AN EXPONENTIAL BOUND ON THE STRONG LAW OF LARGE NUMBERS FOR LINEAR STOCHASTIC PROCESSES WITH ABSOLUTELY CONVERGENT COEFFICIENTS

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1. Introduction. Let $\{\xi_i: -\infty < i < \infty\}$ be a doubly infinite sequence of independent, identically distributed random variables which possess a moment generating function $M(t)$ over an open interval I_M. If $\{a_i: 1 \leqq i < \infty\}$ is a sequence of numbers for which $\sum_{i=1}^{\infty} |a_i| < \infty$, then the linear process

$$\left\{X_k = \sum_{i=1}^{\infty} a_i \, \xi_{k-i}: 1 \leqq k < \infty\right\}$$

possesses moments of all orders.

Let

$$\eta = E(\xi_0), \qquad \mu = \eta \sum_{i=1}^{\infty} a_i \quad \text{and} \quad S_n = \sum_{k=1}^{n} X_k .$$

The purpose of this paper is to establish the following theorem.

THEOREM. *For every* $\epsilon > 0$ *there exist constants* A *and* $\rho < 1$ *such that*

$$P\{|n^{-1}S_n - \mu| \geqq \epsilon \quad \text{for some} \quad n \geqq m\} \leqq A\rho^m.$$

2. Preliminaries. The following lemma will be needed for the proof of the theorem.

LEMMA. *Let* $\{b_i: 1 \leqq i < \infty\}$ *be a sequence of numbers for which* $\sum_{i=1}^{\infty}|b_i| < \infty$ *and* $\sum_{i=1}^{\infty} b_i > 0$. *If* $S_n = \sum_{k=1}^{n} X_k$ *where* $X_k = \sum_{i=1}^{\infty} b_i \xi_{k-i}$, *and if* $\eta < 0$, *then there exist constants* C *and* $\gamma < 1$ *such that* $P\{S_n \geqq 0\} \leqq C\gamma^n$.

PROOF. Let $X_{k,r} = \sum_{i=1}^{r} b_i \xi_{k-i}$, $S_{n,r} = \sum_{k=1}^{n} X_{k,r}$ and take $r > n$. By a rearrangement of terms, $S_{n,r}$ may be put in the form

$$S_{n,r} = \sum_{k=1-r}^{n-r-1} B_{1-k,r}\xi_k + \sum_{k=n-r}^{-1} B_{1-k,n-k}\xi_k + \sum_{k=0}^{n-1} B_{1,n-k}\xi_k ,$$

where $B_{m,n} = \sum_{i=m}^{n} b_i$. Hence, the moment generating function of $S_{n,r}$ is

$$M_{S_{n,r}}(t) = \prod_{k=1}^{n-1} M(B_{1+r-k,r}t) \prod_{k=1}^{r-n} M(B_{1+k,n+k}t) \prod_{k=1}^{n} M(B_{1,k}t). \tag{1}$$

Since the series $\sum_{i=1}^{\infty} b_i$ converges, the partial sums $B_{i,j}$ are uniformly bounded in i and j for $i \leqq j$. Thus $M_{S_{n,r}}(t)$ exists on an open interval I_{MS} which is independent of r and n.

It will now be shown that for each n, $M_{S_{n,r}}(t)$ converges to a function $\lambda(t) > 0$ on I_{MS} as r tends to infinity.

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The first product in Equation 1 tends to 1 as r becomes infinite, since the partial sums $B_{1+r-k,r}$ tend to zero for all k and $M(t)$ is continuous at the origin.

The second product converges (to a non zero limit) for all n and every $t \varepsilon I_{MS}$. To prove this we apply the test for the absolute convergence of an infinite product (see, e.g., page 15 of [2]). Write

$$M(B_{1+k,n+k}t) = 1 + M'(\theta_{n,k}(t)B_{1+k,n+k}t)B_{1+k,n+k}t$$

where $|\theta_{n,k}| \leqq 1$. It is then sufficient to show that

$$\sum_{k=1}^{\infty} |M'(\theta_{n,k}(t)B_{1+k,n+k}t)B_{1+k,n+k}t| < \infty$$

for each n and for every closed subinterval of I_{MS} which contains the origin. Since $M'(t)$ is continuous, it is clear that $M'(\theta_{n,k}(t)B_{1+k,n+k}t)$ is bounded uniformly in n, k and in t in the aforementioned closed subintervals of I_{MS}. Hence, we need only show $\sum_{k=1}^{\infty} |B_{1+k,n+k}| < \infty$ for all n.

Let $C_k = \max_{1+k \leqq i \leqq n+k} |b_i|$. Then the sequence $\{C_k\}$ coincides with a subsequence of $\{|b_i|\}$ except for at most n repetitions of each term. Thus,

$$\sum_{k=1}^{\infty} |B_{1+k,n+k}| \leqq \sum_{k=1}^{\infty} \sum_{i=1+k}^{n+k} |b_i| \leqq n \sum_{k=1}^{\infty} C_k \leqq n^2 \sum_{i=1}^{\infty} |b_i| < \infty.$$

Since the last product in Equation 1 is independent of r, the convergence of $M_{S_{n,r}}(t)$ is established.

In order to conclude the proof of the lemma it will be shown that the moment generating function of S_n exists and coincides with $\lambda(t)$ on I_{MS}. Let $z = t + iu$ and $M_{S_{n,r}}(z) = Ee^{zS_{n,r}}$. Then $M_{S_{n,r}}(z)$ is a bilateral Laplace-Stieltjes transform which, since $|M_{S_{n,r}}(z)| \leqq M_{S_{n,r}}(t)$, is analytic in the semi-infinite strip $\sigma = \{z:t \varepsilon I_{MS}\}$. The convergence of $M_{S_{n,r}}(t)$ implies that $M_{S_{n,r}}(z)$ is bounded uniformly in r and in z for t in every closed subinterval of I_{MS} which contains the origin. Hence, by Vitali's theorem ([2] page 168), $M_{S_{n,r}}(z)$ converges uniformly to a limit $\lambda(z)$ for every region bounded by a contour in σ.

The function $\lambda(z)$ is then analytic in σ and, since σ contains the imaginary axis, $\lim_{r\to\infty} M_{S_{n,r}}(iu) = \lambda(iu)$ for all u. Also it is easily seen that $\text{l.i.m.}_{r\to\infty} S_{n,r} = S_n$, where l.i.m. denotes limit in the mean of order 2. Hence, by the Lévy continuity theorem, $\lambda(iu) = M_{S_n}(iu) = Ee^{iuS_n}$. But then for all z in σ the coefficient of $z^m/m!$ in the power series expansion of $\lambda(z)$ is the mth moment of S_n about the origin. It follows that $M_{S_n}(t)$ exists and is equal to $\lambda(t)$ on I_{MS}.

We have shown that

$$M_{S_n}(t) = \prod_{k=1}^{\infty} M(B_{1+k,n+k}t) \prod_{k=1}^{n} M(B_{1,k}t).$$

Since $\sum_{i=1}^{\infty} b_i > 0$, there exists an integer N such that for all $k \geqq N$, $\epsilon < B_{1,k} < \delta$ for some $\epsilon > 0$ and $\delta < \infty$. Select $t^* > 0$ in I_{MS} so that

$$\gamma = \max[M(\epsilon t^*), M(\delta t^*)] < 1.$$

This is possible since $M(0) = 1$ and $M'(0) = \eta < 0$. Then, since $M(t)$ is also convex, $M(\mu t^*) \leqq \gamma$ for $\epsilon \leqq \mu \leqq \delta$. The conclusion of the lemma now follows from the well known inequality $P\{S_n \geqq 0\} \leqq M_{S_n}(t^*)$ where we may take $C = \max\{1/\gamma^{N-1}, \sup_{n \geqq N} \prod_{k=1}^{\infty} M(B_{1+k,n+k}t^*)\}$.

3. Proof of the theorem. Let $\{a_i: 1 \leqq i < \infty\}$ be an arbitrary sequence for which $\sum_{i=1}^{\infty} |a_i| < \infty$ and let the value of $\eta = E(\xi_0)$ be arbitrary. Now,

$$P\left\{\left|\frac{1}{n} S_n - \mu\right| \geqq \epsilon \quad \text{for some} \quad n \geqq m\right\} \leqq \sum_{n=m}^{\infty} P\left\{\left|\frac{1}{n} S_n - \mu\right| \geqq \epsilon\right\}$$

$$\leqq \sum_{n=m}^{\infty} [P\{(S_n - n\mu - n\epsilon) \geqq 0\} + P\{(-S_n + n\mu - n\epsilon) \geqq 0\}]$$

$$\leqq \frac{A_1}{1-\rho_1}\rho_1^m + \frac{A_2}{1-\rho_2}\rho_2^m \leqq 2 \max\left(\frac{A_1}{1-\rho_1}, \frac{A_2}{1-\rho_2}\right)[\max(\rho_1, \rho_2)]^m$$

provided $\max(\rho_1, \rho_2) < 1$. Thus the theorem will be proved if it can be shown that, for $\epsilon > 0$, $S_n - n\mu - n\epsilon$ and $-S_n + n\mu - n\epsilon$ can be translated into sums of the form considered in the lemma.

It suffices to concentrate on the expression $S_n - n\mu - n\epsilon$ since the arguments are the same for both. Write $X_k - \mu = \sum_{i=1}^{\infty} a_i\theta_{k-i}$ where the random variables $\theta_i = \xi_i - \eta$ have zero expectation. We now analyse three cases.

Case I. $\sum_{i=1}^{\infty} a_i > 0$. Set $\epsilon' = \epsilon / \sum_{i=1}^{\infty} a_i$. Then

$$X_k - \mu - \epsilon = \sum_{i=1}^{\infty} a_i(\theta_i - \epsilon')$$

where $E(\theta_0 - \epsilon') < 0$. The theorem now follows from the Lemma.

Case II. $\sum_{i=1}^{\infty} a_i < 0$. Write

$$X_k - \mu - \epsilon = \sum_{i=1}^{\infty} (-a_i)(-\theta_i - \epsilon'')$$

where $\epsilon'' = -\epsilon / \sum_{i=1}^{\infty} a_i$ and again apply the Lemma.

Case III.

$$\sum_{i=1}^{\infty} a_i = 0. \quad \text{Let} \quad \sum_{i=1}^{\infty} a_i = \sum{}^{+} a_i + \sum{}^{-} a_i$$

where $\sum^{+} a_i$ is the sum of the positive terms and $\sum^{-} a_i$ the sum of the negative terms of the series. Similarly, let $X_k^+ = \sum^{+} a_i\xi_{k-i}$, $X_k^- = \sum^{-} a_i\xi_{k-i}$, $\mu^+ = \eta \sum^{+} a_i$ and $\mu^- = \eta \sum^{-} a_i$. Then if $S_n^+ = \sum_{k=1}^{n} X_k^+$ and $S_n^- = \sum_{k=1}^{n} X_k^-$,

$$P\{S_n - n\mu - n\epsilon \geqq 0\} \leqq P\{S_n^+ - n\mu^+ - \tfrac{1}{2}n\epsilon \geqq 0\} + P\{S_n^- - n\mu^- - \tfrac{1}{2}n\epsilon \geqq 0\}. \tag{2}$$

The two terms on the right hand side of this inequality may be dealt with under Cases I and II except when one of the sums, $\sum^{+} a_i$ or $\sum^{-} a_i$, contains a finite number of terms. In this event, the corresponding process

$$\{X_{k,r} = \textstyle\sum^{\pm} a_i \xi_{k-i}: 1 \leqq k < \infty\}$$

is an r dependent process of identically distributed random variables, where r is the number of terms in the sum. Then $S_{n,r} = \sum_{k=1}^{n} X_{k,r}$ may be written in the form $S_{n,r} = \sum_{j=1}^{r} Z_{n,j}$ where $Z_{n,j}$ is the sum of independent, identically distributed random variables obtained by taking every $(r + 1)$st term of $S_{n,r}$ starting with the jth. It is well known (e.g. from [1]) that the existence of $M(t)$ and the condition $\mu < 0$ are sufficient to guarantee an exponential bound for $P\{Z_{n,j} \geqq 0\}$; $1 \leqq j \leqq r$. The bound for $S_{n,r}$ is then easily obtained from the inequality

$$P\{S_{n,r} \geqq 0\} \leqq \sum_{j=1}^{r} P\{Z_{n,j} \geqq 0\}.$$

Corollary. *If the sequence $\{a_i\}$ is doubly infinite with $\sum_{i=-\infty}^{\infty} |a_i| < \infty$, the conclusion of the theorem applies to the linear process*

$$\{X_k = \textstyle\sum_{i=-\infty}^{\infty} a_i \xi_{k-i}: 1 \leqq k < \infty\}.$$

Proof. Write $X_k = X_{k1} + X_{k2}$ where $X_{k1} = \sum_{i=-\infty}^{0} a_i \xi_{k-i}$ and $X_{k2} = \sum_{i=1}^{\infty} a_i \xi_{k-i}$. Then an inequality analogous to Inequality 2 reduces this to two applications of the theorem.

REFERENCES

[1] Herman Chernoff, "A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations," *Ann. Math. Stat.*, Vol. 23 (1952), pp. 492–507.

[2] E. C. Titchmarsh, *The Theory of Functions*, 2nd ed., Oxford University Press, Cambridge, 1939.