5. Conclusion of the proof. To complete the demonstration of the theorem, write

$$\phi(X_n) - E(\phi(X_n) \mid X_{n-k}) = Z_n^{(1)} + Z_n^{(2)} + \cdots + Z_n^{(k)}, \qquad n > k.$$

Thus, by proposition 2,

$$\lim_N \left| \frac{1}{N} \sum_{n=1}^{N} \phi(X_n) - \frac{1}{N} \sum_{n=k+1}^{N} E(\phi(X_n) \mid X_{n-k}) \right| = 0, \quad \text{a.s. } P_x.$$

Or, neglecting at most k terms,

$$\lim_N \left| \frac{1}{N} \sum_{n=1}^{N} \phi(X_n) - \frac{1}{N} \sum_{n=1}^{N} E(\phi(X_{n+k}) \mid X_n) \right| = 0, \quad \text{a.s. } P_x,$$

so that, for fixed M,

$$\lim_N \left| \frac{1}{N} \sum_{n=1}^{N} \phi(X_n) - \frac{1}{N} \sum_{k=1}^{N} \left[\frac{1}{M} \sum_{k=1}^{M} E(\phi(X_{n+k}) \mid X_n) \right] \right| = 0, \quad \text{a.s. } P_x.$$

By proposition 1, for any $\epsilon > 0$, we may choose M such that

$$\left| \frac{1}{M} \sum_{k=1}^{M} E(\phi(X_{n+k}) \mid X_n) - E_\pi \phi(X_1) \right| \leqq \epsilon,$$

and for such an M we have

$$\lim_N \left| \frac{1}{N} \sum_{n=1}^{N} \phi(X_n) - E_\pi \phi(X_1) \right| \leqq \epsilon \quad \text{a.s. } P_x)$$

proving the theorem.

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EMPTINESS IN THE FINITE DAM

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1. Summary: The paper discusses the general problem of emptiness in the finite dam and considers the probability that, starting with an arbitrary storage, the dam dries up before it fills completely. Some exact results are given both for discrete and continuous inputs. An interesting relation between this probability and the asymptotic distribution function of the dam content has also been obtained.

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2. Introduction: This paper is based on the storage system model given by Moran [5]. The storage, Z_t, of a dam of finite capacity, k, is defined for discrete time t ($t = 0, 1, 2, \cdots$) as the dam content just after an instantaneous release at time t, and just before an input, X_t, flows into it over the time interval $(t, t + 1)$. The model is subject to the conditions

(i) the inputs X_t during the intervals $(t, t + 1)$ are independently and identically distributed;

(ii) there is an overflow, max $(Z_t + X_t - k, 0)$, during the interval $(t, t + 1)$, while min $(k, Z_t + X_t)$ is left in the dam just before the release occurs;

(iii) the amount of water released at time $t + 1$ is min $(m, Z_t + X_t)$, where m is a constant $< k$.

It has been shown that the processes (Z_t) and $(Z_t + X_t)$ are both Markov chains, and the problem of obtaining their stationary distributions has been dealt with by Moran [5], [6], Gani [2], Gani and Prabhu [3] and Prabhu [7], [8].

This paper deals with the problem of finding the probability that, given an arbitrary initial storage and the distribution of the input (X_t), the dam dries up before it fills completely. It also shows that this probability bears an elegant relationship with the asymptotic distribution function of the dam content. D. G. Kendall [4] derived the time required by an infinite dam to dry up; Prabhu [8] dealt with the probability of emptiness at a given storage level for the finite dam, but for $m = 1$. Here, the problem for any m is dealt with.

3. The Probability of Emptiness–Discrete Input: If the release rules given in Section 2 operate, we have

$$Z_{t+1} = \begin{cases} Z_t + X_t - m & \text{if } m < Z_t + X_t < k, \\ 0 & \text{if } Z_t + X_t \leqq m, \\ k - m & \text{if } Z_t + X_t \geqq k. \end{cases} \tag{1}$$

Let $\{g_j\}$ be the probability distribution of X_t, so that

$$\Pr \{X_t = j\} = g_j, \qquad (j = 0, 1, \cdots). \tag{2}$$

Let V_i be the conditional probability that, starting with storage i, the dam becomes empty before it fills completely. It is easy to derive the following:

$$V_i = \begin{cases} \sum_{j=0}^{m-i} g_j + \sum_{j=1}^{k-m-1} g_{j+m-1} V_j & (i \leqq m), \\ \sum_{j=i-m}^{k-m-1} g_{j+m-1} V_j & (m < i \leqq k - m - 1). \end{cases} \tag{3}$$

We note that the states 0 and $k - m$ are absorbing, so that $V_i = 1$ for $i \leqq 0$, $V_{k-m+r} = 0$ for $r \geqq 0$.

3.1. *Geometric Input*: Consider an input distribution of the geometric type,

$$g_j = ab^j \qquad (b = 1 - a, j = 0, 1, \cdots). \tag{4}$$

Applying the transformation $V_i = 1 - b^{-i}\phi_i$ to (3), after substituting (4) in (3), we get

$$(5)\qquad \phi_i = \begin{cases} \alpha & (i \leqq m), \\ \alpha - \lambda \sum_{j=1}^{i-m-1} \phi_j & (i > m), \end{cases}$$

where

$$(6)\qquad \lambda = ab^m,$$
$$\alpha = b^k + \lambda \sum_{j=1}^{k-m-1} \phi_j .$$

We solve for ϕ_i successively for the ranges $(m, 2m)$, $(2m, 3m)$, $\cdots$ in terms of the unknown constant α. For instance, for $m < i \leqq 2m$, we get

$$\phi_i = \alpha[1 - \lambda(i - m - 1)].$$

Let $k = (N + 1)m + U$, where $0 \leqq U < m$. We get the general expression

$$(7)\qquad \phi_i = \alpha \sum_{q=0}^{n} (-\lambda)^q \binom{i - qm - 1}{q}, \qquad (nm < i \leqq (n + 1)m, \; n = 0, 1, \cdots N + 1).$$

We solve for α from (6) and (7):

$$(8)\qquad \alpha = b^k \Big/ \left[\sum_{q=0}^{N+1} (-\lambda)^q \binom{k - qm - 1}{q} \right].$$

From (7) we have

$$(9)\qquad V_i = 1 - \alpha b^{-i} \sum_{q=0}^{n} (-\lambda)^q \binom{i - qm - 1}{q} \qquad (nm < i \leqq (n + 1)m; \; n = 0, 1, \cdots N).$$

In many cases, it may be enough to know the bounds within which V_i should lie, and these bounds are given by Feller ([1], inequalities 8.11, 8.12 on p. 303). Prabhu [8] has obtained the bounds for $m = 1$. For general m, if we put $E(U_t) = E(X_t - m) = \rho - m$, where ρ is the mean input, we have

$$(10)\qquad \begin{aligned} &(Z_0^{k-1} - Z_0^{m+i-1})/(Z_0^{k-1} - 1) \leqq V_i \leqq 1 && (\rho < m), \\ &(Z_0^{m+i-1} - Z_0^{k-1})/(1 - Z_0^{k-1}) \leqq V_i \leqq Z_0^i && (\rho > m), \\ &1 - (m + i - 1)/(k - 1) \leqq V_i \leqq 1 && (\rho = m), \end{aligned}$$

where Z_0 is the unique positive root (other than unity) of the equation $\sum_j Z^j \Pr(U_t = j) = 1$, i.e. $\sum_{j=0}^{\infty} Z^j g_j = Z^m$, and $Z_0 \gtrless 1$ according as $\rho \gtrless m$.

4. Continuous Input: It would be instructive to study the continuous analogue of (3). If $V(y)$ is the continuous analogue of V_i, the equations (3) become

$$(11)\qquad V(y) = \begin{cases} G(m - y) + \int_0^{k-m} V(t)\, dG(t + m - y) & (0 < y \leqq m), \\ \int_{y-m}^{k-m} V(t)\, dG(t + m - y) & (m \leqq y < k - m), \end{cases}$$

where $G(x) = \Pr(X_t \leqq x)$.

4.1. *Exponential Input*: Consider an exponential input of the type

$$dG(x) = \mu e^{-\mu x}\, dx, \qquad (0 < x < \infty; \mu > 0). \tag{12}$$

By applying the transformation $V(t) = 1 - e^{\mu t}\phi(t)$ and substituting (12) in (11), we get

$$\phi(y) = \begin{cases} \alpha & (y \leqq m), \\ \alpha - \lambda \int_0^{y-m} \phi(t)\, dt & (y > m), \end{cases} \tag{13}$$

where

$$\begin{aligned} \lambda &= \mu e^{-\mu m} \\ \alpha &= e^{-\mu k} + \lambda \int_0^{k-m} \phi(t)\, dt. \end{aligned} \tag{14}$$

Suppose $k = (N + 1)m + U$, where $0 \leqq U < m$. We can solve for $\phi(y)$ successively for the ranges $(m, 2m)$, $(2m, 3m)$, etc.

For $nm < y \leqq (n + 1)m$, we have

$$\phi(y) = \alpha \sum_{q=0}^{n} (-\lambda)^q \frac{(y - qm)^q}{q!} \qquad (n = 0, 1, \cdots N + 1). \tag{15}$$

α is determined as follows:

$$\begin{aligned} \alpha &= e^{-\mu k} + \lambda \int_0^{k-m} \phi(t)\, dt \\ &= e^{\mu k} + \lambda \left[\int_0^{m} \phi(t)\, dt + \cdots + \int_{Nm}^{k-m} \phi(t)\, dt \right], \end{aligned}$$

so that

$$\alpha = e^{-\mu k} \Big/ \sum_{q=0}^{N+1} \frac{(-\lambda)^q (k - qm)^q}{q!}. \tag{16}$$

Thus, we have

$$V(y) = \begin{cases} 1 - e^{\mu y}\alpha & (y \leqq m), \\ 1 - e^{\mu y}\alpha \sum_{q=0}^{n} (-\lambda)^q \frac{(y - qm)^q}{q!} & (nm < y \leqq (n + 1)m;\ n = 0, 1, \cdots N). \end{cases} \tag{17}$$

We have the boundary conditions: $V(0) = 1$, $V(k - m + r) = 0$ for $r \geqq 0$. From (17) we find $V(+0) = 1 - \alpha$ and $V(k - m - 0) > 0$, indicating that there are points of discontinuities at $y = 0$ and $y = k - m$.

4.2. *Gamma Input*: Consider a gamma input

$$dG(x) = (\mu^p/(p - 1)!)e^{-\mu x}x^{p-1}\, dx \quad (0 < x < \infty; \mu > 0; p = 1, 2, \cdots). \tag{18}$$

Again, applying the transformation $V(t) = 1 - e^{\mu t}\phi(t)$ to (11), we get

$$(19) \qquad \phi(y) = \begin{cases} \sum_{\gamma=0}^{p-1} \frac{\alpha_\gamma y^\gamma}{\gamma!} & (y \leqq m), \\ \sum_{\gamma=0}^{p-1} \frac{\alpha_\gamma y^\gamma}{\gamma!} - \lambda \int_0^{y-m} \phi(t) \frac{(y-m-t)^{p-1}}{(p-1)!} dt & (y > m), \end{cases}$$

where

$$(20) \qquad \lambda = (-1)^{p-1}\mu^p e^{-\mu m},$$

$$(21) \qquad \alpha_\gamma = e^{-\mu k}(-\mu)^\gamma \sum_{s=0}^{p-\gamma-1} \frac{(\mu k)^s}{s!} + \mu^p e^{-\mu m}(-1)^\gamma \int_0^{k-m} \phi(t) \frac{(t+m)^{p-\gamma-1}}{(p-\gamma-1)!} dt \qquad (\gamma = 0, 1, \cdots p-1).$$

We get

$$(22) \quad \phi(y) = \sum_{\gamma=0}^{p-1} \alpha_\gamma \sum_{q=0}^{n} (-\lambda)^q \frac{(y-qm)^{qp+\gamma}}{(qp+\gamma)!} \quad (nm < y \leqq (n+1)m; \; n = 0, 1, \cdots N+1),$$

where $k = (N+1)m + U$, as in Section 4.1.

Prabhu [7] obtained (22) while deriving the distribution of dam storage. The α_γ's can be obtained by his method ([7], eqn. (13)).

Finally, we have

$$(23) \quad V(y) = \begin{cases} 1 - e^{\mu y} \sum_{\gamma=0}^{p-1} \frac{\alpha_\gamma y^\gamma}{\gamma!} & (y \leqq m), \\ 1 - e^{\mu y} \sum_{\gamma=0}^{p-1} \alpha_\gamma \sum_{q=0}^{n} (-\lambda)^q \frac{(y-qm)^{qp+\gamma}}{(qp+\gamma)!} & (nm < y \leqq (n+1)m, n \geqq 0). \end{cases}$$

$V(y)$ has two points of discontinuity at $y = 0$, $y = k - m$ since the boundary conditions are $V(0) = 1$, $V(k - m + r) = 0$ for $r \geqq 0$.

5. Relationship with the Asymptotic Distribution of Dam Content: If $H(y)$ is the stationary c.d.f. of the dam content $Z_t + X_t$ we get the following integral equation ([7], eqn. (2)) for continuous input:

$$(24) \qquad H(y) = \begin{cases} -\int_m^{m+y} H(t)\, dG(m+y-t) & (y < k-m), \\ G(y-k+m) - \int_m^k H(t)\, dG(m+y-t) & (y \geqq k-m). \end{cases}$$

By applying the transformation $H(k - y) = 1 - e^{\mu y}\phi(y)$ to the above, for exponential and gamma inputs, (12) and (18), we obtain the same integral equations in $\phi(y)$, (13) and (19), as were obtained by applying the transformation $V(y) = 1 - e^{\mu y}\phi(y)$ to the integral equations for $V(y)$. We, therefore, obtain

$$(25) \qquad V(y) = H(k-y).$$

We may verify that (25) holds good for discrete input also.

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