REMARK CONCERNING TWO-STATE SEMI-MARKOV PROCESSES

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Let $\{X_1, Y_1, X_2, Y_2, \cdots\}$ be a sequence of independent non-negative random variables where the X's have a common distribution function F and the Y's, a common distribution function G. Define

$$S_0 = 0,$$

$$S_k = \sum_{i=1}^{k} (X_i + Y_i), \qquad k = 1, 2, \cdots,$$

$$S'_k = S_k + X_{k+1}, \qquad k = 0, 1, 2, \cdots,$$

and, for each t, $0 < t < \infty$,

$$Z(t) = 1 \qquad \text{if } S_k < t \leqq S'_k \text{ for some } k = 0, 1, \cdots,$$
$$= 0 \qquad \text{otherwise.}$$

$Z(t)$ is a two-state Semi-Markov Process (see Lévy [2], Pyke [3] and [5], Smith [6]). Such a process arises in work sampling, and also in counter models as treated, e.g., by Pyke [4].

Let $P(t) = P(Z(t) = 1)$. From a result of Smith ([7], Theorem 1) it follows that, if $H = F * G$ (the distribution function of $X + Y$) is non-lattice, then

$$\lim_{t\to\infty} P(t) = \begin{cases} \dfrac{EX}{EX + EY} & \text{if } EX < \infty, \quad EY < \infty \\ 0 & EX < \infty, \quad EY = \infty. \end{cases}$$

Our remark is directed toward the behavior of $P(t)$ for large t, allowing that *both* EX and EY can be infinite. It consists in the following observation:

$$\text{(1)} \qquad \lim_{T\to\infty} \frac{1}{T} \int_0^T P(t)\, dt = \lim_{s\to 0} \frac{1 - f(s)}{1 - f(s)g(s)}, \qquad s > 0,$$

if either of the limits exist, where $f(s)$ and $g(s)$ are the Laplace-Stieltjes transforms of F and G. The truth of this remark is established by starting from a convolution representation of $P(t)$ (see [4]), taking Laplace transforms, and bringing to bear the Abelian and Tauberian theorems on p. 182 and p. 192 of [9].

If $f(s)$ and $g(s)$ are known, (1) is directly applicable. The following examples demonstrate the applicability of (1) for other situations.

(a) Suppose $g(s) = f^k(s)$ for some $k = 1, 2, \cdots$. Then the limit in (1) exists and is equal to $1/(k + 1)$.

Received October 3, 1960; revised November 3, 1960.

[1] Work sponsored by the Office of Naval Research under Contract number Nonr 266 (55) (Nr 042-099).

(b) Suppose, as $t \to \infty$,

$$\int_0^t (1 - F(x)\,dx \sim \frac{A}{\Gamma(2-\alpha)}\,t^{1-\alpha}, \qquad 0 < \alpha \leqq 1, \qquad A > 0,$$

$$\int_0^t (1 - G(x)\,dx \sim \frac{B}{\Gamma(2-\beta)}\,t^{1-\beta}, \qquad 0 < \beta \leqq 1, \qquad B > 0.$$

It can be shown, using the Abelian theorem on p. 182 of [9], that the limit in (1) is $A/(A + B)$ (if $\alpha = \beta$), 1 (if $\alpha > \beta$), and 0 (if $\alpha < \beta$), a result also obtainable from [8].

The limit (1) could be studied from the point of view of Darling and Kac [1]. Possibly, their results would yield conditions on F and G for (1) to hold.

The behavior of $P(t)$ itself, for large t, does not seem to be ascertainable by the method given here.

REFERENCES

[1] Darling, D. and M. Kac, "On occupation times for Markoff processes," *Trans. Amer. Math. Soc.*, Vol. 84 (1957), pp. 444–458.

[2] Lévy, P., "Processus semi-Markoviens," *Proc. Int. Congr. Math.*, Amsterdam, Vol. 3 (1954), pp. 416–426.

[3] Pyke, Ronald, "Markov renewal processes: definitions and preliminary properties," Technical Report under Contract Nonr 266(59) Columbia University, 1958. To appear in *Ann. Math. Stat.*

[4] Pyke, Ronald, "On renewal processes related to type I and type II counter models," *Ann. Math. Stat.*, Vol. 29 (1958), pp. 737–754.

[5] Pyke, Ronald, "Markov renewal processes with finitely many states," Technical Report under Nonr 266(59), Columbia University, 1958. To appear in *Ann. Math. Stat.*

[6] Smith, W. L. "Regenerative stochastic processes," *Proc. Royal Soc. London*, Ser. A., Vol. 232 (1955), pp. 6–31.

[7] Smith, W. L., "Asymptotic renewal theorems," *Proc. Roy. Soc. Edinburgh*, Ser. A., Vol. 64, Part I (1954), pp. 9–48.

[8] Takács, L., "On a sojourn time problem," *Teor. Veroyatnost. i Primenen*, Vol. 3 (1958), pp. 61–69.

[9] Widder, D. V., *The Laplace Transform*, Princeton University Press, Princeton, 1946.

AN EXAMPLE OF AN ANCILLARY STATISTIC AND THE COMBINATION OF TWO SAMPLES BY BAYES' THEOREM

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1. Origin of the example. In [1], an example was given in which a fiducial distribution served as a distribution *a priori* to be combined with a different set of data (not capable of yielding probability statements), by Bayes' Theorem. In [2], it was shown that this procedure of combining samples, when each sample yielded a fiducial distribution, could lead to a contradiction. In [3], an attempt

Received November 21, 1960.