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REFERENCES

1. RICHARD BELLMAN, *Dynamic Programming*, Princeton University Press, Princeton, N. J., 1957.
2. DAVID BLACKWELL, "On the functional equation of dynamic programing," *J. Math. Analysis and Applications*, Vol. 2 (1961).
3. DAVID G. KENDALL, "Some problems in the theory of dams," *J. Roy. Stat. Soc.*, Ser. B, Vol. 19, (1957), pp. 181–233.

FIRST PASSAGE TIME FOR A PARTICULAR GAUSSIAN PROCESS

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1. Introduction. Let $x(t)$ be a stationary Gaussian process with $Ex(t) = 0$ and $E[x(t)x(t')] = \rho(t - t')$. Denote by $Q_a(T \mid x_0)\, dT$ the conditional probability that for $t > 0$, $x(t)$ first assumes the value a in the interval $T \leqq t \leqq T + dT$ given that $x(0) = x_0$. It is well known that the determination of the first passage time probability $Q_a(T \mid x_0)\, dT$ is not an easy matter in general. To the author's knowledge, $Q_a(T \mid x_0)$ is known explicitly for stationary Gaussian processes with continuous spectral densities only in the Markovian case $\rho(\tau) = e^{-|\tau|}$. See [1], [2], [3] and [4]. This note points out that an elementary solution exists for the process with covariance

$$\rho(\tau) = \begin{cases} 1 - |\tau|, & |\tau| \leqq 1 \\ 0, & |\tau| \geqq 1 \end{cases} \tag{1}$$

for $0 \leqq T \leqq 1$.

2. Markoff-Like Property. The determination of the first passage time probability density $Q_a(T \mid x_0)$ for the process with covariance (1) follows from a peculiar Markoff-like property it possesses which may be described roughly as follows. Let $0 < t_1 < t_2 < 1$ be two instants in the unit interval. Denote the open interval (t_1, t_2) by A and the set $(0, t_1) \cup (t_2, 1)$ by B. Then for the process at hand, given the values of $x(t_1)$ and $x(t_2)$, events defined on A are statistically independent of events defined on B.

More precisely, we show the following. Let

$$0 < t_1 < t_2 < \cdots < t_k < \cdots < t_l < \cdots < t_n < 1.$$

Then

$$\begin{aligned} &p(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{l-1}, x_{l+1}, \cdots, x_n \mid x_k, x_l) \\ &\quad = p(x_1, \cdots, x_{k-1}, x_{l+1}, \cdots, x_n \mid x_k, x_l)\, p(x_{k+1}, \cdots, x_{l-1} \mid x_k, x_l). \end{aligned} \tag{2}$$

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Here we have set $x_i = x(t_i)$, $i = 1, 2, \cdots, n$, and have followed the time-honored (but often deplored) practice of denoting the conditional probability density of x_i given x_j by $p(x_i \mid x_j)$. The assumption of separability for the process leads from (2) to the statement of conditional independence of events defined on A and B.

Equation (2) can be established readily by a direct calculation. Let $z_1 = x_1 + x_n$, $z_j = x_j - x_{j-1}$, $j = 2, 3, \cdots, n$. One easily verifies from (1) that $Ez_i = 0$, $Ez_iz_j = 0$, $Ez_1^2 = 2[2 - (t_n - t_1)]$, $Ez_j^2 = 2(t_j - t_{j-1})$, $i = 1, 2, \cdots, n, j = 2, 3, \cdots, n$. The Jacobian $\dfrac{\partial(z_1, \cdots, z_n)}{\partial(x_1, \cdots, x_n)}$ has the value 2. Therefore,

$$p(x_1, \cdots, x_n) = 2(2\pi)^{-\frac{1}{2}n}[2(2 - t_n + t_1)]^{-\frac{1}{2}}\prod_2^n [2(t_j - t_{j-1})]^{-\frac{1}{2}} \cdot\exp -\frac{1}{2}\left[\frac{(x_1 + x_n)^2}{2(2 - t_n + t_1)} + \sum_2^n \frac{(x_j - x_{j-1})^2}{2(t_j - t_{j-1})}\right]. \tag{3}$$

The factors occurring in (2) are ratios of probability densities each of the form (3). Direct substitution results in the verification of (2).

Let $0 < t_1 < t_2 < t_3 < 1$. The process at hand has the curious property that

$$p(x_2, x_4 \mid x_1, x_3) = p(x_2 \mid x_1, x_3)p(x_4 \mid x_1, x_3)$$

if $t_3 < t_4 \leqq t_1 + 1$ or if $t_4 \geqq t_3 + 1$, but this conditional independence does not hold if $t_1 + 1 < t_4 < t_3 + 1$.

We note in passing that the process under consideration can also be written as $x(t) = y(t + 1) - y(t)$, where $y(t)$ is the Wiener process, and that the Markoff-like property just derived can be obtained from known properties of the Wiener process.

3. First Passage Time. The first passage time probability density for this process can be derived from an integral relation of the sort used by Siegert [4]. The process can pass from a value $x_0 > a$ at time $t = 0$ to a value $x_T \leqq a$ at time $t = T$ only if at some time θ, with $0 < \theta \leqq T$, the process assumes the value a for the first time. If then $R_a(x_T \mid x_0, \theta)\, dx_T$ is the conditional probability that $x_T \leqq x(T) \leqq x_T + dx_T$ given that $x(0) = x_0$ and given that for $t \geqq 0$ the process first assumes the value a for $\theta \leqq t \leqq \theta + d\theta$, we have

$$p(x_T \mid x_0) = \int_0^T d\theta\, Q_a(\theta \mid x_0)R_a(x_T \mid x_0, \theta), \qquad x_0 > a \geq x_T.$$

If now $T \leq 1$, $R_a(x_T \mid x_0, \theta) = p(x_T \mid x_0, x_\theta = a)$ because of the Markoff-like property of $x(t)$ already described. We have then

$$p(x_T \mid x_0) = \int_0^T d\theta\, Q_a(\theta \mid x_0)p(x_T \mid x_0, x_\theta = a), \qquad 0 \leqq T \leqq 1, \tag{4}$$

a relationship in which $Q_a(\theta \mid x_0)$ is the only quantity not known.

Equation (3) can be used to determine the conditional densities appearing in (4). After substituting for these quantities and cancelling some nonzero factors, one finds

$$e^{\frac{1}{2}x_0^2}\frac{\exp\left[-\frac{1}{2}\frac{(x_T-x_0)^2}{2T}\right]}{(2\pi T)^{\frac{1}{2}}} = \int_0^T d\theta(2-\theta)^{\frac{1}{2}}\exp\left[\frac{(x_0+a)^2}{4(2-\theta)}\right] \cdot Q_a(\theta \mid x_0)\frac{\exp\left[-\frac{1}{2}\frac{(x_T-a)^2}{2(T-\theta)}\right]}{[2\pi 2(T-\theta)]^{\frac{1}{2}}}.$$

Integrate on x_T from $-\infty$ to a to obtain

$$\pi^{-\frac{1}{2}}e^{\frac{1}{2}x_0^2}\int_{-\infty}^{(a-x_0)/(2T)^{\frac{1}{2}}} e^{-\frac{1}{2}u^2}\,du = \int_0^T d\theta(2-\theta)^{\frac{1}{2}}\exp\left[\frac{(x_0+a)^2}{4(2-\theta)}\right]Q_a(\theta \mid x_0)\tfrac{1}{2}.$$

Then $Q_a(T \mid x_0)$ can be obtained directly by differentiation with respect to T. A similar derivation can be carried out under the assumption $x_0 < a$. The combined result is

$$Q_a(T \mid x_0) = \frac{|x_0 - a|\exp\left\{-\frac{1}{2}\frac{[x_0(1-T)-a]^2}{T(2-T)}\right\}}{T[2\pi T(2-T)]^{\frac{1}{2}}}, \quad x_0 \neq a, \quad 0 < T \leqq 1.$$

The author has been unable to obtain an expression for $Q_a(T \mid x_0)$ valid for $T > 1$.

REFERENCES

[1] DARLING, D. A., AND SIEGERT, A. J. F., "The first passage problem for a continuous Markov process," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 624–639.

[2] RICE, S. O., "Distribution of the duration of fades in radio transmission," *Bell System Tech. J.*, Vol. 37 (1958), pp. 620–623, 630–631.

[3] SIEGERT, ARNOLD J. F., "On the zeros of Markoffian random functions," Rand Corporation Memorandum RM-447, September 5, 1950.

[4] SIEGERT, A. J. F., "On the first passage time probability problem," *Phys. Rev.*, Vol. 81 (1951), pp. 617–623.

A NOTE ON THE ERGODIC THEOREM OF INFORMATION THEORY[1]

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The purpose of this note is to extend the result of Breiman [1], [2] to an infinite alphabet, or equivalently, the result of Carleson [3] to convergence with probability one.

Let $\{\cdots, x_{-1}, x_0, x_1, \cdots\}$ be a stationary stochastic process taking values in a countable "alphabet" $\{a_i, i = 1, 2, \cdots\}$. Let

$$p(a_{i_1}, \cdots, a_{i_n}) = \mathcal{P}\{x_k = a_{i_k}, k = 1, \cdots, n\},$$

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