 = G(u_G+) - G(u_G)$, *then*

$$(3.37) \qquad t(n) = n \frac{\int_{l_G}^{u_G} u\omega'(u)\, du}{\omega(u_G) - \omega(l_G)} + \frac{\ln \dfrac{\zeta_1(c - l_G)}{\zeta_2(u_G - c)}}{\omega(u_G) - \omega(l_G)} + o(1).$$

PROOF: As in Theorem 3.3 we must find an asymptotic expansion for the solution $t(n)$ of

$$(3.38) \quad I(n) = \int_0^\infty (\lambda - c) \exp\left\{ t(n)\omega(\lambda) - n \int_\gamma^\lambda u\omega'(u)\, du \right\} dG(\lambda) = 0.$$

For $\lambda < l_G$, consider

$$(3.39) \qquad \tau_1(\lambda, n) = \frac{t(n)}{n} (\omega(\lambda) - \omega(l_G)) + \int_\lambda^{l_G} u\omega'(u)\, du.$$

Integrating by parts, and applying the mean value theorem and Theorem 3.2, we have $\limsup_{n\to\infty} \tau_1(\lambda, n) \leqq l_G (\omega(\lambda) - \omega(l_G)) + l_G \omega(l_G) - \lambda\omega(\lambda) - (l_G - \lambda)\omega(\lambda^*)$ where $\lambda < \lambda^* < l_G$. Therefore, since $\omega(\lambda)$ is increasing, we have $\limsup_{n\to\infty} \tau_1(\lambda, n) < 0$ for each $\lambda < l_G$. Hence, for each $\lambda < l_G$

$$(3.40) \quad \frac{(\lambda - c) \exp\left\{ t(n)\omega(\lambda) - n \int_\gamma^\lambda u\omega'(u)\, du \right\}}{(l_G - c) \exp\left\{ t(n)\omega(l_G) - n \int_\gamma^{l_G} u\omega'(u)\, du \right\}}$$

$$= \frac{\lambda - c}{l_G - c} \exp \{n\tau_1(\lambda, n)\} \to 0,$$

as $n \to \infty$, so that

$$\int_0^{l_G+} \frac{(\lambda - c)\exp\left\{t(n)\omega(\lambda) - n\int_\gamma^\lambda u\omega'(u)\,du\right\}}{(l_G - c)\exp\left\{t(n)\omega(l_G) - n\int_\gamma^{l_G} u\omega'(u)\,du\right\}} \to \zeta_1 \tag{3.41}$$

as $n \to \infty$, by the dominated convergence theorem. This, however, is equivalent to

$$\begin{aligned}\int_0^{l_G+} &(\lambda - c)\exp\left\{t(n)\omega(\lambda) - n\int_\gamma^\lambda u\omega'(u)\,du\right\} dG(\lambda)\\ &= \zeta_1(l_G - c)\exp\left\{t(n)\omega(l_G) - n\int_\gamma^{l_G+} u\omega'(u)\,du\right\}(1 + o(1)).\end{aligned} \tag{3.42}$$

Similarly

$$\limsup_{n\to\infty}\left\{\frac{t(n)}{n}(\omega(\lambda) - \omega(u_G)) - \int_{u_G}^\lambda u\omega'(u)\,du\right\} < 0 \tag{3.43}$$

for each $\lambda > u_G$, and

$$\begin{aligned}\int_{u_G}^\infty &(\lambda - c)\exp\left\{t(n)\omega(\lambda) - n\int_\gamma^\lambda u\omega'(u)\,du\right\} dG(\lambda)\\ &= \zeta_2(u_G - c)\exp\left\{t(n)\omega(u_G) - n\int_\gamma^{u_G} u\omega'(u)\,du\right\}(1 + o(1)).\end{aligned} \tag{3.44}$$

Therefore we must determine $t(n)$ so that

$$\begin{aligned}\zeta_1&(c - l_G)\exp\left\{t(n)\omega(l_G) - n\int_\gamma^{l_G} u\omega'(u)\,du\right\}(1 + o(1))\\ &= \zeta_2(u_G - c)\exp\left\{t(n)\omega(u_G) - n\int_\gamma^{u_G} u\omega'(u)\,du\right\}(1 + o(1)).\end{aligned} \tag{3.45}$$

Taking logarithms, we obtain (3.37) as desired.

4. Asymptotic characterization of the Bayes risk and the Bayes sample size.

THEOREM 4.1: *If $F(x \mid \lambda) \in \mathfrak{F}_1$ (with $k = 1$) and $G(\lambda) \in \mathcal{G}_1$, then the Bayes risk for fixed n and N is given by*

$$\begin{aligned}R(\delta^*, n, N) = n&\left((s_1 - r_1)E(\Lambda) + s_2 - r_2 + (r_1 - a_1)\int_0^c (\lambda - c)\,dG(\lambda)\right)\\ &+ N\left(r_1 E(\Lambda) + r_2 + (a_1 - r_1)\int_0^c (\lambda - c)\,dG(\lambda)\right)\\ &\qquad + (N - n)\frac{(r_2 - a_2)(\alpha + \beta c)G'(c)}{2\alpha n} + (N - n)o\left(\frac{1}{n}\right)\end{aligned} \tag{4.1}$$

and the Bayes sample size is

$$n^*(N) = \begin{cases} N, & A_G \leqq 0, \\ N^{1/2}\left(\dfrac{(r_2 - a_2)(\alpha + \beta c)G'(c)}{2\alpha A_G}\right)^{1/2} + o(N^{1/2}), & A_G > 0, \end{cases} \tag{4.2}$$

where

$$A_G = (s_1 - r_1)E(\Lambda) + s_2 - r_2 + (r_1 - a_1)\int_0^c (\lambda - c)\, dG(\lambda). \tag{4.3}$$

PROOF: By applying (1.20) to (1.19) we see that

$$\begin{aligned} R(\delta^*, n, N) &= n((s_1 - r_1)E(\Lambda) + s_2 - r_2) \\ &\quad + N(r_1E(\Lambda) + r_2) + (N - n)(a_1 - r_1)E\{(\Lambda - c)L(\Lambda, n)\}, \end{aligned} \tag{4.4}$$

where

$$L(\lambda, n) = E\{\delta^*(S_n) \mid \Lambda = \lambda\} = P\{S_n \leqq t(n) \mid \Lambda = \lambda\}, \tag{4.5}$$

where $t(n)$ is defined by (1.24).

Letting $\tau(n) = [t(n)]$, where $[x]$ indicates the largest integer less than equal to x, we may write $\tau(n) = cn + \varphi(n)$, where $\varphi(n) = O(1)$ by Theorem 3.3. Now, noting that $F^{(n)}(m \mid \lambda) \to 0$ as $\lambda \to b$ for $m < b$, we may apply Theorem 2.5 to obtain

$$\begin{aligned} L(\lambda, n) &= (\tau(n) + 1)r^{(n)}(\tau(n) + 1) \\ &\quad \cdot \int_\lambda^b \frac{t^{\tau(n)}}{(\alpha + \beta t)^{\tau(n)+1}} \exp\left\{-n\int_0^t \frac{\alpha\, du}{\alpha + \beta u}\right\} dt, \end{aligned} \tag{4.6}$$

for values of n large enough so that $0 < \tau(n) < b$. Hence

$$\begin{aligned} E[(\Lambda - c)L(\Lambda, n)] &= (\tau(n) + 1)r^{(n)}(\tau(n) + 1) \int_0^{b+} (\lambda - c) \int_\lambda^b \\ &\quad \cdot \frac{t^{\tau(n)}}{(\alpha + \beta t)^{\tau(n)+1}} \exp\left\{-n \int_0^t \frac{\alpha\, du}{(\alpha + \beta u)}\, du\right\} dt\, dG(\lambda) \\ &= (\tau(n) + 1)r^{(n)}(\tau(n) + 1) \int_0^b \frac{t^{\tau(n)}}{(\alpha + \beta t)^{\tau(n)+1}} \\ &\quad \cdot \exp\left\{-n \int_0^t \frac{\alpha\, du}{\alpha + \beta u}\right\} \int_0^t (\lambda - c)\, dG(\lambda)\, dt. \end{aligned} \tag{4.7}$$

As before, let

$$h(t) = c \ln\left(\frac{t}{\alpha + \beta t}\right) - \int_0^t \frac{\alpha\, du}{\alpha + \beta u}. \tag{4.8}$$

Then

$$E[(\Lambda - c)L(\Lambda, n)] = (\tau(n) + 1)r^{(n)}(\tau(n) + 1)I(n) \tag{4.9}$$

where

$$I(n) = \int_0^b K(t, n) \exp\{nh(t)\}\, dt, \tag{4.10}$$

and

$$K(t, n) = \frac{t^{\varphi(n)}}{(\alpha + \beta t)^{\varphi(n)+1}} \int_0^t (\lambda - c)\, dG(\lambda). \tag{4.11}$$

Now

$$\frac{\partial K(c, n)}{\partial t} = \frac{c^{\varphi(n)}}{(\alpha + \beta c)^{\varphi(n)+1}} \left(\frac{\alpha\varphi(n) - \beta c}{c(\alpha + \beta c)}\right) \int_0^c (\lambda - c)\, dG(\lambda), \tag{4.12}$$

and

$$\begin{aligned} \frac{\partial^2 K(c, n)}{\partial t^2} &= \frac{c^{\varphi(n)}}{(\alpha + \beta c)^{\varphi(n)+1}} \\ &\left[G'(c) + \frac{\alpha^2(\varphi^2(n) - \varphi(n)) - 4\alpha\beta c\varphi(n) + 2\beta^2 c^2}{c^2(\alpha + \beta c)^2} \int_0^c (\lambda - c)\, dG(\lambda)\right], \end{aligned} \tag{4.13}$$

so that

$$K(t, n) = K(c, n) + (t - c)\frac{\partial}{\partial t} K(c, n) + \frac{1}{2}\frac{\partial^2}{\partial t^2} K(c, n) + R(t, n) \tag{4.14}$$

where, by an argument similar to that used in Theorem 3.3,

$$R(t, n) = o((t - c)^2)$$

uniformly in n. Furthermore by Lemma 3.1

$$I(n) = \int_{c-\epsilon}^{c+\epsilon} K(t, n) \exp\{nh(t)\}\, dt + o(\exp\{nh(c)\}n^{-m}) \tag{4.15}$$

for all $m \geqq 0$. Hence, substituting (4.14) in (4.15), treating the remainder of (4.14) as was done in Theorem 3.3, and applying Lemma 3.2, we obtain

$$\begin{aligned} I(n) = \frac{\exp\{nh(c)\}}{\sqrt{n}} \left(-\frac{2}{h^{(2)}(c)}\right)^{1/2} \Bigg\{ & K(c, n)\,\Gamma\left(\frac{1}{2}\right) \\ &+ \frac{\Gamma\left(\frac{5}{2}\right)}{n} \left[K(c, n)\left(-\frac{2}{h^{(2)}(c)}\right)^2 \left[\frac{h^{(4)}(c)}{4!} - \frac{5(h^{(3)}(c))^2}{72h^{(2)}(c)}\right]\right. \\ &\left. + \frac{\partial K(c, n)}{\partial t}\left(-\frac{2}{h^{(2)}(c)}\right)^2 \left(\frac{h^{(3)}(c)}{3!}\right) + \frac{1}{3}\frac{\partial^2 K(c, n)}{\partial t^2}\left(-\frac{2}{h^{(2)}(c)}\right)\right] + o\left(\frac{1}{n}\right)\Bigg\}. \end{aligned} \tag{4.16}$$

From (4.8) we have

$$\begin{aligned} h^{(2)}(c) &= -\frac{\alpha}{c(\alpha+\beta c)}, \\ h^{(3)}(c) &= \frac{2\alpha^2+4\alpha\beta c}{c^2(\alpha+\beta c)}, \\ h^{(4)}(c) &= -6\left(\frac{\alpha^3+3\alpha^2\beta c+3\alpha\beta^2c^2}{c^3(\alpha+\beta c)^3}\right), \end{aligned} \tag{4.17}$$

and hence

$$I(n) = \left(\frac{2\pi c(\alpha+\beta c)}{\alpha n}\right)^{1/2} \exp\{nh(c)\} \frac{c^{\varphi(n)}}{(\alpha+\beta c)^{\varphi(n)+1}}$$

$$\int_0^c (\lambda - c)\, dG(\lambda) \left\{1 + \frac{1}{n}\left(\frac{\alpha(\varphi^2(n)+\varphi(n))}{2c(\alpha+\beta c)} + \frac{\alpha^2+\alpha\beta c+\beta^2c^2}{12\alpha c(\alpha+\beta c)} + \frac{G'(c)}{2\alpha\int_0^c(\lambda-c)\,dG(\lambda)}\right) + o\left(\frac{1}{n}\right)\right\}. \tag{4.18}$$

Furthermore by (4.8)

$$\exp\{nh(c)\} = \begin{cases} \left(\frac{c}{\alpha+\beta c}\right)^{cn}\left(\frac{\alpha}{\alpha+\beta c}\right)^{\frac{\alpha n}{\beta}}, & \beta \neq 0, \\ \left(\frac{c}{\alpha}\right)^{cn} \exp\{-cn\}, & \beta = 0, \end{cases} \tag{4.19}$$

and from (2.16) we have

$$(cn+\varphi(n)+1)r^{(n)}(cn+\varphi(n)+1)$$

$$= \begin{cases} \dfrac{\beta^{cn+\varphi(n)+1}\,\Gamma\left(\frac{n\alpha}{\beta}+cn+\varphi(n)+1\right)}{(cn+\varphi(n))!\,\Gamma\left(\frac{n\alpha}{\beta}\right)}, & \beta > 0, \\ \dfrac{(n\alpha)^{cn+\varphi(n)+1}}{(cn+\varphi(n))!}, & \beta = 0, \\ \dfrac{(-\beta)^{cn+\varphi(n)+1}(nb)!}{(cn+\varphi(n))!\,(nb-cn-\varphi(n)-1)!}, & -\frac{\alpha}{\beta} = b > 0 \end{cases}$$

(4.20)

$$= \begin{cases} c^{-(cn+\varphi(n)+1/2}(\alpha+\beta c)^{\frac{n\alpha}{\beta}+cn+\varphi(n)+1/2}\,\alpha^{-\frac{n\alpha}{\beta}+1/2} \\ \cdot\sqrt{\frac{n}{2\pi}}\left[1 - \frac{\alpha(\varphi^2(n)+\varphi(n))}{2nc(\alpha+\beta c)} - \frac{\alpha^2+\beta^2c^2+\alpha\beta c}{12n\alpha c(\alpha+\beta c)} + 0\left(\frac{1}{n}\right)\right], & \beta \neq 0, \\ \exp\{cn\}\,\alpha^{cn+\varphi(n)+1}\sqrt{\frac{n}{c\pi}}\,c^{-(cn+\varphi(n)+1/2)} \\ \cdot\left(1 - \frac{\varphi^2(n)+\varphi(n)}{2nc} - \frac{1}{12cn} + o\left(\frac{1}{n}\right)\right), & \beta = 0, \end{cases}$$

by Stirling's formula, using the form $\ln n! = (n + \frac{1}{2}) \ln n + \frac{1}{2} \ln 2\pi - n + 1/12n + o(1/n)$. Combining (4.19) and (4.20), we obtain

$$
\begin{aligned}
(N - n)(a_1 - r_1)E\{(\Lambda - c)L(\Lambda, n)\} &= N(a_1 - r_1) \int_0^c (\lambda - c)\, dG(\lambda) \\
&+ \frac{(r_2 - a_2)(\alpha + \beta c)}{2n\alpha} G'(c) + n(r_1 - a_1) \int_0^c (\lambda - c)\, dG(\lambda) \qquad (4.21)\\
&+ \frac{(a_2 - r_2)(\alpha + \beta c)G'(c)}{2\alpha} + (N - n)\, o\left(\frac{1}{n}\right),
\end{aligned}
$$

which upon substitution in (4.4) yields (4.1).

It is easily verified that, as long as there are sets with positive probability on each side of c, the Bayes decision rule leads to an incorrect decision with positive probability whenever n is finite. Hence, for each finite n

$$E[(\Lambda - c)L(\Lambda, n)] > \int_0^c (\lambda - c)\, dG(\lambda). \qquad (4.22)$$

By (4.21), however, $E[(\Lambda - c)L(\Lambda, n)] \to \int_0^c (\lambda - c)\, dG(\lambda)$, as $n \to \infty$. Suppose that the Bayes sample size $n^*(N) = O(1)$ as $N \to \infty$. Then there exists an integer m such that

$$E\{(\Lambda - c)L(\Lambda, n^*(N))\} > E\{(\Lambda - c)L(\Lambda, m))\}. \qquad (4.23)$$

Referring to (4.4) and recalling that $a_1 > r_1$ by assumption we see that this implies that

$$R(\delta^*, n^*(N), N) > R(\delta^*, m, N), \qquad (4.24)$$

for all sufficiently large values of N. This, however, contradicts the assertion that $n^*(N)$ is the Bayes sample size. Similarly, for any subsequence $N_k \to \infty$ the assertion that $n^*(N_k) = O(1)$ leads to a contradiction. Hence $n^*(N) \to \infty$ as $N \to \infty$.

Now for simplicity of notation we write (4.1) as

$$R(\delta^*, n, N) = A_G\, n + B_G\, N + C_G(N - n) \left(\frac{1}{n} + o\left(\frac{1}{n}\right)\right), \text{as } n \to \infty. \qquad (4.25)$$

In order to characterize the Bayes sample size we must consider two cases.

(Case i; $A_G \leqq 0$): If $A_G \leqq 0$, the risk is clearly minimized by taking n as large as possible (i.e., equal to N) since $C_G > 0$ by assumption so that $C_G((1/n^*) + o(1/n^*)) > 0$ for large N since $n^*(N) \to \infty$.

(Case ii; $A_G > 0$): Let the Bayes sample size be written as

$$n^*(N) = AN^{1/2} + \xi(N) \qquad (4.26)$$

where

$$A = \left(\frac{C_G}{A_G}\right)^{1/2} \qquad (4.27)$$

and let

$$n(N) = AN^{1/2}. \tag{4.28}$$

Now as $N \to \infty$,

$$\begin{aligned} R(\delta^*, n^*(N), N) &- R(\delta^*, n(N), N) \\ &= A_G\,\xi(N) + C_G N \left(\frac{1}{AN^{1/2} + \xi(N)} - \frac{1}{AN^{1/2}}\right)(1 + o(1)) \\ &= \frac{A_G\,\xi^2(N) + o(N^{1/2}\xi(N))}{AN^{1/2} + \xi(N)}, \end{aligned} \tag{4.29}$$

which is positive for arbitrarily large values of N unless $\xi(N) = o(N^{1/2})$. If this expression is positive then the risk using $n(N)$ is less than that using $n^*(N)$ which contradicts the assertion that $n^*(N)$ is the Bayes sample size. Hence $\xi(N) = o(N^{1/2})$.

THEOREM 4.2: *If* $F(x \mid \lambda)\ \varepsilon\ \mathfrak{F}_1$ *(with* $k = 1$*)* $G(\lambda)\ \varepsilon\ \mathcal{G}_2$ *and* A_G *is defined by* (4.3), *then the Bayes sample size is given by*

$$n^*(N) = \begin{cases} N, & A_G \leqq 0, \\ K \ln N - \dfrac{K}{2} \ln \ln N + O(1), & A_G > 0. \end{cases} \tag{4.30}$$

(The definition of K is lengthy and is contained in the proof.)

PROOF: We rewrite (4.4) as

$$\begin{aligned} R(\delta^*, n, N) &= N(r_1 E(\Lambda) + r_2) + n((s_1 - r_1)E(\Lambda) + s_2 - r_2) \\ &+ (N - n)(a_1 - r_1)\left\{\int_0^c (\lambda - c)\, dG(\lambda) + \int_0^c (c - \lambda)(1 - L(\lambda, n))\, dG(\lambda) \right. \\ &\left. + \int_c^\infty (\lambda - c) L(\lambda, n)\, dG(\lambda)\right\}. \end{aligned}$$

Now referring to (3.37) let $\tau(n) = [t(n)] = K_1 n + \varphi(n) = O(1)$ and

$$K_1 = \frac{\int_{l_G}^{U_G} u\omega'(u)\, du}{\omega(u_G) - \omega(l_G)} \qquad (\text{clearly } l_G < K_1 < u_G).$$

For $\lambda < K_1$ we may apply a well known result of asymptotic expansion theory (cf. [10]) to (2.15) to obtain

$$\begin{aligned} 1 - L(\lambda, n) &= (\tau(n) + 1) r^{(n)}(\tau(n) + 1)\left(\frac{\lambda}{\alpha + \beta\lambda}\right)^{\varphi(n)} \frac{1}{\alpha n(K_1 - \lambda)} \\ &\cdot \exp\left\{n\left(K_1 \ln \frac{\lambda}{\alpha + \beta\lambda} - \int_0^\lambda \frac{\alpha\, du}{\alpha + \beta u}\right)\right\}(1 + o(1)). \end{aligned} \tag{4.31}$$

Similarly, we obtain for $\lambda > K_1$

$$
\begin{aligned}
L(\lambda, n) &= (\tau(n) + 1)r^{(n)}(\tau(n) + 1)\left(\frac{\lambda}{\alpha + \beta\lambda}\right)^{\varphi(n)} \\
&\quad \cdot \frac{1}{\alpha n(\lambda - K_1)} \exp\left\{n\left(K_1 \ln\left(\frac{\lambda}{\alpha + \beta\lambda}\right) - \int_0^\lambda \frac{\alpha\, du}{\alpha + \beta u}\right)\right\}(1 + o(1)).
\end{aligned} \tag{4.32}
$$

Now, for $\lambda < K_1$, $K_1 \ln(\lambda/(\alpha + \beta\lambda)) - \int_0^\lambda \alpha\, du/(\alpha + \beta u)$ is increasing in λ, and for $\lambda > K_1$ it is decreasing. Hence we have $1 - L(\lambda, n) = o(1 - L(l_G, n))$ for $\lambda < l_G$, and $L(\lambda, n) = o(L(u_G, n))$ for $\lambda > u_G$ as $n \to \infty$. Let λ^* be whichever of l_G and u_G maximizes $K_1 \ln \lambda/(\alpha + \beta\lambda) - \int_0^\lambda \alpha\, du/(\alpha + \beta u)$, and let ζ^* be the weight assigned to that point by $G(\lambda)$. At this stage we discuss in detail the case where

$$
K_1 \ln \frac{l_G}{\alpha + \beta l_G} - \int_0^{l_G} \frac{\alpha\, du}{\alpha + \beta u} \neq K_1 \ln\left(\frac{u_G}{\alpha + \beta u_G}\right) - \int_0^{u_G} \frac{\alpha\, du}{\alpha + \beta u}. \tag{4.33}
$$

The case where equality holds can be treated in a similar manner and leads to the same Bayes sample size, as will be noted at the conclusion of the proof. By using expression (4.2) for $(\tau(n) + 1)r^{(n)}(\tau(n) + 1)$, we have

$$
\begin{aligned}
R(\delta^*, n, N) &= N(r_1 E(\Lambda) + r_2) + n((s_1 - r_1)E(\Lambda) + (s_2 - r_2)) \\
&\quad + (N - n)(a_1 - r_1)\Bigg\{\int_0^c (\lambda - c)\, dG(\lambda) \\
&\quad + \left[\frac{\lambda^*(\alpha + \beta K_1)}{K_1(\alpha + \beta\lambda^*)}\right]^{\varphi(n)} \frac{(\alpha + \beta K_1)\lambda^*\zeta^*(\lambda^* - c)}{(\lambda^* - K_1)\sqrt{2\pi\alpha K_1(\alpha + \beta K_1)n}} \\
&\quad \cdot \begin{cases} \exp\left\{n\left(K_1 \ln \dfrac{\lambda^*}{K_1} + \dfrac{\alpha + \beta K_1}{\beta} \ln\left(\dfrac{\alpha + \beta K_1}{\alpha + \beta\lambda^*}\right)\right)\right\}, & \beta \neq 0 \\ \exp\left\{n\left(K_1 \ln \dfrac{\lambda^*}{K_1} + K_1 - \lambda^*\right)\right\}, & \beta = 0 \end{cases} \\
&\quad \cdot (1 + o(1))\Bigg\}.
\end{aligned} \tag{4.34}
$$

Let

$$
\gamma(n) = \left(\frac{\lambda^*(\alpha + \beta K_1)}{K_1(\alpha + \beta\lambda^*)}\right)^{\varphi(n)} \left(\frac{(\alpha + \beta K_1)\lambda^*\zeta^*(\lambda^* - c)}{(\lambda^* - K_1)\sqrt{2\pi\alpha K_1(\alpha + \beta K_1)}}\right), \tag{4.35}
$$

$$
-\frac{1}{K} = \begin{cases} K_1 \ln \dfrac{\lambda^*}{K_1} + \dfrac{\alpha + \beta K_1}{\beta} \ln \dfrac{\alpha + \beta K_1}{\alpha + \beta\lambda^*}, & \beta \neq 0, \\ K_1 \ln \dfrac{\lambda^*}{K_1} + K_1 - \lambda^*, & \beta = 0, \end{cases} \tag{4.36}
$$

and note that $K > 0$. We then have, from (4.34)

$$
\begin{aligned}
R(\delta^*, n, N) &= N\left(r_1 E(\Lambda) + r_2 + (a_1 - s_1)\int_0^c (\lambda - c)\, dG(\lambda)\right) \\
&\quad + n\left((s_1 - r_1)E(\Lambda) + s_2 - r_2 + (r_1 - a_1)\int_0^c (\lambda - c)\, dG(\lambda)\right) \qquad (4.37) \\
&\quad + (a_1 - s_1)(N - n)\frac{\gamma(n)}{\sqrt{n}} \exp\left\{-\frac{n}{K}\right\}(1 + o(1))
\end{aligned}
$$

as $n \to \infty$. Now, by the argument given in the proof of Theorem 4.1, $n^*(N) \to \infty$ as $N \to \infty$. As in Theorem 4.1 we have two cases, (i) $A_G \leqq 0$ and (ii) $A_G > 0$. Case (i) is treated exactly as before. Case (ii), however, requires a slightly different argument as follows: Let $n(N) = K \ln N - (K/2) \ln \ln N$ and write $n^*(N) = n(N) + \xi(N)$. Noting that

$$
\frac{N - n(N)}{\sqrt{n(N)}} \exp\left\{-\frac{1}{K} n(N)\right\} \to \frac{1}{\sqrt{K}} \quad \text{as } N \to \infty, \qquad (4.38)
$$

we have

$$
\begin{aligned}
&R(\delta^*, n^*(N), N) - R(\delta^*, n(N), N) \\
&= A_G \xi(N) + (a_1 - s_1)\left(\frac{N - n(N) - \xi(N)}{\sqrt{n(N) + \xi(N)}} \gamma(n(N) + \xi(N))\right. \qquad (4.39) \\
&\quad \left.\cdot \exp\left\{-\frac{1}{K}(n(N) + \xi(N))\right\}(1 + o(1)) + O(1)\right).
\end{aligned}
$$

If $\xi(N)$ or any subsequence $\to +\infty$, the exponential term is bounded and the linear term becomes infinite. If $\xi(N) \to -\infty$, then

$$
\begin{aligned}
&\frac{N - n(N) - \xi(N)}{\sqrt{n(N) + \xi(N)}} \exp\left\{-\frac{1}{K}(n(N) + \xi(N))\right\} \qquad (4.40) \\
&\qquad \geqq \frac{N - n(N)}{\sqrt{n(N)}} \exp\left\{-\frac{1}{K} n(N)\right\} \exp\left\{-\frac{1}{K}\xi(N)\right\}.
\end{aligned}
$$

Recalling (4.38), and noting that $\gamma(n)$ is positive and bounded away from zero, we see that the exponential term of (4.39) becomes infinite and dominates the linear term. Therefore (4.39) is positive if $\xi(N)$ or any subsequence becomes infinite, hence $\xi(N) = O(1)$.

If equality holds in (4.33), we observe that letting

$$
\begin{aligned}
\gamma(n) &= \left(\frac{l_G(\alpha + \beta K_1)}{K_1(\alpha + \beta l_G)}\right)^{\varphi(n)} \left(\frac{(\alpha + \beta K_1) l_G \zeta_1(c - l_G)}{(K_1 - l_G)\sqrt{2\pi\alpha K_1(\alpha + \beta K_1)}}\right) \qquad (4.41) \\
&\quad + \left(\frac{u_G(\alpha + \beta K_1)}{K_1(\alpha + \beta u_G)}\right)^{\varphi(n)} \left(\frac{(\alpha + \beta K_1) u_G \zeta_2(u_G - c)}{(u_G - K_1)\sqrt{2\pi\alpha K_1(\alpha + \beta K_1)}}\right)
\end{aligned}
$$

leads to (4.37), and hence to (4.30).

THEOREM 4.3: *If $F(x \mid \lambda) \varepsilon \mathfrak{F}_2$ (with $\eta = 1$), $G(\lambda) \varepsilon \mathcal{G}_1$ and A_G is defined by (4.3), then the Bayes sample size is given by*

$$(4.42) \qquad n^*(N) = \begin{cases} N, & A_G \leqq 0, \\ N^{1/2}\left(\dfrac{(r_2 - a_2)cG'(c)}{2A_G}\right)^{1/2} + o(N^{1/2}), & A_G > 0. \end{cases}$$

PROOF: The method employed here is similar to that used in the proof of Theorem 4.2. By Theorem 2.6

$$(4.43) \qquad 1 - L(\lambda, n) = \frac{(cn + \psi(n))^n}{\Gamma(n)} \int_0^\lambda \frac{1}{t} \exp\left\{nh(t) - \frac{\psi(n)}{t}\right\} dt$$

where $h(t) = \ln(1/t) - (c/t)$. We note that this $h(t)$ satisfies the conditions of the lemmas on asymptotic expansions and a simple calculation shows

$$(4.44) \qquad \begin{aligned} R(\delta^*, n, N) &= n\left((s_1 - r_1)E(\Lambda) + s_2 - r_2 + (r_1 - a_1)\int_0^c (\lambda - c)dG(\lambda)\right) \\ &\quad + N\left(r_1 E(\Lambda) + r_2 + (a_1 - r_1)\int_0^c (\lambda - c)\, dG(\lambda)\right) \\ &\quad + (a_1 - r_1)(N - n)\left(\frac{c^2G'(c)}{2n} + o\left(\frac{1}{n}\right)\right). \end{aligned}$$

This now may be written as in (4.25) and exactly the same argument proves the theorem.

THEOREM 4.4: *If $F(x \mid \lambda) \varepsilon \mathfrak{F}_2$ (with $\eta = 1$), $G(\lambda) \varepsilon \mathcal{G}_2$ and A_G is defined by (4.3), then*

$$(4.45) \qquad n^*(N) = \begin{cases} N, & A_G \leqq 0, \\ K \ln N - \dfrac{K}{2} \ln \ln N + O(1), & A_G > 0. \end{cases}$$

PROOF: We define λ^* as being whichever of l_G and u_G maximizes $-(K_1/\lambda) + \ln(1/\lambda)$ and let

$$(4.46) \qquad -\frac{1}{K} = \ln \frac{K_1}{\lambda^*} - \left(\frac{K_1}{\lambda^*} - 1\right).$$

The remainder of the proof parallels that of Theorem 4.2.

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Added in proof: The following two articles survey other publications concerned with the economic aspects of acceptance sampling.

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