ERGODOCITY OF QUEUES IN SERIES[1]

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1. Introduction. We are interested here in determining when a queueing system consisting of several queues in series is ergodic. To define what is meant by queues in series let us consider the case where there are two servers. The definition of two queues in series is given as follows: The nth individual arriving to the queueing system enters, at his time of arrival, a queue (queue 1) in front of the first server. He waits there until all the individuals in front of him have been served in the first server at which time he begins his service. Upon completion of his service the nth individual enters a queue (queue 2) in front of the second server, waits there until all the individuals in front of him have completed their service in the second server, and at that time he begins his own service. Queue 1 and queue 2 are now said to be **in series.** Putting matters more concisely we can say that two queues are in series if the output of the first queue is the input of the second queue.

To define what we mean by the ergodicity of this queueing system, let W_n be the waiting time in queue 1 of the nth individual and let W_n^* denote his waiting time in queue 2. The queueing system is said to be ergodic if the joint distribution of (W_n, W_n^*) converges, as $n \to \infty$, to a probability distribution. Assuming existence of first moments for the two service time random variables and the interarrival random variable (the sequence of interarrival time random variables is assumed to be a sequence of independent and identically distributed random variables and the same is assumed for each of the two sequences of service time random variables) we are able, in Theorems 1 and 2 below, to characterize when $\{(W_n, W_n^*)\}$ has a limiting probability distribution.

The method we use to characterize ergodicity is first to show (Lemma 2 below) that the distribution function of (W_n, W_n^*) converges to a limit as $n \to \infty$ though the limit may not be a probability distribution function. This is the easy part of the argument. The second part of the argument is to show under appropriate conditions (see Theorem 1) that (W_n, W_n^*) is bounded in probability so that as $n \to \infty$ no probability escapes to infinity and this yields the fact that the limit shown to exist in the first part is a bonafide probability distribution. The last part of the characterization lies in showing that when the conditions for Theorem 1 are not satisfied then either W_n or W_n^* goes to $+\infty$ in probability (Theorem 2). This outline of the argument is quite the same as the outline of the argument used by Kiefer and Wolfowitz [3] for the queueing system they consider. The details of the first part are strongly related to those in [3]. The

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means whereby the second and third parts of the argument are accomplished depends on knowledge of the behavior of the maximum of partial sums of independent and identically distributed random variables. This was first utilized by Lindley [4] in his treatment of the one server queueing system.

Burke [2], Reich [5] and others (see [2] and [5] for other references) have considered queues in series when the service time random variables and interarrival random variables are exponentially distributed. Akaike [1] considers a problem of ergodic behavior of queues in series related to the one we treat here. Akaike assumes that all the random variables in sight take on values which are integral multiples of some fixed positive number so that the waiting time process is a discrete process (our random variables have no such restriction). Furthermore he assumes that the nth customer cannot enter the second queue before customer $n + 1$ arrives at the first queue; thus if customer n finishes service at server 1 before $n + 1$ arrives, customer n must wait until customer $n + 1$ arrives before entering the second queue. Our assumptions are that the nth customer enters the second queue immediately after finishing service at server 1.

We have only talked about the case of two servers which gives rise to two queues in series. It is simple to see how to define s queues in series when there are s servers—this giving rise to s different waiting times to worry about. All our previous remarks for the two-server case are valid for the s-server case. We have separated the treatment of the two-server case from that of the s-server case in order to avoid confusing notational problems with the principal ideas.

2. The Two-Server Case. In this section we will consider the case where there are two queues in series, the output of the first queue being the input to the second queue.

Let τ_n be the time at which the nth individual enters the system. Let R_n denote the service time in the first server of individual n and let ρ_n be the service time of individual n in the second server. Let $g_{n+1} = \tau_{n+1} - \tau_n$. We assume that each of the three sequences $\{R_n ; n \geqq 1\}$, $\{g_n ; n \geqq 2\}$, $\{\rho_n ; n \geqq 1\}$ is a sequence of independent and identically distributed random variables and that the three sequences are mutually independent. Furthermore we assume that the R_n's, g_n's, and ρ_n's are non-negative random variables and that $ER_1 < \infty$, $Eg_2 < \infty$, $E\rho_1 < \infty$.

Let W_n be the waiting time in the first queue of the nth person and let W_n^* be the waiting time in the second queue of the nth person. The waiting time, of course, is the time between arrival at the queue and the beginning of service. To establish a relationship between (W_n, W_n^*) and (W_{n+1}, W_{n+1}^*) observe that the nth individual leaves the first server (enters the second queue) at time $\tau_n + W_n + R_n$ and leaves the second server at time $\tau_n + W_n + R_n + W_n^* + \rho_n$, while the $(n + 1)$th individual arrives at the first queue at time τ_{n+1} and at the second queue at time $\tau_{n+1} + W_{n+1} + R_{n+1}$. Thus the $(n + 1)$th person waits 0 time in the first queue if $\tau_n + W_n + R_n \leqq \tau_{n+1}$ i.e., if $W_n + R_n - g_{n+1} \leqq 0$, and waits $W_n + R_n - g_{n+1}$ if the last quantity is positive. Stated more concisely

$$W_{n+1} = \max [0, W_n + R_n - g_{n+1}]. \tag{2.1}$$

Similarly,

$$W^*_{n+1} = \max [0, W^*_n + \rho_n - R_{n+1} + R_n - g_{n+1} + W_n - W_{n+1}]. \tag{2.2}$$

(2.1) and (2.2) are valid for all $n \geqq 1$ with $W_1 = W_1^* = 0$.

Let $Z_n = (W_n, W_n^*)$. $\{Z_n\}$ is not a Markov process but putting $Y_n = (Z_n, R_n)$ provides us with a sequence $\{Y_n\}$ which is a Markov process with stationary transition probabilities. These considerations will enable us to prove Lemma 2 below which is the first step in characterizing when $\{Z_n\}$ is an ergodic process.

Let $t = (t_1, t_2)$, $x = (x_1, x_2)$ with t_1, t_2, x_1, x_2 all nonnegative numbers.

LEMMA 1: *$P\{Z_n \leqq t \mid Z_1 = x, R_1 = r\} \leqq P\{Z_n \leqq t \mid Z_1 = 0, R_1 = r\}$ for all n, x, t, r.*

PROOF: Fix a point ω in the sample space of $R_2, \cdots, R_n, g_2, \cdots, g_n, \rho_1, \cdots \rho_{n-1}$, and let

$$W_1(\omega, x) = x_1, \qquad W_1^*(\omega, x) = x_2$$

$$W_j(\omega, x) = \max [0, W_{j-1}(\omega, x) + R_{j-1}(\omega) - g_j(\omega)]$$

$$W_j^*(\omega, x) = \max [0, W_{j-1}^*(\omega, x) + \rho_{j-1}(\omega) - R_j(\omega) + R_{j-1}(\omega) - g_j(\omega) + W_{j-1}(\omega, x) - W_j(\omega, x)],$$

for $2 \leqq j \leqq n$. It is clear that $W_j(\omega, 0) \leqq W_j(\omega, x)$ for each j. Observing that

$$\begin{aligned} R_{j-1}(\omega) + W_{j-1}(\omega, x) - W_j(\omega, x) &= R_{j-1}(\omega) + W_{j-1}(\omega, x) - \max [0, W_{j-1}(\omega, x) + R_{j-1}(\omega) - g_j(\omega)] \\ &= \min [W_{j-1}(\omega, x) + R_{j-1}(\omega), g_j(\omega)], \end{aligned}$$

we have

$$W_j^*(\omega, x) = \max [0, W_{j-1}^*(\omega, x) + \rho_{j-1}(\omega) - R_j(\omega) - g_j(\omega) + \min [W_{j-1}(\omega, x) + R_{j-1}(\omega), g_j(\omega)]]$$

and it follows easily that $W_j^*(\omega, 0) \leqq W_j^*(\omega, x)$ for all $2 \leqq j \leqq n$. Lemma 1 is now seen to be true.

LEMMA 2: *$P\{Z_n \leqq t \mid Z_1 = 0\} \to F(t)$ as $n \to \infty$ where F is a two-dimensional distribution function whose variation over two-dimensional space may be less than one i.e., F may not be a probability distribution function.*

PROOF: Let $H(x, r) = P\{Z_2 \leqq x, R_2 \leqq r \mid Z_1 = 0\}$. Then

$$\begin{aligned} &P\{Z_{n+1} \leqq t \mid Z_1 = 0\} \\ &\quad = \int P\{Z_{n+1} \leqq t \mid Z_2 = x, R_2 = r, Z_1 = 0\}\, dH(x, r). \end{aligned} \tag{2.3}$$

Since $\{Y_n\}$ $(Y_n = (Z_n, R_n))$ is a stationary Markov process and because of

Lemma 1

$$(2.4)\quad \begin{aligned} &P\{Z_{n+1} \leqq t \mid Z_2 = x, R_2 = r, Z_1 = 0\} \\ &\quad = P\{Z_n \leqq t \mid Z_1 = x, R_1 = r\} \leqq P\{Z_n \leqq t \mid Z_1 = 0, R_1 = r\}. \end{aligned}$$

Let H^* be the distribution function of R_1 and, therefore, of R_2. Then, using (2.4) in (2.3), we have

$$\begin{aligned} P\{Z_{n+1} \leqq t \mid Z_1 = 0\} &\leqq \int P\{Z_n \leqq t \mid Z_1 = 0, R_1 = r\}\, dH(x, r) \\ &= \int P\{Z_n \leqq t \mid Z_1 = 0, R_1 = r\}\, dH^*(r) = P\{Z_n \leqq t \mid Z_1 = 0\}. \end{aligned}$$

Thus $P\{Z_n \leqq t \mid Z_1 = 0\}$ is a monotone sequence and therefore converges to a limit which we call $F(t)$. The above-mentioned properties of F are easily deduced.

THEOREM 1: *If* $Eg_2 > \max(ER_1, E\rho_1)$ *then* F *(defined in Lemma 2) is a probability distribution.*

PROOF: Because of Lemma 2 we need only show that, under the conditions stated here, $\{Z_n\}$ is bounded in probability, i.e., for all n

$$(2.5)\qquad P\{Z_n \leqq t \mid Z_1 = 0\} \geqq 1 - \eta(t)$$

where $\eta(t) \to 0$ as $t \to \infty$. If we can prove that $\{W_n\}$ and $\{W_n^*\}$ are each bounded in probability then (2.5) will be established. Lindley [4] has shown that $\{W_n\}$ is bounded in probability so it remains only to consider $\{W_n^*\}$.

For $j = 1, 2, \cdots$ let

$$(2.6)\qquad S_j = \sum_{i=1}^{j} (R_i - g_{i+1})$$

and let $S_0 = 0$. By iterating (2.1) and using the fact that $W_1 = 0$ we have

$$(2.7)\qquad W_{n+1} = \max_{0 \leqq j \leqq n} [S_n - S_j].$$

Hence

$$(2.8)\quad R_n - g_{n+1} + W_n = R_n - g_{n+1} + \max_{0 \leqq j \leqq n-1} [S_{n-1} - S_j] = \max_{0 \leqq j \leqq n-1} [S_n - S_j].$$

For $k \geqq 0$, let

$$(2.9)\qquad B_k = \max_{0 \leqq j \leqq k} (-S_j).$$

Then (2.7), (2.8), and (2.9) yield

$$(2.10)\qquad R_n - g_{n+1} + W_n - W_{n+1} = B_{n-1} - B_n.$$

Using (2.10) in (2.2) gives

$$(2.11)\qquad W_{n+1}^* = \max[0, W_n^* + \rho_n - R_{n+1} + B_{n-1} - B_n].$$

Put

$$T_k = \sum_{i=1}^{k} (\rho_i - R_{i+1}) \quad \text{for} \quad k \geqq 1 \quad \text{and} \quad T_0 = 0. \tag{2.12}$$

Iterating (2.11) (use $W_1^* = 0$) we have

$$\begin{aligned} W_{n+1}^* &= \max_{0 \leqq k \leqq n} [T_n - T_k + B_k - B_n] \\ &= \max_{0 \leqq k \leqq n} [T_n - T_k + \max_{0 \leqq j \leqq k} (-S_j) - B_n] \\ &= \max_{0 \leqq j \leqq k \leqq n} [T_n - T_k - S_j - B_n]. \end{aligned} \tag{2.13}$$

Let $\epsilon \geqq 0$ (we shall specify ϵ later). Then

$$\begin{aligned} W_{n+1}^* &= \max_{0 \leqq j \leqq k \leqq n} [T_n - T_k - (n-k)\epsilon + (n-k)\epsilon - S_j - B_n] \\ &\leqq \max_{0 \leqq j \leqq k \leqq n} [T_n - T_k - (n-k)\epsilon] + \max_{0 \leqq j \leqq k \leqq n} [(n-k)\epsilon - S_j - B_n] \\ &= \max_{0 \leqq k \leqq n} [T_n - T_k - (n-k)\epsilon] + \max_{0 \leqq j \leqq n} [(n-j)\epsilon - S_j - B_n]. \end{aligned} \tag{2.14}$$

Define $\xi_i = \rho_i - R_{i+1} - \epsilon$ and let $U_k = \sum_{i=1}^{k} \xi_i$. Thus U_k is the kth partial sum of independent and identically distributed random variables. Then

$$\max_{0 \leqq k \leqq n} [T_n - T_k - (n-k)\epsilon] = \max_{0 \leqq k \leqq n} (U_n - U_k) = A_n \quad \text{(say)} \tag{2.15}$$

has the same distribution as $\max_{0 \leqq k \leqq n} U_k$. If ϵ is such that

$$E\rho_1 - ER_1 - \epsilon < 0 \tag{2.16}$$

then, it is well known, $\max_{0 \leqq k \leqq n} U_k \to$ a finite random variable with probability one (w.p.1) which implies that $\{A_n\}$ is bounded in probability.

Observe that $B_n = \max_{0 \leqq j \leqq n} (-S_j) \geqq -S_n$. Hence

$$\max_{0 \leqq j \leqq n} [(n-j)\epsilon - S_j - B_n] \leqq \max_{0 \leqq j \leqq n} [S_n - S_j + (n-j)\epsilon] = C_n \quad \text{(say)}. \tag{2.17}$$

Let $V_j = \sum_{i=1}^{j} (R_i - g_{i+1} + \epsilon)$. Then V_j is the jth partial sum of independent and identically distributed random variables. Thus, as before, C_n has the same distribution as $\max_{0 \leqq j \leqq n} V_j$ and, if ϵ is such that

$$ER_1 - Eg_2 + \epsilon < 0, \tag{2.18}$$

then $\{C_n\}$ is bounded in probability. Since $W_{n+1}^* < A_n + C_n$ we have only to verify that ϵ can be chosen to satisfy (2.16) and (2.18) in order to conclude that $\{W_n^*\}$ is bounded in probability.

If $E\rho_1 < ER_1$ then the choice $\epsilon = 0$ gives (2.16) and the condition of the Theorem guarantees (2.18). If $E\rho_1 \geqq ER_1$ take

$$\epsilon = [(E\rho_1 + Eg_2)/2] - ER_1$$

ϵ is clearly positive and (2.16) and (2.18) are satisfied because $Eg_2 > E\rho_1$. This concludes the proof of Theorem 1.

Theorem 2 which we now prove shows the necessity of the condition of Theorem 1 when first moments are assumed to exist.

THEOREM 2: (a) *If* $ER_1 \geqq Eg_2$ *then* $F(t) \equiv 0$.
(b) *If* $E\rho_1 \geqq Eg_2$ *then* $F(t) \equiv 0$.

PROOF: (a) is due to Lindley [4] who proved that, in this case, $W_n \to +\infty$ in probability. For (b) we might as well assume in addition that $ER_1 < Eg_2$ otherwise we can use (a).

If $ER_1 < Eg_2$ then

$$-S_n - B_n = -\max_{0 \leqq j \leqq n} [S_n - S_j] \tag{2.19}$$

is bounded in probability. From (2.13)

$$\begin{aligned} W^*_{n+1} &= \max_{0 \leqq j \leqq k \leqq n} [T_n - T_k - S_j - B_n] \\ &= \max_{0 \leqq j \leqq k \leqq n} [T_n + S_n - T_k - S_j] - S_n - B_n . \end{aligned} \tag{2.20}$$

Now, recalling that $R_1 > 0$ w.p.1,

$$\begin{aligned} \max_{0 \leqq j \leqq k \leqq n} [T_n + S_n - T_k - S_j] &\geqq \max_{0 \leqq k \leqq n} [T_n + S_n - T_k - S_k] \\ &= \max_{0 \leqq k \leqq n} \left[\sum_{i=k+1}^{n} (\rho_i - g_{i+1}) + R_{k+1} - R_{n+1} \right. \\ &\geqq \max_{0 \leqq k \leqq n} \left[\sum_{i=k+1}^{n} (\rho_i - g_{i+1}) \right] - R_{n+1} \end{aligned} \tag{2.21}$$

$\{R_{n+1}\}$ is, of course, bounded in probability but, because $E\rho_1 - Eg_2 \geqq 0$,

$$\max_{0 \leqq k \leqq n} \left[\sum_{i=k+1}^{n} (\rho_i - g_{i+1}) \right] \to +\infty \quad \text{in probability.} \tag{2.22}$$

(2.22), (2.21) and (2.19) show that $W^*_{n+1} \to +\infty$ in probability which proves that $F(t) \equiv 0$. This proves Theorem 2.

It is interesting to note that if $ER_1 \geqq Eg_2$ and $E\rho_1 < ER_1$ then, although $W_n \to +\infty$ in probability, W^*_n has a legitimate limiting distribution. This is because we can show (as in Lemma 2) that $P\{W^*_n \leqq t_1 \mid Z_1 = 0\}$ has a limit and because of (2.13)

$$\begin{aligned} W^*_{n+1} &\leqq \max_{0 \leqq k \leqq n} [T_n - T_k] + \max_{0 \leqq k \leqq n} [B_k - B_n] \\ &= \max_{0 \leqq k \leqq n} [T_n - T_k] \end{aligned}$$

which is bounded in probability.

Just as in Kiefer and Wolfowitz [3] we can write down an integral equation for the limiting distribution of Y_n. Under the conditions of Theorem 1 this integral equation will have a unique probability distribution as a solution. The uniqueness argument in [3] is rather delicate but in this problem the difficulty is easily disposed of because of the ease in seeing (by means of (2.13) for example) that the limiting distribution must be independent of the starting point (W_1, W_1^*).

3. The s-Server case. The question of ergodicity in the case of s servers can be handled in essentially the same fashion as in Section 2 where we had 2 servers. We shall be brief in those places where the generalization of the ideas in Section 2 is transparent.

For $\sigma = 1, \cdots, s$ let R_n^σ be the service time in server σ of the nth person. Let $R_{n+1}^0 = \tau_{n+1} - \tau_n$ where τ_n is the time at which the nth person arrives to the first queue. Let $t_n^\sigma = R_n^\sigma - R_{n+1}^{\sigma-1}$, $\sigma = 1, \cdots, s$. Let W_n^σ be the waiting time in the σth queue of the nth person. It is easily verified that, for all $1 \leqq p \leqq s$,

$$W_{n+1}^p = \max\left[0, W_n^p + t_n^p + \sum_{\sigma=1}^{p-1} [t_n^\sigma + W_n^\sigma - W_{n+1}^\sigma]\right]. \tag{3.1}$$

Let $T_k^\sigma = \sum_{i=1}^k t_i^\sigma$ and let $D_k^p = \max^* [-T_{j_1}^1 - \cdots - T_{j_p}^p]$ where max* is maximum over all $0 \leqq j_1 \leqq \cdots \leqq j_p \leqq k$. Let $H_n^p = \sum_{\sigma=1}^p [t_n^\sigma + W_n^\sigma - W_{n+1}^\sigma]$. To obtain a manageable expression for W_{n+1}^p we will show that $H_n^p = D_{n-1}^p - D_n^p$ for all p, n. Observe first that this is true when $p = 1$ and all n. Assume now that $H_n^p = D_{n-1}^p - D_n^p$ for all n. We will show that $H_n^{p+1} = D_{n-1}^{p+1} - D_n^{p+1}$ for all n.

From (3.1), the induction hypothesis, and iteration

$$\begin{aligned} W_{k+1}^{p+1} &= \max[0, W_k^{p+1} + t_k^{p+1} + H_k^p] = \max[0, W_k^{p+1} + t_k^{p+1} + D_{k-1}^p - D_k^p] \\ &= \max_{0 \leqq j \leqq k} [T_k^{p+1} - T_j^{p+1} + D_j^p - D_k^p]. \end{aligned} \tag{3.2}$$

Hence, using (3.2) for $k = n - 1$ and $k = n$

$$\begin{aligned} t_n^{p+1} + W_n^{p+1} - W_{n+1}^{p+1} &= t_n^{p+1} + \max_{0 \leqq j \leqq n-1} [T_{n-1}^{p+1} - T_j^{p+1} + D_j^p - D_{n-1}^p] \\ &\quad - \max_{0 \leqq j \leqq n} [T_n^{p+1} - T_j^{p+1} + D_j^p - D_n^p] \\ &= \max_{0 \leqq j \leqq n-1} [-T_j^{p+1} + D_j^p] - \max_{0 \leqq j \leqq n} [-T_j^{p+1} + D_j^p] \\ &\quad + D_n^p - D_{n-1}^p = D_{n-1}^{p+1} - D_n^{p+1} + D_n^p - D_{n-1}^p. \end{aligned}$$

Thus

$$H_n^{p+1} = t_n^{p+1} + W_n^{p+1} - W_{n+1}^{p+1} + H_n^p = D_{n-1}^{p+1} - D_n^{p+1}$$

which is what we wanted to show. Since H_n^p is what we say it is (3.2) is valid for all k and p (of course since there are only s servers we have no use for W_k^p where $p > s$).

Returning to (3.1) we remark that it is easy to verify just as in Section 2 that $Y_n = (W_n^1, \cdots, W_n^s, R_n^1, \cdots, R_n^{s-1})$ is the nth random variable in a stationary Markov process and that

$$(3.3) \quad P\{W_n^1 \leqq \alpha_1, \cdots, W_n^s \leqq \alpha_s \mid W_1^\sigma = 0, \sigma = 1, \cdots, s\} \to F(\alpha_1, \cdots, \alpha_s)$$

where F is an s-dimensional distribution function but not necessarily a probability distribution function.

Let $\mu_\sigma = ER_i^\sigma \quad \sigma = 0, 1, \cdots, s.$

THEOREM 3: *If*

$$(3.4) \quad \max_{1 \leqq \sigma \leqq s} \mu_\sigma < \mu_0$$

then F is a probability distribution.

PROOF: As in Theorem 1 we only have to show that each $\{W_n^\sigma\}$ is bounded in probability. It is easy to see by a trivial induction argument that we only have to verify that $\{W_n^s\}$ is bounded in probability. Actually the argument we give is legitimate when s is replaced by p for any $1 \leqq p \leqq s$. In any case we will only consider $\{W_n^s\}$.

To begin with observe that

$$(3.5) \quad -D_n^{s-1} \leqq T_n^1 + \cdots + T_n^{s-1}.$$

Hence from (3.2) with $k = n$, $p = s - 1$

$$(3.6) \quad \begin{aligned} W_{n+1}^s &\leqq \max_{0 \leqq j \leqq n} [T_n^1 + \cdots + T_n^s - T_j^s + D_j^{s-1}] \\ &= \max_{0 \leqq j_1 \leqq \cdots \leqq j_s \leqq n} [T_n^s - T_{j_s}^s + \cdots + T_n^1 - T_{j_1}^1] \end{aligned}$$

Let $s_0 = s$ and define s_i to be the largest $\sigma < s_{i-1}$ ($\sigma \geqq 0$) with the property that

$$(3.7) \quad \mu_\sigma - \mu_{s_{i-1}} > 0.$$

$\sigma = 0$ satisfies (3.7) because of (3.4) so that s_1 is well-defined. Let k be the first i such that $s_i = 0$. Then it is easy to check that

$$(3.8) \quad \mu_0 = \mu_{s_k} > \mu_{s_{k-1}} > \cdots > \mu_{s_0} = \mu_s$$

and that for $s_i < \sigma < s_{i-1}$

$$(3.9) \quad \mu_\sigma \leqq \mu_{s_{i-1}} < \mu_{s_i}.$$

Define, for $i = 1, \cdots, k$,

$$(3.10) \quad \begin{aligned} U_n^i &= \max_{0 \leqq j_1 \leqq \cdots \leqq j_s \leqq n} \left[\sum_{\sigma = s_i + 1}^{s_{i-1}} (T_n^\sigma - T_{j_\sigma}^\sigma) \right] \\ &= \max_{0 \leqq j_{s_i+1} \leqq \cdots \leqq j_{s_{i-1}} \leqq n} \left[\sum_{\sigma = s_i + 1}^{s_{i-1}} (T_n^\sigma - T_{j_\sigma}^\sigma) \right]. \end{aligned}$$

Because of (3.6) we have

$$W_{n+1}^{s} \leqq U_n^1 + \cdots + U_n^k \tag{3.11}$$

and, therefore, in order to show that $\{W_n^s\}$ is bounded in probability, we have only to verify that each $U_n^i (i = 1, \cdots, k)$ is bounded in probability.

The verification that each U_n^i is bounded can be summarized in the following lemma.

LEMMA: *For $m = 1, \cdots, M$ let $\{X_i^m, i = 1, \cdots\}$ be a sequence of independent and identically distributed random variables with*

$$EX_1^m = \lambda_m - \lambda_{m-1} \tag{3.12}$$

where

$$\lambda_0 > \lambda_M \geqq \max_{0<a<M} \lambda_a > \min_{0<a<M} \lambda_a \geqq 0. \tag{3.13}$$

(It is not assumed that $\{X_i^m\}$ and $\{X_i^{m'}\}$ are independent of one another). Let $S_k^m = \sum_{i=1}^k X_i^m$ and let $\psi_n = \max^* [\sum_{m=1}^M (S_n^m - S_{j_m}^m)]$ where max* *is maximum over all $j_1, \cdots, j_M$ with $0 \leqq j_1 \leqq \cdots \leqq j_M \leqq n$. Then ψ_n is bounded in probability.*

It is easy to verify that U_n^i can be taken as ψ_n and that the conditions of the lemma are satisfied ((3.9) giving (3.13)) so that the proof of the lemma is the last step in proving Theorem 3.

PROOF OF LEMMA: Let $\gamma = \min' [\lambda_i - \lambda_j]$ where min′ is minimum over all $0 \leqq i, j \leqq M$ with $\lambda_i - \lambda_j > 0$. Let $\delta = \gamma/M$. δ is, of course, strictly positive. For $2 \leqq m \leqq M$ define $\epsilon_m = \lambda_M - \lambda_{m-1} + (M - m + 1)\delta$ and let $\epsilon_{M+1} = 0$ and $\epsilon_1 = 0$. Then

$$\epsilon_2, \cdots, \epsilon_M \text{ are positive,} \tag{3.14}$$

$$\lambda_m - \lambda_{m-1} + \epsilon_{m+1} - \epsilon_m = -\delta, \qquad 2 \leqq m \leqq M \tag{3.15}$$

$$\lambda_1 - \lambda_0 + \epsilon_2 = \lambda_M - \lambda_0 + [(M-1)/M]\gamma < (-1/M)(\lambda_0 - \lambda_M) < 0. \tag{3.16}$$

Letting $j_{M+1} = n$ and taking note of the fact that $\epsilon_{M+1} = 0$ we have

$$\sum_{m=1}^{M} [(n - j_{m+1})\epsilon_{m+1} - (n - j_m)\epsilon_m] = 0 \tag{3.17}$$

where $0 \leqq j_1 \leqq j_2 \leqq \cdots \leqq j_M \leqq j_{M+1} = $ n. Now, using (3.17),

$$\psi_n = \max^* \left[\sum_{m=1}^{M} (S_n^m - S_{j_m}^m + (n - j_{m+1})\epsilon_{m+1} - (n - j_m)\epsilon_m) \right]$$

$$\leqq \sum_{m=1}^{M} \max^* [S_n^m - S_{j_m}^m + (n - j_{m+1})\epsilon_{m+1} - (n - j_m)\epsilon_m] \tag{3.18}$$

$$\leqq \sum_{m=1}^{M} \max_{0 \leqq j_m \leqq n} [S_n^m - S_{j_m}^m + (n - j_m)(\epsilon_{m+1} - \epsilon_m)].$$

Each of the terms in the summation on the right hand side of (3.18) is bounded in probability since

$$\max_{0 \leqq j_m \leqq n} [S_n^m - S_{j_m}^m + (n - j_m)(\epsilon_{m+1} - \epsilon_m)] = \max_{0 \leqq k \leqq n} [\xi_n^m - \xi_k^m]$$

where $\xi_k^m = \sum_{i=1}^k (X_i^m + \epsilon_{m+1} - \epsilon_m)$ and $E(X_i^m + \epsilon_{m+1} - \epsilon_m) < 0$ due to to (3.15) and (3.16). This concludes the proof of the lemma and, therefore, the theorem.

THEOREM 4: *If* $\max_{1 \leqq \sigma \leqq s} \mu_\sigma \geqq \mu_0$ *F is identically* 0.

PROOF: Letting p be the first $\sigma > 1$ with $\mu_\sigma \geqq \mu_0$ we need only show that $W_n^p \to +\infty$ in probability. Using (3.2)

$$\begin{aligned} W_{n+1}^p &= \max_{0 \leqq j \leqq n} [T_n^p - T_j^p + D_j^{p-1} - D_n^{p-1}] \\ &= \max_{0 \leqq j_1 \leqq \cdots j_p \leqq n} [T_n^p - T_{j_p}^p - T_{j_{p-1}}^{p-1} - \cdots - T_{j_1}^1 - D_n^{p-1}] \\ &= \max_{0 \leqq j_1 \leqq \cdots \leqq j_p \leqq n} [T_n^p + \cdots + T_n^1 - T_{j_p}^p - \cdots - T_{j_1}^1] \\ &\quad - \max_{0 \leqq j_1 \leqq \cdots \leqq j_p \leqq n} [T_n^1 + \cdots + T_n^{p-1} - T_{j_1}^1 - \cdots - T_{j_{p-1}}^{p-1}]. \end{aligned} \tag{3.19}$$

The last term on the right hand side of (3.19) is bounded in probability because $\max_{1 \leqq \sigma \leqq p-1} \mu_\sigma < \mu_0$. Looking at the first term on the right hand side of (3.19) we have

$$\begin{aligned} &\max_{0 \leqq j_1 \leqq \cdots \leqq j_p \leqq n} [T_n^p + \cdots + T_n^1 - T_{j_p}^p - \cdots - T_{j_1}^1] \\ &\qquad = \max_{0 \leqq k \leqq n} \left[\sum_{j=k+1}^n \sum_{\sigma=1}^p (R_j^\sigma - R_{j+1}^{\sigma-1}) \right] \\ &\qquad = \max_{0 \leqq k \leqq n} \left[\sum_{j=k+1}^n (R_j^p - R_j^0) + \sum_{\sigma=1}^p (R_{k+1}^{\sigma-1} - R_{n+1}^{\sigma-1}) \right] \\ &\qquad \geqq \max_{0 \leqq k \leqq n} \left[\sum_{j=k+1}^n (R_j^p - R_j^0) \right] - \sum_{\sigma=1}^p R_{n+1}^{\sigma-1}. \end{aligned}$$

The last term written is bounded in probability while the preceding term goes to $+\infty$ in probability because $\mu_p \geqq \mu_0$. It is then quite clear that W_{n+1}^p must go to $+\infty$ in probability.

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