A BOUND FOR THE LAW OF LARGE NUMBERS FOR DISCRETE MARKOV PROCESSES

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1. Summary. An exponential bound is obtained for the law of large numbers for $S_n = \sum_{k=1}^n f(X_k)$ where $\{X_k: k = 1, 2, \cdots\}$ is a discrete parameter Markov process satisfying Doeblin's condition and f is a bounded, real-valued, measurable function.

2. Introduction. Let $(\mathfrak{X}, \mathfrak{A}, P)$ be an arbitrary probability space and $p(x, A)$ a stationary transition probability function which we shall assume satisfies Doeblin's condition [1]. As a matter of convenience we assume there exists only one ergodic set. We denote by π the unique stationary measure and by ν_x the initial measure concentrating all the probability at the point $x \in \mathfrak{X}$. Let

$$\{X_k : k = 1, 2, \cdots\}$$

be the discrete Markov process determined by $p(x, A)$ and an arbitrary initial distribution. Denote by f an arbitrary bounded, real-valued, measurable function on $\mathfrak{X}$ and let $\mu = \int f(x)\pi(dx)$.

The purpose of this note is to prove the following

Theorem. *For every $\epsilon > 0$ there exist two constants, C and $\gamma < 1$, such that for all m and any initial distribution*

$$P\left\{\left|\frac{1}{n} S_n - \mu\right| \geqq \epsilon \quad \text{for some} \quad n \geqq m\right\} \leqq C\gamma^m.$$

An explicit bound was obtained by a more complicated proof in [2] for the case when $\mathfrak{X}$ is finite.

3. Proof of the theorem. We will need the following

Lemma. *If $\mu < 0$ then there exist two constants A and $\rho < 1$ such that for all n and any initial distribution*

$$P\{S_n \geqq 0\} \leqq A\rho^n.$$

Proof. Let $E_x e^{tS_n}$ denote the expected value of e^{tS_n} with respect to the initial measure ν_x. Define

$$\phi(n, t) = \sup_{x \in \mathfrak{X}} E_x e^{tS_n}.$$

If $n = k + l$, then

$$\begin{aligned} Ee^{tS_n} &= E\{E(\exp [tS_k + t\textstyle\sum_{j=k+1}^n f(X_j)] \mid X_1, \cdots, X_k)\} \\ &= E\{e^{tS_k} E(\exp [t\textstyle\sum_{j=k+1}^n f(X_j)] \mid X_k)\} \leqq \phi(k, t)\phi(l, t). \end{aligned}$$

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Consider any integer d and for $n \geqq d$ write $n = md + l$ where $0 \leqq l \leqq d - 1$. Then

$$\phi(n, t) \leqq \phi(md, t)\phi(l, t) \leqq [\phi(d, t)]^m \phi(l, t).$$

Therefore $(Ee^{tS_n})^{1/n} \leqq [\phi(d, t)]^{m/n}[\phi(l, t)]^{1/n}$. Now let $n \to \infty$ and it follows that

$$\lim_n \sup (Ee^{tS_n})^{1/n} \leqq [\phi(d, t)]^{1/d}.$$

Next we show that there exists a $t_0 > 0$ and an integer d_0 such that $\phi(d_0, t_0) < 1$. From Doeblin's condition we have that

$$(1/n) \sum_{k=1}^{n} p^{(k)}(x, A) \to \pi(A) \quad \text{uniformly in } x \text{ and } A$$

and thus, since $|S_n/n| \leqq M$ where $|f| \leqq M$, it follows that

$$E_x(S_n/n) \to \mu < 0 \quad \text{uniformly in } x.$$

Thus we can find an integer d_0 so that

$$E_x(S_{d_0}/d_0) \leqq \delta < 0 \quad \text{for all } x.$$

Further note that for $t < 1$

$$E_x e^{tS_{d_0}} \leqq E_x\{1 + td_0(S^{d_0}/d_0) + t^2M^2 d_0^2 e^{Md_0}\}.$$

Thus there exists a sufficiently small $t_0 > 0$ so that

$$E_x e^{t_0 S_{d_0}} \leqq 1 + t_0 d_0 \delta + t_0^2 M^2 d_0^2 e^{Md_0} < 1$$

for all x. Hence $\phi(d_0, t_0) < 1$ and since $P(S_n \geqq 0) \leqq Ee^{t_0 S_n}$ we have shown that

$$P(S_n \geqq 0) \leqq A\rho^n \quad \text{where} \quad \rho = \{[\phi(d_0, t_0)]^{1/d_0} + \epsilon\}$$

with $\epsilon > 0$ chosen so that $\rho < 1$ and the Lemma is proved.

The Theorem is an immediate consequence of the Lemma since

$$\begin{aligned} P\{|(1/n)S_n - \mu| \geqq \epsilon \text{ some } n \geqq m\} &\leqq \sum_{n=m}^{\infty} P\{|(1/n)S_n - \mu| \leqq \epsilon\} \\ &\leqq \sum_{n=m}^{\infty} \{P[(S_n - n\mu - n\epsilon) \geqq 0] + P[(-S_n + n\mu - n\epsilon) \geqq 0]\} \\ &\leqq \frac{A_1}{1-\rho_1}\rho_1^m + \frac{A_2}{1-\rho_2}\rho_2^m \\ &\leqq 2 \max\left(\frac{A_1}{1-\rho_1}, \frac{A_2}{1-\rho_2}\right)[\max(\rho_1, \rho_2)]^m. \end{aligned}$$

REFERENCES

[1] J. L. DOOB, *Stochastic Processes*, John Wiley and Sons, New York, 1953.
[2] MELVIN KATZ, JR., AND A. J. THOMASIAN, "An exponential bound for functions of a Markov chain," *Ann. Math. Stat.*, Vol. 31 (1960), pp. 470–474.