THE EXISTENCE AND CONSTRUCTION OF BALANCED INCOMPLETE BLOCK DESIGNS[1]

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1. Introduction. Given a set E of v elements, and given positive integers, k, l ($l \leqq k \leqq v$) and λ, we understand by a *tactical configuration* $C[k, l, \lambda, v]$ (briefly, *configuration*) a system of subsets of E, having k elements each, such that every subset of E having l elements is contained in exactly λ sets of the system.

A necessary condition [13, 9] for the existence of a configuration $C[k, l, \lambda, v]$ is that

$$\text{(i)} \qquad \lambda \binom{v-h}{l-h} \Big/ \binom{k-h}{l-h} = \text{integer}, \qquad h = 0, 1, \cdots, l-1.$$

Clearly, $\lambda \binom{v}{l} \Big/ \binom{k}{l}$ is the number of elements of $C[k, l, \lambda, v]$ and

$$\lambda \binom{v-h}{l-h} \Big/ \binom{k-h}{l-h}$$

is the number of those elements of $C[k, l, \lambda, v]$ that contain h fixed elements of E.

A *balanced incomplete block design* (BIBD), $B[k, \lambda, v]$, ($k \leqq v$) is a configuration $C[k, 2, \lambda, v]$ with $l = 2$. The elements of $B[k, \lambda, v]$ are called *blocks*.

In the usual terminology, a BIBD is an arrangement of v elements in b blocks of k elements each so that every element occurs in r blocks and every pair of elements occurs λ times in all [8].

From (i) follows:

A necessary condition for the existence of a BIBD is

$$\text{(ii)} \qquad \lambda(v-1) \equiv 0 (\text{mod } (k-1)) \quad \text{and} \quad \lambda v(v-1) \equiv 0(\text{mod } k(k-1)).$$

In the sequel we shall consider (ii) as a condition on v for fixed k and λ.

Steiner triple systems [17] are BIBD with $k = 3, \lambda = 1$. It has been proved by Reiss [15] and by Moore [12] that in this case condition (ii) is also sufficient for the existence of a BIBD. Bose [1] proved that condition (ii) is also sufficient in the case $k = 3, \lambda = 2$.

On the other hand, there are known cases in which condition (ii) is not sufficient. A BIBD with $k = n + 1, \lambda = 1$ and $v = n^2 + n + 1$ is a finite projective plane of order n. For such planes condition (ii) is clearly satisfied; it was how-

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Received February 23, 1960; revised August 10, 1960.

[1] The work on this paper was done at the Mathematics Research Center, United States Army, Madison, Wisconsin, sponsored by the United States Army, under Contract No. Da-11-022-ORD-2059.

ever, already conjectured by Euler [5], and proved by Tarry [20], that no projective plane of order $n = 6$ exists. Bruck and Ryser [2] have proved moreover that no finite projective plane exists if $n \equiv 1$ or 2 (mod 4) and the square-free part of n contains at least one prime factor of the form $4m + 3$.

The purpose of this paper is to prove that condition (ii) is sufficient for the existence of a BIBD for $k = 3$ and 4 (and every λ) and also for $k = 5, \lambda = 1,4$ and 20.[2] The proof is given by induction on v for any pair of fixed values of k and λ, and it enables effective construction of the designs. The induction works also for larger values of k, but in these cases the existence of designs for initial values of v remains undetermined.

Tactical configurations $C[k, l, 1, v]$ with $\lambda = 1$ have been introduced by Moore [13] as tactical systems $S[k, l, v]$. From (i) it follows that a necessary condition for the existence of a system $S[k, l, v]$ is that

$$\text{(iii)} \qquad \binom{v-h}{l-h} \Big/ \binom{k-h}{l-h} = \text{integer}, \qquad h = 0, 1, \cdots, l-1.$$

This condition again is not always sufficient, as the nonexistence of a finite projective plane of order $n = 6$ shows. So far, it has been proved that (iii) is sufficient for $k = 3$, $l = 2$ (the mentioned Steiner triple systems) and for $k = 4$, $l = 3$ [9]. In the present paper, sufficiency of (iii) is also proved for $k = 4$, $l = 2$, (6.5), and for $k = 5$, $l = 2$,[3] (7.10). No other general sufficient conditions on the existence of systems $S[k, l, v]$ are known so far; the special cases of systems known to exist may be found listed in [23].

For some detailed information on incomplete balanced block designs and for bibliography, see the excellent survey by Hall [8].

Considering the rather tedious proofs of combinatorial character a special subdivision into sections has been adopted. Every subsection denoted by two figures consists of one of the following: a definition (e.g., (2.1)), a theorem (5.1), a proposition (3.4), a lemma (5.3), or a proof of a part of a theorem (5.5). Some of these subsections contain auxiliary lemmas which for reference are denoted by three figures (e.g., (5.3.1.)).

2. T-systems.

(2.1) DEFINITION. Let a class of m mutually disjoint sets τ_i, $i = 0, 1, \cdots, m-1$ having t elements each be given. If it is possible to form a system of t^2 m-tuples (i.e., sets having m elements each) in such a way that

(i) each m-tuple has exactly one element in common with each of the sets τ_i, $i = 0, 1, \cdots, m-1$, and

(ii) every two m-tuples have at most one element in common, then we denote the above system of m-tuples by $T_0[m, t]$.

The class of all numbers t for which systems $T_0[m, t]$ exist will be denoted by $T_0(m)$.

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[2] With the possible exception of B [5, 1, 141].

[3] *Ibid.*

(2.2) Definition. If a system $T_0[m, t]$ exists and if moreover there are in the system at least e subsystems $(0 \leqq e \leqq t)$ each consisting of t mutually disjoint m-tuples, then we denote such a system by $T_e[m, t]$.

The class of all numbers t for which systems $T_e[m, t]$ exist will be denoted by $T_e(m)$.

As a direct consequence of the definitions we obtain

(2.3) *Let a system $T_e[m, t]$ $(0 \leqq e \leqq t)$ be given and let $A \in \tau_i$, $B \in \tau_j$, $i < j$, then there exists exactly one m-tuple of $T_e[m, t]$ containing both elements A and B.*

(2.4) *If $e \geqq d$, then $T_e(m) \subset T_d(m)$, i.e., $t \in T_e(m)$ implies $t \in T_d(m)$.*

(2.5) *$t \in T_1(m)$ is possible only if $t \geqq m$.*

We shall now prove

(2.6) *If t is a power of a prime, then $t \in T_t(t)$.*

For $t = p^\alpha$ (p prime, α a positive integer) finite projective planes $PG[2, p^\alpha]$ have been constructed with $t + 1$ points on a line [21]. Through every point in infinity go—besides the line in infinity—t otherwise mutually disjoint lines. Omit the line and the points in infinity and choose any t mutually disjoint lines of the remaining Euclidean plane $EG[2, p^\alpha]$ as the sets τ_i, $i = 0, 1, \cdots, t - 1$. The remaining lines form a system $T_t[t, t]$; compare [19].

(2.7) *If $t \in T_e(m_1)$ and $m_1 \geqq m_2$, then also $t \in T_e(m_2)$.*

This is obtained by omitting the $m_1 - m_2$ sets τ_j, $j = m_2, m_2 + 1, \cdots, m_1 - 1$.

(2.8) *If $t \in T_e(m)$ and $s \in T_d(m)$, then $ts \in T_{ed}(m)$.*

Consider a 3-dimensional finite lattice of points with integral coordinates $0 \leqq x \leqq m - 1, 0 \leqq y \leqq t - 1, 0 \leqq z \leqq s - 1$. In this lattice the m-tuples of $T_e[m, t]$ may be described as functions $y = y_h(x)$, $h = 0, 1, \cdots, t^2 - 1$, and the m-tuples of $T_d[m, s]$ as functions $z = z_j(x), j = 0, 1, \cdots, s^2 - 1$. For every pair of indices (h, j) we form the m-tuple defined by the pair of functions $y = y_h(x)$, $z = z_j(x), h = 0, 1, \cdots, t^2 - 1, j = 0, 1, \cdots, s^2 - 1$. Taking for τ_i the planes $x = i, i = 0, 1, \cdots, m - 1$, it is easily verified that the conditions of the definition (2.1) are fulfilled and thus the obtained m-tuples form a system $T_0[m, ts]$. In order to show that this system is a $T_{ed}[m, ts]$, we remark that if the functions $y = y_{h_\alpha}(x)$, $\alpha = 0, 1, \cdots, t - 1$, are mutually disjoint and also the functions $z = z_{j_\beta}(x)$, $\beta = 0, 1, \cdots, s - 1$, are such, then also the ts m-tuples given by the pairs of functions $y = y_{h_\alpha}(x)$, $z = z_{j_\beta}(x)$ are mutually disjoint.

From (2.6) and (2.7) by repeated use of (2.8) follows:

(2.9) *Let $t = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, where p_i are primes and α_i positive integers, $i = 1, 2, \cdots n$. If $p_i^{\alpha_i} \geqq m$, $i = 1, 2, \cdots, n$, then $t \in T_t(m)$.*

Proposition (2.9) is equivalent to the theorem proved by MacNeish [10] and later by Mann [11] that under the conditions of (2.9) there exist at least $m - 1$ mutually orthogonal Latin squares.

(2.10) *$t \in T_t(m - 1)$ if and only if $t \in T_0(m)$.*

If $t \in T_0(m)$, then every element of τ_{m-1} belongs to t otherwise mutually disjoint m-tuples. By omission of τ_{m-1} we thus obtain the required system $T_t[m - 1, t]$. If on the other hand $t \in T_t(m - 1)$, then, for every subsystem of t mutually disjoint $(m - 1)$-tuples, we adjoin a fixed element to all the $(m - 1)$-

tuples of such subsystem. Denoting by τ_{m-1} the set of these additional elements, we obtain a system $T_0[m, t]$.

From (2.10) and (2.5) follows:

(2.11) *$t \varepsilon T_0(m)$ is possible only if $t \geqq m - 1$.*

(2.12) *If $t \varepsilon T_s(m)$ and $s \varepsilon T_0(m)$, then $ts \varepsilon T_{s^2}(m)$.*

Consider a 3-dimensional finite lattice of points with integral coordinates $0 \leqq x \leqq m - 1$, $0 \leqq y \leqq t - 1$, $0 \leqq z \leqq s - 1$. In this lattice denote by $y = y_i^{(j)}(x)$, $i = 0, 1, \cdots, t - 1$, the functions corresponding to the t m-tuples of the jth subsystem of mutually disjoint m-tuples of $T_s[m, t]$, $j = 0, 1, \cdots, s - 1$, and by $y = y_h(x)$, $h = 0, 1, \cdots, t^2 - ts - 1$, the functions corresponding to the remaining m-tuples of $T_s[m, t]$. By $z = z_k(x)$, $k = 0, 1, \cdots, s^2 - 1$, denote the functions corresponding to the m-tuples of $T_0[m, s]$. Now form the pairs of functions

(i) $y = y_i^{(j)}(x)$, $z = z_k(x) + j \pmod s$, $(i = 0, 1, \cdots, t - 1; j = 0, 1, \cdots, s - 1; k = 0, 1, \cdots, s^2 - 1)$,

(ii) $y = y_h(x)$, $z = z_k(x)$, $(h = 0, 1, \cdots, t^2 - ts - 1; k = 0, 1, \cdots, s^2 - 1)$, obtaining their values in the yz plane. These functions are m-tuples, any two of which have at most one element in common. Moreover for every fixed k, $k = 0, 1, \cdots, s^2 - 1$, the ts functions (i) are mutually disjoint, for different j's are namely the functions $z = z_k(x) + j$ disjoint and for fixed j and different i's — the functions $y = y_i^{(j)}(x)$.

From (2.10), (2.4) and (2.12) follows

(2.13) *If $t \varepsilon T_t(m)$ and $m - 1 \varepsilon T_{m-1}(m - 1)$, then $t(m - 1) \varepsilon T_{(m-1)^2}(m)$.*

3. B-systems.

(3.1) Definition. Let a set E having v elements be given; further let $K = \{k_i\}_{i=1}^n$ be a finite set of integers $3 \leqq k_i \leqq v$, $i = 1, 2, \cdots, n$, and λ a positive integer. If it is possible to form a system of blocks (subsets of E) in such a way that

(i) the number of elements in each block is some $k_i \varepsilon K$ and

(ii) every (unordered) pair of elements of E is contained in exactly λ blocks, then we shall denote such a system by $B[K, \lambda, v]$.

The class of all numbers v for which systems $B[K, \lambda, v]$ exist will be denoted by $B(K, \lambda)$.

If $K = \{k\}$ consists of one number k only we shall write $B[k, \lambda, v]$ and $B(k, \lambda)$ instead of $B[\{k\}, \lambda, v]$ and $B(\{k\}, \lambda)$ respectively.

The systems $B[k, \lambda, v]$ are the BIBD introduced in Section 1.

(3.2) Definition. If a system $B[K, \lambda, v]$ exists and if moreover there exists an element $A \varepsilon E$ and a number $m \varepsilon K$ such that $(m - 1)$ divides $(v - 1)$ and the set $E - \{A\}$ can be split into $(v - 1)/(m - 1)$ mutually disjoint subsets E_i, $i = 1, 2, \cdots, (v - 1)/(m - 1)$, each having $(m - 1)$ elements, in such a way that each of the sets $E_i \cup \{A\}$, $i = 1, 2, \cdots, (v - 1)/(m - 1)$, appears exactly λ times as a block in the system $B[K, \lambda, v]$, then we denote such system by $B_m[K, \lambda, v]$, and the class of all numbers v for which systems $B_m[K, \lambda, v]$ exist by $B_m(K, \lambda)$.

(3.3) DEFINITION. If a system $B[k, \lambda, v]$ exists and if moreover there exists a number $m \varepsilon K$ such that m divides v, and the set E can be split into v/m mutually disjoint subsets E_i, $i = 1, 2, \cdots, v/m$, each having m elements and each appearing exactly λ times as a block in the system $B[K, \lambda, v]$, then we denote such system by $B'_m[K, \lambda, v]$.

As an immediate consequence of the definitions we have:

(3.4) *From* $v \varepsilon B_m(K, \lambda)$ *follows* $v \varepsilon B(K, \lambda)$.

(3.5) *From* $v \varepsilon B'_m(K, \lambda)$ *follows* $v \varepsilon B(K, \lambda)$.

(3.6) *From* $v \varepsilon B(k, 1)$ *follows* $v \varepsilon B_k(k, 1)$.

(3.7) *If* $K' \subset K$ *then from* $v \varepsilon B(K', \lambda)$ *follows* $v \varepsilon B(K, \lambda)$.

(3.8) *If* λ' *is a factor of* λ *or if* $\lambda' = 1$ *then from* $v \varepsilon B(K, \lambda')$, $v \varepsilon B_m(K, \lambda')$ *and* $v \varepsilon B'_m(K, \lambda')$ *follow* $v \varepsilon B(K, \lambda)$, $v \varepsilon B_m(K, \lambda)$ *and* $v \varepsilon B'_m(K, \lambda)$ *respectively.*

(3.9) *If* $v \varepsilon B(K', \lambda')$ *and if for every* $k' \varepsilon K'$, $k' \varepsilon B(K, \lambda'')$, *then* $v \varepsilon B(K, \lambda)$, *where* $\lambda = \lambda'\lambda''$.

We shall now prove the following proposition

(3.10) *If* $v = (m - 1)u + 1$, *where* $u \varepsilon B(K', \lambda')$ *and if for every* $k' \varepsilon K'$, $(m - 1)k' + 1 \varepsilon B_m(K, \lambda'')$, *then* $v \varepsilon B_m(K, \lambda)$, *where* $\lambda = \lambda'\lambda''$.

Consider a 2-dimensional finite lattice of points (x, y) with integral coordinates $0 \leqq x \leqq u - 1$, $0 \leqq y \leqq m - 2$ and a point A. The total number of points is clearly v. Denote

$$(A, i) = \{A, (i, y): \ 0 \leqq y \leqq m - 2\}.$$

Now for every block β of the system $B[K', \lambda', u]$ consider the set $\bigcup_{i \varepsilon \beta}(A, i)$. On this set we may construct a system $\tilde{B}(\beta) = B_m[K, \lambda'', (m - 1)\bar{\bar{\beta}} + 1]$, ($\bar{\bar{\beta}}$ is the number of elements in β) in such a way that each of the sets (A, i), $i \varepsilon \beta$, appears in $\tilde{B}(\beta)$ as block exactly λ'' times. We construct now a system $B_m[K, \lambda, v]$ as follows: take all the blocks of all the systems $\tilde{B}(\beta)$, $\beta \varepsilon B[K', \lambda', u]$,—except of the blocks (A, i), $i = 0, 1, \cdots, u - 1$,—as often as they appear, and the blocks (A, i), $i = 0, 1, \cdots, u - 1$, λ times each. It is easily checked that the number of elements in each block is a number of $K(m \varepsilon K$ by definition) and that each pair appears in exactly λ blocks.

In the same way it can be proved:

(3.11) *If* $v = mu$ *where* $u \varepsilon B(K', \lambda')$ *and if for every* $k' \varepsilon K'$, $mk' \varepsilon B'_m(K, \lambda'')$, *then* $v \varepsilon B'_m(K, \lambda)$, *where* $\lambda = \lambda'\lambda''$.

Putting in (3.10): $K = \{k\}$ and $m = k$ we obtain

(3.12) *If* $v = (k - 1)u + 1$, *where* $u \varepsilon B(K', \lambda')$ *and if for every* $k' \varepsilon K'$, $(k - 1)k' + 1 \varepsilon B_k(k, \lambda'')$, *then* $v \varepsilon B_k(k, \lambda)$, *where* $\lambda = \lambda'\lambda''$.

Further we prove:

(3.13) *Let* $t, s, s + 1 \varepsilon B(K, 1)$, $t \varepsilon T_q(s)$ *and* $q \varepsilon B(K, 1)$ *or* $q = 0$ *or* 1; *then* $u = st + q \varepsilon B(K, 1)$.

Consider a 2-dimensional lattice of points with integral coordinates $0 \leqq x \leqq t - 1$, $0 \leqq y \leqq s - 1$ and $0 \leqq x \leqq q - 1$, $y = s$. Take all the s-tuples of $T_q[s, t]$; there are among them q subsystems of t mutually disjoint s-tuples each and we adjoin to all the s-tuples of the jth subsystem, $j = 0, 1, \cdots, q - 1$, the point $x = j$, $y = s$. We form now $B[K, 1, u]$ taking the blocks of the qt systems $B[K, 1, s + 1]$ on all so obtained $(s + 1)$-tuples, the blocks of the $t(t - q)$ systems $B[K, 1, s]$ on the remaining s-tuples of $T_q[s, t]$, and also all the blocks of the systems $B[K, 1, t]$ on each of the lines $y = i$, $i = 0, 1, \cdots, s - 1$, and if $q > 1$ — the blocks of the system $B[K, 1, q]$ on the line $y = s$.

By the same proof we may obtain the more general result:

(3.14) *Let* $t, s, s + 1 \varepsilon B(K, \lambda)$, $t \varepsilon T_q(s)$ *and* $q \varepsilon B(K, \lambda)$ *or* $q = 0$ *or* 1; *then* $u = st + q \varepsilon B(K, \lambda)$.

The following propositions may also be proved in a similar way:

(3.15) *Let* $t + 1, s \varepsilon B(K, \lambda)$ *and* $t \varepsilon T_0(s)$, *then* $u = st + 1 \varepsilon B(K, \lambda)$.

(3.16) *Let* $t + 1, s, s + 1 \varepsilon B(K, \lambda)$, $t \varepsilon T_q(s)$ *and* $q + 1 \varepsilon B(K, \lambda)$ *or* $q = 0$, *then* $u = st + q + 1 \varepsilon B(K, \lambda)$.

4. Block designs with $v = p^\alpha$.

(4.1) Let E be a set of $v = p^\alpha$ elements (p prime, α a positive integer). We may denote the elements of E as marks in a Galois field (see e.g., [3] pp. 242–288) and more specifically as polynoms $\sum_{i=0}^{\alpha-1} a_i x^i$, $a_i = 0, 1, \cdots, p - 1$; $i = 0, 1, \cdots, \alpha - 1$. In order to shorten the notation we shall in the sequel denote such marks by (g),

$$g = \sum_{i=0}^{\alpha-1} a_i x^i, \quad a_i = 0, 1, \cdots, p - 1, i = 0, 1, \cdots, \alpha - 1.$$

Putting $x^\alpha = \sum_{i=0}^{\alpha-1} c_i x^i$, where $x^\alpha - \sum_{i=0}^{\alpha-1} c_i x^i = 0$ is an irreducible equation in the field and taking all coefficients modulo p, (for $\alpha = 1$ we take for x a primitive root of p) we are able to reduce any polynom to a mark in the Galois field and in the sequel such reduction will always supposed to be performed.

For $v = p^\alpha$, BIBD may in some cases be constructed in a simple way as the following propositions show (compare also [1, 6, 16]).

(4.2) *If* $v = p^\alpha$, *then* $v \varepsilon B(k, k(k - 1))$.

The blocks are:

$$\{(g + x^\beta), (g + x^{\beta+1}), \cdots, (g + x^{\beta+k-1})\}, \qquad \beta = 0, 1, \cdots, v - 2.$$

Considering that g obtains the values of all the marks of the Galois field it is sufficient to show that for a fixed g each non-zero mark of the field appears exactly $k(k - 1)$ times as difference between the elements of the blocks. Now for each pair of integers γ, δ, $(0 \leqq \gamma \leqq k - 1, 0 \leqq \delta \leqq k - 1, \gamma \neq \delta)$ the differences $(g + x^{\beta+\gamma}) - (g + x^{\beta+\delta}) = x^\beta(x^\gamma - x^\delta)$ run for $\beta = 0, 1, \cdots, v - 2$ through all the non-zero marks of our field. The number of the pairs γ, δ being $k(k - 1)$ our assertion is proved.

As a further check we remark that the number of blocks in the design should be $\lambda v(v - 1)/(k(k - 1))$. In our case $\lambda = k(k - 1)$ and the number of blocks is as it should be $v(v - 1)$.

In the same way it may be proved:

(4.3) *If* $v = p^\alpha$, *and* q *is the greatest common factor of* $(v - 1)$ *and* k, *then* $v \varepsilon B(k, k(k - 1)/q)$.

The blocks are:

$$\{(g + x^{\beta+\gamma+\delta}) : \gamma = 0, (v-1)/q, 2(v-1)/q, \cdots, (q-1)(v-1)/q;$$
$$\delta = 0, 1, \cdots, k/q - 1\}, \qquad \beta = 0, 1, \cdots, (v-1)/q - 1.$$

(4.4) *If $v = p^\alpha$, q is the greatest common factor of $(v-1)$ and k, and 2 is a common factor of $(v-1)$ and $(k-1)$, then $v \varepsilon B(k, k(k-1)/(2q))$.*

The blocks are:

$$\{(g + x^{\beta+\gamma+\delta}) : \gamma = 0, (v-1)/q, \ 2(v-1)/q, \cdots, (q-1)(v-1)/q;$$
$$\delta = 0, 1, \cdots, k/q - 1\}, \qquad \beta = 0, 1, \cdots, (v-1)/(2q) - 1.$$

(4.5) *If $v = p^\alpha$ and q is the greatest common factor of $(v-1)$ and $(k-1)$, then $v \varepsilon B(k, k(k-1)/q)$.*

The blocks are:

$$\{(g), (g + x^{\beta+\gamma+\delta}) : \gamma = 0, (v-1)/q, \ 2(v-1)/q, \cdots, (q-1)(v-1)/q;$$
$$\delta = 0, 1, \cdots, (k-1)/q - 1\}, \qquad \beta = 0, 1, \cdots, (v-1)/q - 1.$$

(4.6) *If $v = p^\alpha$, q is the greatest common factor of $(v-1)$ and $(k-1)$, and 2 is a common factor of $(v-1)$ and k, then $v \varepsilon B(k, k(k-1)/(2q))$.*

The blocks are:

$$\{(g), (g + x^{\beta+\gamma+\delta}) : \gamma = 0, (v-1)/q, 2(v-1)/q, \cdots, (q-1)(v-1)/q;$$
$$\delta = 0, 1, \cdots, (k-1)/q - 1\}, \qquad \beta = 0, 1, \cdots, (v-1)/(2q) - 1.$$

5. Block designs: $k = 3$.

(5.1) THEOREM. *A necessary and sufficient condition for the existence of* BIBD *of v elements, with $k = 3$ and any λ is that*

(i) $\lambda(v-1) \equiv 0 (\text{mod } 2)$ *and* $\lambda v(v-1) \equiv 0 (\text{mod } 6)$.

PROOF. The necessity of (i) follows from (ii) Section 1. It remains to prove its sufficiency. From (i) follows that

if $\lambda \equiv 1$ or $5 (\text{mod } 6)$,	then $v \equiv 1$ or $3 (\text{mod } 6)$;
if $\lambda \equiv 2$ or $4 (\text{mod } 6)$,	then $v \equiv 0$ or $1 (\text{mod } 3)$;
if $\lambda \equiv 3 (\text{mod } 6)$,	then $v \equiv 1 (\text{mod } 2)$;
if $\lambda \equiv 0 (\text{mod } 6)$,	there are no restrictions on v.

Consequently by (3.8) it remains to be shown that

(5.2) for every $v \geqq 3$,

$v \equiv 1$ or $3 (\text{mod } 6)$	implies $v \varepsilon B(3, 1)$,
$v \equiv 0$ or $1 (\text{mod } 3)$	implies $v \varepsilon B(3, 2)$,
$v \equiv 1 (\text{mod } 2)$	implies $v \varepsilon B(3, 3)$
and for every v,	$v \varepsilon B(3, 6)$ holds.

The proof of (5.2) will be given with the help of the following lemmas:

(5.3) *If $u \equiv 0$ or $1 (\text{mod } 3)$ and $u \geqq 3$, then $u \varepsilon B(K_3^1, 1)$, where $K_3^1 = \{3, 4, 6\}$.*

The proof of this lemma is given by induction. Note that by (2.9),

$t \varepsilon T_t(3)$ whenever $t \equiv 0, 1$ or $3 (\text{mod } 4)$ and by (2.13), $t \varepsilon T_4(3)$ when $t \equiv 2 (\text{mod } 4)$ and $t \geqq 6$. Consequently $3 \varepsilon T_3(3)$ and for $t \geqq 4$, $t \varepsilon T_4(3)$. Now for for $u \varepsilon K_3^1$ our proposition is trivial and for $u = 7$ we have:

(5.3.1)*[4] $7 \varepsilon B(3, 1)$, (compare (4.4), the projective plane $PG[2, 2]$).

Elements: (i), $(i = 0, 1, \cdots, 6)$.

Blocks: $\{(i + 3^0), (i + 3^2), (i + 3^4)\}$.

For other values of u, i.e. $u \geqq 9$ makes use of (3.13) putting $K = K_3^1$, $s = 3$ and taking the values of q and t as follows:

for $u \equiv 0 (\text{mod } 9)$, $q = 0$, $t = \frac{1}{3}u$;
$u \equiv 1 (\text{mod } 9)$, $q = 1$, $t = \frac{1}{3}(u - 1)$;
$u \equiv 3 (\text{mod } 9)$, $q = 0$, $t = \frac{1}{3}u$;
$u \equiv 4 (\text{mod } 9)$, $q = 1$, $t = \frac{1}{3}(u - 1)$;
$u \equiv 6 (\text{mod } 9)$, $q = 3$, $t = \frac{1}{3}(u - 3)$;
$u \equiv 7 (\text{mod } 9)$, $q = 4$, $t = \frac{1}{3}(u - 4)$.

(5.4) *If* $u \geqq 3$, *then* $u \varepsilon B(K_3^2, 1)$ *where* $K_3^2 = \{3, 4, 5, 6, 8, 11, 14\}$.

The proof is again by induction and we make again use of (2.9) noting that $t \varepsilon T_t(3)$ whenever $t \equiv 0, 1$, or $3 (\text{mod } 4)$ and that $t \varepsilon T_t(4)$ for $t = 4, 5$ and 7. Now for $u \varepsilon K_3^2$ the proposition is trivial, for $u = 7$ see (5.3.1) and for other values of u we insert in (3.13) $K = K_3^2$ and take the values of q, s and t asfollows:

for $9 \leqq u \leqq 10$, $q = u - 9$, $s = 3$, $t = 3$;
$12 \leqq u \leqq 13$, $q = u - 12$, $s = 3$, $t = 4$;
$15 \leqq u \leqq 20$, $u \neq 17$, $q = u - 15$, $s = 3$, $t = 5$;
$u = 17$, $q = 1$, $s = 4$, $t = 4$;
$21 \leqq u \leqq 28$, $u \neq 23$, $q = u - 21$, $s = 3$, $t = 7$;
$u = 23$, $q = 3$, $s = 4$, $t = 5$;
$u = 29$, $q = 1$, $s = 4$, $t = 7$;
$30 \leqq u \leqq 36$, $q = u - 27$, $s = 3$, $t = 9$;
$37 \leqq u \leqq 44$, $q = u - 33$, $s = 3$, $t = 11$;
$45 \leqq u \leqq 50$, $q = u - 39$, $s = 3$, $t = 13$;
$u \geqq 51$, $q \equiv u (\text{mod } 12)$, $3 \leqq q \leqq 14$, $s = 3$, $t = \frac{1}{3}(u - q)$.

We now proceed to prove (5.2):

(5.5)* *If* $v \equiv 1$ *or* $3 (\text{mod } 6)$, *then* $v \varepsilon B(3, 1)$, (see also [15, 12, 18]).

For $v = 3$ this is trivial. For $v \geqq 7$ we may write $v = 2u + 1$ where u satisfies the conditions of (5.3). Putting in (3.12): $k = 3$, $K' = K_3^1$, $\lambda' = \lambda'' = 1$, it remains by (5.3) and (3.6) to be shown that $2u + 1 \varepsilon B(3, 1)$ for $u \varepsilon K_3^1$. For $u = 3$ see (5.3.1) and for $u = 4$ and 6 we have:

(5.5.1)* $9 \varepsilon B(3, 1)$, (the Euclidean plane $EG[2, 3]$).

Elements: (i, j), $(i = 0, 1, 2; j = 0, 1, 2)$.

Blocks: $\{(0, j), (1, j), (2, j)\}$, $\{(i, 0), (i, 1), (i, 2)\}$,
$\{(i, 0), (i + 2^\beta, 1), (i + 2^{\beta+1}, 2)\}$, $\beta = 0, 1$.

[4] The propositions denoted by * have been known. They are proved here for the sake of completeness and partly because the new method of proof seemed to be interesting.

(5.5.2)* $13 \varepsilon B(3, 1)$, (compare (4.4)).

Elements: (i), $(i = 0, 1, \cdots, 12)$.

Blocks: $\{(i + 2^{\beta}), (i + 2^{\beta+4}), (i + 2^{\beta+8})\}$, $\beta = 0, 1$.

(5.6)* *If* $v \equiv 0$ *or* $1 (\text{mod } 3)$, *then* $v \varepsilon B(3, 2)$, (see also [1]).

Putting in (3.9): $K' = K_3^1$, $K = \{3\}$, $\lambda' = 1$, $\lambda'' = 2$ the proposition follows from (5.3) provided that $v \varepsilon B(3, 2)$ for $v \varepsilon K_3^1$. For $v = 3$ this is trivial and for $v = 4$ and 6 we have:

(5.6.1)* $4 \varepsilon B(3, 2)$, (compare (4.3)).

Elements: (g), $(g = a_0 + a_1 x; a_i = 0, 1; i = 0, 1); x^2 = x + 1$.[5]

Blocks: $\{(g + x^0), (g + x^1), (g + x^2)\}$.

(5.6.2)* $6 \varepsilon B(3, 2)$.

Elements: (i, j), $(i = 0, 1, 2; j = 0, 1)$

Blocks: $\{(i, j + 1), (i, j), (i + 2^0, j)\}$, $\{(i, 0), (i + 2^0, 1), (i + 2^1, 1)\}$, $\{(0, 0), (2^0, 0), (2^1, 0)\}$.

(5.7) *For every* v, $v \varepsilon B(3, 6)$ *holds.*

Putting in (3.9): $K' = K_3^2$, $K = \{3\}$, $\lambda' = 1$, $\lambda'' = 6$, it remains by (5.4) to be shown that $v \varepsilon B(3, 6)$ for $v \varepsilon K_3^2$. For $v = 3$ this is trivial and for $v = 4$ and 6 this follows from (5.6.1) and (5.6.2) respectively. For other values of v we have:

(5.7.1) $5 \varepsilon B(3, 3)$, (compare (4.5)).

Elements: (i), $(i = 0, 1, 2, 3, 4)$.

Blocks: $\{(i), (i + 2^{\beta}), (i + 2^{\beta+2})\}$, $\beta = 0, 1$.

(5.7.2) $8 \varepsilon B(3, 6)$, (compare (4.2)).

Elements: (g), $(g = a_0 + a_1 x + a_2 x^2; a_i = 0, 1; i = 0, 1, 2); x^3 = x + 1$.

Blocks: $\{(g + x^{\beta}), (g + x^{\beta+1}), (g + x^{\beta+2})\}$, $\beta = 0, 1, \cdots, 6$.

(5.7.3) $11 \varepsilon B_3(3, 3)$.

Put in (3.12): $u = 5$, $K' = \{3\}$, $k = 3$, $\lambda' = 3$, $\lambda'' = 1$ and make use of (5.7.1) and (5.3.1) with (3.6).

(5.7.4) $14 \varepsilon B(3, 6)$.

Elements: (i), $(i = 0, 1, \cdots, 12)$ and (A).

Blocks: $\{(1 + 2^{\beta}), (i + 2^{\beta+4}), (i + 2^{\beta+8})\}$, $\beta = 0, 1$, taken 5 times,
$\{(i + 2^1), (i + 2^5), (i + 2^9)\}$,
$\{(A), (i + 2^{2\gamma}), (i + 2^{2\gamma+6})\}$, $\gamma = 0, 1, 2$.

(5.8) *If* $v \equiv 1 (\text{mod } 2)$, *then* $v \varepsilon B(3, 3)$.

For $v = 3$ this is trivial, for $v = 5$ see (5.7.1). For $v \geqq 7$ we have $v = 2u + 1$, where u satisfies the conditions of (5.4). Putting in (3.12): $k = 3$, $K' = K_3^2$, $\lambda' = 1$, $\lambda'' = 3$ it remains by (5.4) to be shown that $2u + 1 \varepsilon B_3(3, 3)$ for $u \varepsilon K_3^2$. Making use of (3.6) and (3.8) this follows for $u = 3$, 4, 5 and 6 from (5.3.1), (5.5.1), (5.7.3) and (5.5.2) respectively. For $u = 8$ we have

(5.8.1)* $8 \varepsilon B(4, 3)$, (see e.g. [3] p. 429 and [7]).

Elements: (i, j), $(i = 0, 1, 2, 3; j = 0, 1.)$

Blocks: $\{(0, b_0), (1, b_1), (2, b_2), (3, b_3)\}$, $\Sigma b_i = 0 (\text{mod } 2)$,
$\{(i, 0), (i, 1), (i', 0), (i', 1)\}$, $i < i'$.

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[5] The primitive marks throughout this paper are taken from [3] p. 262 and the primitive roots from [19].

Now 17 $\varepsilon\ B_3(3, 3)$ follows from (3.12) by putting $u = 8$, $K' = \{4\}$, $k = 3$, $\lambda' = 3$, $\lambda'' = 1$ and applying on (5.8.1) and (5.5.1). To prove 23 $\varepsilon\ B_3(3, 3)$ put in (3.12): $u = 11$, $K' = \{3\}$, $k = 3$, $\lambda' = 3$, $\lambda'' = 1$, then use (5.7.3) and (5.3.1). For $u = 14$ we show

(5.8.2) 14 $\varepsilon\ B(\{3, 4\}, 3)$.

Elements: (i), $(i = 0, 1, \cdots, 12)$ and (A).

Blocks: $\{(i + 2^\gamma), (i + 2^{\gamma+1}), (i + 2^{\gamma+5}), (i + 2^{\gamma+6})\}$, $\gamma = 0, 1$,
$\{(A), (i + 2^0), (i + 2^4), (i + 2^8)\}$, $\{(i + 2^1), (i + 2^5), (i + 2^9)\}$.

29 $\varepsilon\ B_3(3, 3)$ is obtained by putting in (3.12): $u = 14$, $K' = \{3, 4\}$, $k = 3$, $\lambda' = 3$, $\lambda'' = 1$ and applying to (5.8.2), (5.3.1) and (5.5.1).

6. Block designs: $k = 4$.

(6.1) THEOREM. *A necessary and sufficient condition for the existence of* BIBD *of v elements, with $k = 4$ and any λ is that*

(i) $\lambda(v - 1) \equiv 0 (\mathrm{mod}\ 3)$ *and* $\lambda v(v - 1) \equiv 0 (\mathrm{mod}\ 12)$.

PROOF. The necessity of (i) follows from (ii) Section 1. In order to prove its sufficiency we remark that from (i) follows that

if $\lambda \equiv 1$ or $5 (\mathrm{mod}\ 6)$,	then $v \equiv 1$ or $4 (\mathrm{mod}\ 12)$;
if $\lambda \equiv 2$ or $4 (\mathrm{mod}\ 6)$,	then $v \equiv 1 (\mathrm{mod}\ 3)$;
if $\lambda \equiv 3 (\mathrm{mod}\ 6)$,	then $v \equiv 0$ or $1 (\mathrm{mod}\ 4)$;
if $\lambda \equiv 0 (\mathrm{mod}\ 6)$,	there are no restrictions on v.

Consequently by (3.8) it remains to be shown that

(6.2) for every $v \geqq 4$,

$v \equiv 1$ or $4 (\mathrm{mod}\ 12)$	implies $v\ \varepsilon\ B(4, 1)$,
$v \equiv 1 (\mathrm{mod}\ 3)$	implies $v\ \varepsilon\ B(4, 2)$,
$v \equiv 0$ or $1 (\mathrm{mod}\ 4)$	implies $v\ \varepsilon\ B(4, 3)$
and for every v,	$v\ \varepsilon\ B(4, 6)$ holds.

The proof of (6.2) is analogous to that of (5.2) and will be given with the help of the following lemmas:

(6.3) *If* $u \equiv 0$ *or* $1 (\mathrm{mod}\ 4)$ *and* $u \geqq 4$, *then* $u\ \varepsilon\ B(K_4^1, 1)$ *where* $K_4^1 = \{4, 5, 8, 9, 12\}$.

The proof is given by induction. Note that by (2.9), $t\ \varepsilon\ T_t(4)$ if $t \not\equiv 2 (\mathrm{mod}\ 4)$ and $t \not\equiv 3$ and $6 (\mathrm{mod}\ 9)$, and by (2.13), $t\ \varepsilon\ T_9(4)$ if $t \not\equiv 2 (\mathrm{mod}\ 4)$, $t \equiv 3$ or $6 (\mathrm{mod}\ 9)$ and $t \geqq 12$; consequently $t\ \varepsilon\ T_t(4)$ for $t = 4$, 5 and 8, and $t\ \varepsilon\ T_9(4)$ if $t \equiv 0$ or $1 (\mathrm{mod}\ 4)$ and $t \geqq 9$. Now for $u\ \varepsilon\ K_4^1$ the lemma is trivial and for $u = 13$, 28 and 29 we have:

(6.3.1)* 13 $\varepsilon\ B(4, 1)$, (the projective plane $PG[2, 3]$).

Elements: (i), $(i = 0, 1, \cdots, 12)$.

Blocks: $\{(i + 2^0), (i + 2^1), (i + 2^5), (i + 2^6)\}$.

(6.3.2)* 28 $\varepsilon\ B(4, 1)$, (see [1]).

Elements: (i, j), $(i = 0, 1, \cdots, 6; j = 0, 1, 2, 3)$.
Blocks: $\{(i, 0), (i + 6, 1), (i + 5, 2), (i + 3, 3)\}$,
$\{(i, 0), (i + 5, 1), (i + 3, 2), (i + 6, 3)\}$,
$\{(i, 0), (i, 1), (i, 2), (i, 3)\}$,
$\{(i + 1, 0), (i + 3, 0), (i + 4, 1), (i + 5, 1)\}$,
$\{(i + 1, 1), (i + 3, 1), (i + 4, 2), (i + 5, 2)\}$,
$\{(i + 2, 2), (i + 6, 2), (i + 1, 3), (i + 3, 3)\}$,
$\{(i + 2, 3), (i + 6, 3), (i + 4, 0), (i + 5, 0)\}$,
$\{(i + 2, 0), (i + 6, 0), (i + 1, 2), (i + 3, 2)\}$,
$\{(i + 2, 1), (i + 6, 1), (i + 4, 3), (i + 5, 3)\}$.

It may be of interest to note that in this design the 63 blocks form 9 groups of 7 mutually disjoint quadruples each.

(6.3.3) $29 \varepsilon B(\{4, 5\}, 1)$.

Take 28 elements as in (6.3.2) and an additional element (A). Adjoin this element (A) to each of the 7 (mutually disjoint) quadruples

$$\{(i, 0), (i + 6, 1), (i + 5, 2), (i + 3, 3)\}$$

of (6.3.2) thus forming 7 quintuples. These quintuples together with the remaining 56 quadruples of (6.3.2) form the required block design.

For other values of u we make use of (3.13) putting $K = K_4^1$, $s = 4$ and taking for q and t the values as follows:

for $u \equiv 0 \pmod{16}$, $u \geqq 16$, $q = 0$, $t = \frac{1}{4}u$;
$u \equiv 1 \pmod{16}$, $u \geqq 17$, $q = 1$, $t = \frac{1}{4}(u - 1)$;
$u \equiv 4 \pmod{16}$, $u \geqq 20$, $q = 0$, $t = \frac{1}{4}u$;
$u \equiv 5 \pmod{16}$, $u \geqq 21$, $q = 1$, $t = \frac{1}{4}(u - 1)$;
$u \equiv 8 \pmod{16}$, $u \geqq 24$, $q = 4$, $t = \frac{1}{4}(u - 4)$;
$u \equiv 9 \pmod{16}$, $u \geqq 25$, $q = 5$, $t = \frac{1}{4}(u - 5)$;
$u = 12 \pmod{16}$, $u \geqq 44$, $q = 8$, $t = \frac{1}{4}(u - 8)$;
$u = 13 \pmod{16}$, $u \geqq 45$, $q = 9$, $t = \frac{1}{4}(u - 9)$.

(6.4) *If* $u \geqq 4$, *then* $u \varepsilon B(K_4^2, 1)$ *where*

$$K_4^2 = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 22, 23\}.$$

The proof is by induction. We shall make use of (2.9) and especially of the fact that $t \varepsilon T_t(4)$ if $t \not\equiv 2 \pmod 4$ and $t \not\equiv 3$ and $6 \pmod 9$, further $t \varepsilon T_t(5)$ for $t = 5$, 8 and 13 and $9 \varepsilon T_9(7)$. For $u \varepsilon K_4^2$ the proposition is trivial, for $u = 13$ see (6.3.1) and for $u = 27$ and 31 we have:

(6.4.1)* $31 \varepsilon B(6, 1)$, (the projective plane $PG[2, 5]$).
Elements: (i), $(i = 0, 1, \cdots, 30)$.
Blocks: $\{(i + 3^0), (i + 3^1), (i + 3^2), (i + 3^3), (i + 3^{11}), (i + 3^{19})\}$.

(6.4.2) $27 \varepsilon B(\{4, 5, 6\}, 1)$.
Delete from the block design (6.4.1) any 4 elements no 3 of which are "collinear", e.g., the elements (27), (28), (29) and (30).

For other values of u make use of (3.13) putting $K = K_4^2$ and taking for q,

s and t the values as follows:

u	q	s	t	u	q	s	t
$16 \leqq u \leqq 17$	$u - 16$	4	4	$68 \leqq u \leqq 79$	$u - 64$	4	16
$20 \leqq u \leqq 21$	$u - 20$	4	5	$80 \leqq u \leqq 95$	$u - 76$	4	19
$u = 24$	4	4	5	$96 \leqq u \leqq 103$	$u - 92$	4	23
$25 \leqq u \leqq 26$	$u - 25$	5	5	$104 \leqq u \leqq 119$	$u - 100$	4	25
$28 \leqq u \leqq 29$	$u - 28$	4	7	$120 \leqq u \leqq 143$	$u - 116$	4	29
$u = 30$	5	5	5	$144 \leqq u \leqq 175$	$u - 140$	4	35
$32 \leqq u \leqq 35$	$u - 28$	4	7	$176 \leqq u \leqq 211$	$u - 172$	4	43
$36 \leqq u \leqq 39$	$u - 32$	4	8	$212 \leqq u \leqq 259$	$u - 208$	4	52
$40 \leqq u \leqq 45$	$u - 36$	4	9	$260 \leqq u \leqq 319$	$u - 256$	4	64
$46 \leqq u \leqq 47$	$u - 40$	5	8	$320 \leqq u \leqq 391$	$u - 316$	4	79
$48 \leqq u \leqq 55$	$u - 44$	4	11	$392 \leqq u \leqq 479$	$u - 388$	4	97
$56 \leqq u \leqq 65$	$u - 52$	4	13	$480 \leqq u \leqq 583$	$u - 476$	4	119
$u = 66$	1	5	13	$584 \leqq u \leqq 723$	$u - 580$	4	145
$u = 67$	4	7	9	$u \geqq 724$	$\begin{cases} u \pmod{144} \\ 4 \leqq q \leqq 147 \end{cases}$	4	$\frac{1}{4}(u - q)$

We are now able to prove (6.2).

(6.5) *If* $v \equiv 1$ *or* $4 \pmod{12}$, *then* $v \; \varepsilon \; B(4, 1)$.

For $v = 4$ this is trivial. For $v \geqq 13$ we may put $v = 3u + 1$ where u satisfies the conditions of (6.3). Putting in (3.12): $k = 4, K' = K_4^1, \lambda' = \lambda'' = 1$ it remains by (6.3) and (3.6) to be shown that $3u + 1 \; \varepsilon \; B(4, 1)$ for $u \; \varepsilon \; K_4^1$. For $u = 4$ and 9 this is proved in (6.3.1) and (6.3.2) respectively and for $u = 5$, 8 and 12 we have:

(6.5.1)* $16 \; \varepsilon \; B(4, 1)$, (the Euclidean plane $EG[2, 4]$).

Elements: $(g, j), (g = a_0 + a_1 x; a_i = 0, 1; i = 0, 1; j = 0, 1, 2, 3); x^2 = x + 1$.

Blocks: $\{(0, j), (x^0, j), (x^1, j), (x^2, j)\}, \{(g, 0), (g, 1), (g, 2), (g, 3)\}$,
$\{(g, 0), (g + x^\beta, 1), (g + x^{\beta+1}, 2), (g + x^{\beta+2}, 3)\}, \quad \beta = 0, 1, 2$.

(6.5.2)* $25 \; \varepsilon \; B(4, 1)$, (see [1]).

Elements: $(g), (g = a_0 + a_1 x; a_i = 0, 1, 2, 3, 4; i = 0, 1); x^2 = 2x + 2$.

Blocks: $\{(g), (g + x^{2\gamma}), (g + x^{2\gamma+8}), (g + x^{2\gamma+16})\}, \quad \gamma = 0, 1$.

(6.5.3) $37 \; \varepsilon \; B(4, 1)$.

Elements: $(i), (i = 0, 1, \cdots, 36)$.

Blocks: $\{(i), (i + 2^{12\beta}), (i + 2^{12\beta+11}), (i + 2^{12\beta+14})\}, \quad \beta = 0, 1, 2$.

(6.6) *If* $v \equiv 0$ *or* $1 \pmod 4$, *then* $v \; \varepsilon \; B(4, 3)$.

Putting in (3.9): $K' = K_4^1, K = \{4\}, \lambda' = 1, \lambda'' = 3$, this proposition follows from (6.3) provided that $v \; \varepsilon \; B(4, 3)$ for $v \; \varepsilon \; K_4^1$. For $v = 4$ this is trivial and for $v = 8$ it is proved in (5.8.1). For other values of v we have:

(6.6.1) $5 \; \varepsilon \; B(4, 3)$, (compare (4.3)).

Elements: $(i), (i = 0, 1, 2, 3, 4)$.

Blocks: $\{(i + 2^0), (i + 2^1), (i + 2^2), (i + 2^3)\}$.

(6.6.2) $9 \; \varepsilon \; B(4, 3)$, (compare (4.3)).

Elements: $(g), (g = a_0 + a_1 x; a_i = 0, 1, 2; i = 0, 1); x^2 = 2x + 1$.

Blocks: $\{(g + x^{\beta}), (g + x^{\beta+2}), (g + x^{\beta+4}), (g + x^{\beta+6})\}$, $\beta = 0, 1$.

(6.6.3) $12 \varepsilon B(4, 3)$.

Elements: $(i, j), (i = 0, 1, 2; j = 0, 1, 2, 3)$.

Blocks: $\{(i + 2^0, j), (i + 2^1, j), (i, j + 1), (i + 2^{\gamma}, j + 3)\}$, $\gamma = 0, 1$,
$\{(i + 2^0, j), (i + 2^1, j), (i + 2^0, j + 2), (i + 2^1, j + 2)\}$, $j = 0, 1$,
$\{(i, 0), (i, 1), (i, 2), (i, 3)\}$.

(6.7) *For every v, $v \varepsilon B(4, 6)$ holds.*

Putting in (3.9): $K' = K_4^2$, $K = \{4\}$, $\lambda' = 1$, $\lambda'' = 6$, the proposition follows from (6.4) provided that $v \varepsilon B(4, 6)$ for $v \varepsilon K_4^2$. For $v = 4$ this is trivial and for $v = 5, 8, 9$ and 12 it follows from (6.6.1), (5.8.1), (6.6.2) and (6.6.3) respectively. For other values of v we have:

(6.7.1) $6 \varepsilon B(4, 6)$.

Elements: $(i, j), (i = 0, 1, 2; j = 0, 1)$.

Blocks: $\{(i, j + 1), (i, j), (i + 2^0, j), (i + 2^1, j)\}$,
$\{(i, j), (i + 2^0, j), (i, j + 1), (i + 2^1, j + 1)\}$,
$\{(i + 2^0, 0), (i + 2^1, 0), (i + 2^0, 1), (i + 2^1, 1)\}$.

(6.7.2) $7 \varepsilon B(4, 2)$, (compare (4.6)).

Elements: $(i), (i = 0, 1, \cdots, 6)$.

Blocks: $\{(i), (i + 3^0), (i + 3^2), (i + 3^4)\}$.

(6.7.3) $10 \varepsilon B(4, 2)$.

Elements: $(i, j), (i = 0, 1, 2, 3, 4; j = 0, 1)$.

Blocks: $\{(i, 1), (i + 2^0, 0), (i + 2^1, 0), (i + 2^3, 0)\}$,
$\{(i, 0), (i, 1), (i + 2^0, 1), (i + 2^3, 1)\}$,
$\{(i, 0), (i + 2^3, 0), (i, 1), (i + 2^2, 1)\}$.

(6.7.4) $11 \varepsilon B(4, 6)$, (compare (4.3)).

Elements: $(i), (i = 0, 1, \cdots, 10)$.

Blocks: $\{(i + 2^{\beta}), (i + 2^{\beta+1}), (i + 2^{\beta+5}), (i + 2^{\beta+6})\}$, $\beta = 0, 1, 2, 3, 4$.

(6.7.5) $14 \varepsilon B(4, 6)$, (see also [9]).

Elements: $(i, j), (i = 0, 1, \cdots, 6; j = 0, 1)$.

Blocks: $\{(i, j + 1), (i + 3^0, j), (i + 3^2, j), (i + 3^4, j)\}$, 5 times,
$\{(i, 0), (i + 3^{2\beta}, 0), (i, 1), (i + 3^{2\beta}, 1)\}$, $\beta = 0, 1, 2$.

(6.7.6) $15 \varepsilon B(4, 6)$.

Elements: $(i, j), (i = 0, 1, 2, 3, 4; j = 0, 1, 2)$.

Blocks: $\{(i + 2^0, j), (i + 2^1, j), (i, j + 1), (i + 2^3, j + 1)\}$, 5 times,
$\{(i, j), (i + 2^{\beta}, j), (i, j + 1), (i, j + 2)\}$, $\beta = 0, 1$.

(6.7.7) $18 \varepsilon B(\{4, 5\}, 2)$.

Elements: $(g, j), (g = a_0 + a_1 x; a_i = 0, 1, 2; i = 0, 1; j = 0, 1)$;
$x^2 = 2x + 1$.

Blocks: $\{(g, 1), (g + x^0, 0), (g + x^2, 0), (g + x^4, 0), (g + x^6, 0)\}$,
$\{(g, 0), (g, 1), (g + x^{2\beta+1}, 1), (g + x^{2\beta+3}, 1)\}$, $\beta = 0, 1$,
$\{(g, 0), (g + x^{2\beta+2}, 0), (g + x^{2\beta+6}, 1), (g + x^{2\beta+7}, 1)\}$, $\beta = 0, 1$.

$18 \varepsilon B(4, 6)$ follows from (3.9) with $K' = \{4, 5\}$, $K = \{4\}$, $\lambda' = 2$, $\lambda'' = 3$ applied to (6.6.1).

(6.7.8) $19 \varepsilon B_4(4, 2)$.

Elements: (i, j, h), $(i = 0, 1, 2; j = 0, 1; h = 0, 1, 2)$ and (A).

Blocks: $\{(A), (i, j, 0), (i, j, 1), (i, j, 2)\}$, twice,

(these blocks show that $19 \varepsilon B_4$),

$\{(i + 2^0, j, h), (i + 2^1, j, h), (i, j + 1, h), (i, j, h + 1)\}$,

$\{(i + 2^0, j, h), (i + 2^1, j, h), (i + 2^0, j + 1, h + 1), (i, j + 1, h + 2)\}$,

$\{(i, 0, h), (i, 1, h), (i + 2^0, 0, h + 1), (i + 2^0, 1, h + 1)\}$.

(6.7.9) $22 \varepsilon B_4(4, 2)$.

Put in (3.12): $u = 7$, $K' = \{4\}$, $k = 4$, $\lambda' = 2$, $\lambda'' = 1$ and apply to (6.7.2) and (6.3.1).

(6.7.10) $23 \varepsilon B(4, 6)$, (compare (4.3)).

Elements: (i), $(i = 0, 1, \cdots, 22)$.

Blocks: $\{(i + 5^\beta), (i + 5^{\beta+1}), (i + 5^{\beta+11}), (i + 5^{\beta+12})\}$, $\beta = 0, 1, \cdots, 10$.

(6.8) *If* $v \equiv 1 (\text{mod } 3)$, *then* $v \varepsilon B(4, 2)$.

For $v = 4$ this is trivial, for $v = 7$ and 10 it is proved in (6.7.2) and (6.7.3) respectively. For $v \geqq 13$ we may put $v = 3u + 1$ where u satisfies the conditions of (6.4). Putting in (3.12): $k = 4$, $K' = K_4^2$, $\lambda' = 1$, $\lambda'' = 2$ it remains by (6.4) to be shown that $3u + 1 \varepsilon B_4(4.2)$ for $u \varepsilon K_4^2$. Now for $u = 4, 5, 6, 7, 8, 9$ and 12 this follows from (6.3.1), (6.5.1), (6.7.8), (6.7.9), (6.5.2), (6.3.2) and (6.5.3) respectively; for $u = 10$, 19 and 22 we put in (3.12): $k = 4$, $K' = \{4\}$, $\lambda' = 2$, $\lambda'' = 1$ and apply to (6.3.1) and to (6.7.3), (6.7.8) and (6.7.9) respectively; for $u = 18$ we put in (3.12): $k = 4$, $K' = \{4, 5\}$, $\lambda' = 2$, $\lambda'' = 1$ and apply to (6.7.7), (6.3.1) and (6.5.1). For other values of u namely $u = 11, 14, 15$ and 23 we have:

(6.8.1) $11 \varepsilon B(5, 2)$, (compare (4.4)).

Elements: (i), $(i = 0, 1, \cdots, 10)$.

Blocks: $\{(i + 2^0), (i + 2^2), (i + 2^4), (i + 2^6), (i + 2^8)\}$.

$34 \varepsilon B_4(4, 2)$ follows from (3.12) with $u = 11$, $k = 4$, $K' = \{5\}$, $\lambda' = 2$, $\lambda'' = 1$ applied to (6.5.1).

(6.8.2) $43 \varepsilon B_4(4, 2)$.

Elements: (i, j, h), $(i = 0, 1, \cdots, 6; j = 0, 1; h = 0, 1, 2)$ and (A).

Blocks: $\{(A), (i, j, 0), (i, j, 1), (i, j, 2)\}$, twice,

(these blocks show that $43 \varepsilon B_4$),

$\{(i + 3^{2\beta}, j, h), (i + 3^{2\beta+1}, j, h), (i, j, h + 1 - 2j), (i, j + 1, h + (1 - 2j)(1 + \beta))\}$, $\beta = 0, 1, 2$,

$\{(i, j + 1, h + 2), (i + 3^{3\gamma}, j, h + 1 - 2j), (i + 3^{3\gamma+2}, j, h), (i + 3^{3\gamma+4}, j, h - 1 + 2j)\}$, $\gamma = 0, 1$,

$\{(i + 3^\beta, 0, h), (i + 3^{\beta+3}, 0, h), (i + 3^{\beta+2}, 1, h + 2 - \beta), (i + 3^{\beta+5}, 1, h + 2 - \beta)\}$, $\beta = 0, 1, 2$.

(6.8.3) $46 \varepsilon B_4(4, 2)$.

Elements: (i, j, h), $(i = 0, 1, 2, 3, 4; j = 0, 1, 2; h = 0, 1, 2)$ and (A).

Blocks: $\{(A), (i, j, 0), (i, j, 1), (i, j, 2)\}$, twice,
(these blocks show that $46 \varepsilon B_4$),
$\{(i + 2^\beta, j, h), (i + 2^\beta, j + 1, h + \delta), (i + 2^{\beta+1}, j + 2, h + 2\delta), (i, j + 2, \delta)\}$, $\beta = 0, 1; \delta = 0, 1, 2$,
$\{(i, j, h), (i + 2^0, j, h), (i + 2^1, j, h + 1), (i + 2^2, j, h + 1)\}$.

(6.8.4) $70 \varepsilon B_4(4, 2)$.

Elements: (i, j), $(i = 0, 1, \cdots, 22; j = 0, 1, 2)$ and (A).

Blocks: $\{(A), (i, 0), (i, 1), (i, 2)\}$, twice,
(these blocks show that $70 \varepsilon B_4$),
$\{(i + 5^\beta, j), (i + 5^{\beta+11}, j), (i + 5^{\beta+1}, j + 1), (i + 5^{\beta+12}, j + 1)\}$, $\beta = 0, 1, \cdots, 10$.

7. On block designs $k > 4$. In this section we shall prove some general theorems which will enable us to show by induction the existence of BIBD for some given k and λ and an infinite set of values of v, *provided* that for some fixed finite subset of values of v such designs exist.

To give some example we shall thereafter use those theorems for discussing the case $k = 5$.

(7.1) *Let $a \geqq 2$, $d \geqq 2$ and $m \geqq 2$ be integers and let R be a set of some residue classes modulo d with $0 \varepsilon R$. Then there exists an integer n such that for every u satisfying $u \varepsilon R(\text{mod } d)$ and $u \geqq m$, $u \varepsilon B(K(a, d, R; m, n), 1)$ holds, where $K(a, d, R; m, n) = \{a, a + 1, x : x \varepsilon R(\text{mod } d) \text{ and } m \leqq x < n\}$.*

Let p_i, $i = 1, 2, \cdots, h$, be the primes $p_i \leqq a$ and α_i, $i = 1, 2, \cdots, h$, the smallest integers satisfying $p_i^{\alpha_i} \geqq a$; further let N be the smallest common multiple of $\prod_{i=1}^h p_i^{\alpha_i}$ and d, and δ the smallest integer satisfying $\delta N \geqq m$. We take $n = a(a + \delta)N + m$ and obtain the proof of our proposition by induction. For $u \varepsilon K(a, d, R; m, n)$ the proposition holds trivially and for $u \varepsilon R(\text{mod } d)$, $u \geqq n$ we make use of (3.13) putting $q \equiv u(\text{mod } aN)$, $m \leqq q < m + aN$; $s = a$, $t = a^{-1}(u - q)$ and $K = K(a, d, R; m, n)$. The conditions of (3.13) are satisfied because by definition $a \varepsilon K$ and $a + 1 \varepsilon K$, further $q \equiv u(\text{mod } d)$ because d is a factor of N, also $m \leqq q < m + aN < n$ and consequently $q \varepsilon K$. As for t we have $t \geqq q$ and by (2.9), $t \varepsilon T_t(a)$, consequently by (2.4), $t \varepsilon T_q(a)$; we have also $t \equiv 0(\text{mod } d) \varepsilon R$ (mod d) and by induction assumption we may put $t \varepsilon B(K, 1)$.

In the sequel we shall use (7.1) with the values $a = d = m = k$, $\delta = 1$ exclusively. Now the set $K(k, k, R; k, n)$ has a large number of elements and is therefore inconvenient in applications. We can however by methods of Section 3 and especially proposition (3.13) reduce this set to its subset

$$K(k, R) \subset K(k, k, R; k, n)$$

with relatively few elements. We obtain thus from (7.1):

(7.2) *Let R be a set of some residue classes modulo k with $0 \varepsilon R$. Then there exists a finite set $K(k, R)$ of integers (which includes the integers k and $k + 1$ and whose* *all other elements belong to* $R(\text{mod } k)$) *such that for every u satisfying* $u \varepsilon R(\text{mod } k)$ *and* $u \geqq k$, $u \varepsilon B(K(k, R), 1)$ *holds.*

From (7.2) and from (3.9), (3.12) and (3.11) respectively we obtain (with notation of (7.2)):

(7.3) *If for every* $k' \varepsilon K(k, R)$, $k' \varepsilon B(k, \lambda)$ *holds then for every* $v \varepsilon R(\text{mod } k)$, $v \varepsilon B(k, \lambda)$ *holds as well.*

(7.4) *If* $v = (k - 1)u + 1$ *where* $u \varepsilon R(\text{mod } k)$ *and* $u \geqq k$ *and if for every* $k' \varepsilon K(k, R)$, $(k - 1)\,k' + 1 \,\varepsilon B_k(k, \lambda)$ *holds, then* $v \varepsilon B_k(k, \lambda)$.

(7.5) *If* $v = ku$, *where* $u \varepsilon R(\text{mod } k)$ *and* $u \geqq k$ *and if for every* $k' \varepsilon K(k, R)$, $kk' \varepsilon B_k'(k, \lambda)$ *holds, then* $v \varepsilon B_k'(k, \lambda)$.

(7.6) We shall now use the obtained results for finding conditions under which BIBD with $k = 5$ exist. From (ii) Section 1 follows that the necessary condition for the existence of such designs is

$$\lambda(v - 1) \equiv 0(\text{mod } 4) \quad \text{and} \quad \lambda v(v - 1) \equiv 0(\text{mod } 20).$$

For specific values of λ the necessary conditions imposed on v are accordingly:

(i) for $\lambda \equiv 1, 3, 7, 9, 11, 13, 17$ or $19(\text{mod } 20)$,	$v \equiv 1$ or $5(\text{mod } 20)$;
(ii) for $\lambda \equiv 2, 6, 14$ or $18(\text{mod } 20)$,	$v \equiv 1$ or $5(\text{mod } 10)$;
(iii) for $\lambda \equiv 4, 8, 12$ or $16(\text{mod } 20)$,	$v \equiv 0$ or $1(\text{mod } 5)$;
(iv) for $\lambda \equiv 5$ or $15(\text{mod } 20)$,	$v \equiv 1(\text{mod } 4)$;
(v) for $\lambda \equiv 10(\text{mod } 20)$,	$v \equiv 1(\text{mod } 2)$;
(vi) for $\lambda \equiv 0(\text{mod } 20)$,	every v.

We shall show that in the cases (i), (iii) and (vi) the above necessary conditions are also sufficient.[6] By (3.8) it suffices to prove the following

THEOREM.

$v \equiv 1$ or $5(\text{mod } 20)$	*implies* $v \varepsilon B(5, 1)$,
$v \equiv 0$ or $1(\text{mod } 5)$	*implies* $v \varepsilon B(5, 4)$
and for every v	$v \varepsilon B(5, 20)$ *holds.*

This is proved in (7.10),[7] (7.11) and (7.12) respectively. Regarding the case (iv) it shall be proved in (7.13) that $v \equiv 1(\text{mod } 4)$ implies $v \varepsilon B\ (5, 5)$, provided that $v \varepsilon B_5(5, 5)$ for $v = 4u + 1$, $u \varepsilon K(5, \{0, 1, 2, 3, 4\})$, (see (7.9)). Concerning the case (ii) it has been proved by Nandi [14] (see also [4, 7]) that no BIBD, B[5, 2, 15] exists which shows that in this case the necessary condition is not generally sufficient.

We begin with proving a general result, namely

(7.7) $K(5, R) \subset \{x : 5 \leqq x \leqq 579\}$ *for every R.*

By definition $0 \varepsilon R$. We make use of (3.13) by putting $s = 5$ and taking $5 \leqq q \leqq t$, $t \equiv 0(\text{mod } 5)$, $t \not\equiv 2, 4, 6(\text{mod } 8)$, $t \not\equiv 3, 6(\text{mod } 9)$. For $u \geqq 580$ we

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[6] With the possible exception of $v = 141$ in the case (i).

[7] *Ibid.*

put accordingly the values of q and t as follows:

u	q	t	u	q	t
$580 \leq u \leq 690$	$u - 575$	115	$2851 \leq u \leq 3390$	$u - 2825$	565
$691 \leq u \leq 810$	$u - 675$	135	$3391 \leq u \leq 4050$	$u - 3375$	675
$811 \leq u \leq 960$	$u - 800$	160	$4051 \leq u \leq 4830$	$u - 4025$	805
$961 \leq u \leq 1110$	$u - 925$	185	$4831 \leq u \leq 5790$	$u - 4825$	965
$1111 \leq u \leq 1290$	$u - 1075$	215	$5791 \leq u \leq 6870$	$u - 5725$	1145
$1291 \leq u \leq 1470$	$u - 1225$	245	$6871 \leq u \leq 8160$	$u - 6800$	1360
$1471 \leq u \leq 1680$	$u - 1400$	280	$8161 \leq u \leq 9750$	$u - 8125$	1625
$1681 \leq u \leq 2010$	$u - 1675$	335	$9751 \leq u \leq 10804$	$u - 9725$	1945
$2011 \leq u \leq 2400$	$u - 2000$	400		$u \pmod{1800}$	
$2401 \leq u \leq 2850$	$u - 2375$	475	$u \geq 10805$	$5 \leq q \leq 1804$	$\frac{1}{5}(u - q)$

(7.8) $K(5, \{0, 1\}) = \{5, 6, 10, 11, 15, 16, 20, 35, 36, 40, 70, 71, 75, 76\}$.

We shall prove, that for every $u \geq 5$ satisfying $u \equiv 0$ or $1 \pmod 5$, $u \in B(K(5, \{0, 1\}), 1)$ holds. For $u \in K(5, \{0, 1\})$ the proposition is trivial and for $u = 31$ see (6.4.1). For $u = 21$, 41 and 45 we have:

(7.8.1)* $21 \in B(5, 1)$, (the projective plane $PG[2, 4]$).

Elements: (i, j), $(i = 0, 1, \cdots, 6; j = 0, 1, 2)$.

Blocks: $\{(i + 3^0, j), (i + 3^2, j), (i + 3^4, j), (i, j + 1), (i, j + 2)\}$.

(7.8.2)* $41 \in B(5, 1)$, (see [1]).

Elements: (i), $(i = 0, 1, \cdots, 40)$.

Blocks: $\{(i + 6^{2\beta}), (i + 6^{2\beta+8}), (i + 6^{2\beta+16}), (i + 6^{2\beta+24}), (i + 6^{2\beta+32})\}$, $\beta = 0, 1$.

(7.8.3)* $45 \in B(5, 1)$, (see [1]).

Elements: (g, j), $(g = a_0 + a_1 x; a_i = 0, 1, 2; i = 0, 1; j = 0, 1, 2, 3, 4)$; $x^2 = 2x + 1$.

Blocks: $\{(g, 0), (g, 1), (g, 2), (g, 3), (g, 4)\}$,
$\{(g + x^\beta, j), (g + x^{\beta+4}, j), (g + x^{\beta+2}, j + 1), (g + x^{\beta+6}, j + 1), (g, j + 3)\}$, $\beta = 0, 1$.

For $u = 46, 50, 51$ put in (3.16): $s = 5$, $q = u - 46$, $t = 9$; for $u = 120$, 121 use (3.13) with $s = 10$, $q = u - 110$, $t = 11$; for $u = 151$ use (3.15) with $s = 6$, $t = 25$; for $u = 271$ use (3.13) with $s = 10$, $q = 21$, $t = 25$; and for $u \geq 580$ see (7.7). For other values of u, $u \equiv 0$ or $1 \pmod 5$ we make use of (3.13) with $s = 5$, putting for q and t the following values:

u	q	t	u	q	t
$25 \leq u \leq 30$	$u - 25$	5	$200 \leq u \leq 240$	$u - 200$	40
$55 \leq u \leq 66$	$u - 55$	11	$241 \leq u \leq 270$	$u - 225$	45
$80 \leq u \leq 96$	$u - 80$	16	$275 \leq u \leq 330$	$u - 275$	55
$100 \leq u \leq 116$	$u - 100$	20	$331 \leq u \leq 390$	$u - 325$	65
$125 \leq u \leq 150$	$u - 125$	25	$391 \leq u \leq 426$	$u - 355$	71
$155 \leq u \leq 186$	$u - 155$	31	$430 \leq u \leq 510$	$u - 425$	85
$190 \leq u \leq 196$	$u - 180$	36	$511 \leq u \leq 576$	$u - 505$	101

(7.9) $K(5, \{0, 1, 2, 3, 4\})$

$$= \{x(5 \leqq x \leqq 20), 22, 23, 24, 27, 28, 29, 32, 33, 34, 38, 39\}.$$

We shall prove that for every $u \geqq 5$, $u \varepsilon B(K(5, \{0, 1, 2, 3, 4\}), 1)$ holds. For $u \varepsilon K(5, \{0, 1, 2, 3, 4\})$ the proposition is trivial and for $u = 21$ and 31 see (7.8.1) and (6.4.1) respectively. For $u = 37$, 44, 49 and 58 we have:

(7.9.1) $37 \varepsilon B(\{5, 9\}, 1)$.

Elements: (i, j), $(i = 0, 1, \cdots, 6; j = 0, 1, 2, 3)$ and (A, h), $(h = 0, 1, \cdots, 8)$.

Blocks: Out of the elements (i, j), $(i = 0, 1, \cdots, 6; j = 0, 1, 2, 3)$ form the design (6.3.2) and adjoin the element (A, h) to each of the 7 disjoint quadruples of the hth group, $(h = 0, 1, \cdots, 8)$. Further form the block $\{(A, h) : h = 0, 1, \cdots, 8\}$.

(7.9.2)* $49 \varepsilon B(7, 1)$, (the Euclidean plane $EG[2, 7]$).

Elements: (i, j), $(i = 0, 1, \cdots, 6; j = 0, 1, \cdots, 6)$.

Blocks: $\{(0, j), (1, j), (2, j), (3, j), (4, j), (5, j), (6, j)\}$,
$\{(i, 0), (i, 1), (i, 2), (i, 3), (i, 4), (i, 5), (i, 6)\}$,
$\{(i, 0), (i + 3^{\beta}, 1), (i + 3^{\beta+1}, 2), (i + 3^{\beta+2}, 3), (i + 3^{\beta+3}, 4),$
$(i + 3^{\beta+4}, 5), (i + 3^{\beta+5}, 6)\}$, $\beta = 0, 1, \cdots, 5$.

(7.9.3) $44 \varepsilon B(\{5, 6, 7\}, 1$.

Delete from the design (7.9.2) any 5 elements no 3 of which are collinear, e.g. the elements: (0, 0), (0, 1), (1, 0), (1, 1), (2, 2).

(7.9.4)* $64 \varepsilon B(8, 1)$, (the Euclidean plane EG[2, 8]).

Elements: $(g, j), (g = a_0 + a_1x + a_2x^2; a_i = 0, 1; i = 0, 1, 2;$
$j = 0, 1, \cdots, 7); x^3 = x + 1$.

Blocks: $\{(0, j), (1, j), (x, j), (x^2, j), (1 + x, j), (x + x^2, j)$
$(1 + x + x^2, j), (1 + x^2, j)\}$,
$\{(g, 0), (g, 1), (g, 2), (g, 3), (g, 4), (g, 5), (g, 6), (g, 7)\}$,
$\{(g, 0), (g + x^{\beta}, 1), (g + x^{\beta+1}, 2), (g + x^{\beta+2}, 3), (g + x^{\beta+3}, 4),$
$(g + x^{\beta+4}, 5), (g + x^{\beta+5}, 6), (g + x^{\beta+6}, 7)\}$, $\beta = 0, 1, \cdots, 6$.

(7.9.5) $58 \varepsilon B(\{5, 6, 7, 8\}, 1)$.

Delete from the design (7.9.4) any 6 elements no 4 of which are collinear, e.g. the elements: (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2).

For $u \geqq 580$ see (7.7) and for all other values of u we make use of (3.13) taking for q, s and t the values as shown at top of next page.

We are now able to prove the theorem stated in (7.6):

(7.10) *If* $v \equiv 1$ *or* 5(mod 20) *and* $v \neq 141$, *then* $v \varepsilon B(5, 1)$.

For $v = 5$ the proposition is trivial. For $v \geqq 21$ we may write $v = 4u + 1$ with $u \equiv 0$ or 1(mod 5) and $u \geqq 5$. Putting in (7.4): $k = 5$, $R = \{0, 1\}$, $\lambda = 1$ and considering (3.6) it remains to be shown that $4u + 1 \varepsilon B(5, 1)$ for $u \varepsilon K(5, \{0, 1\})$, (see (7.8)). For $u = 5$, 10 and 11 this is proved in (7.8.1), (7.8.2) and (7.8.3) respectively and for other values of u we prove:

(7.10.1)* $25 \varepsilon B_5'(5, 1)$, (the Euclidean plane $EG[2, 5]$).

Elements: (i, j), $(i = 0, 1, 2, 3, 4; j = 0, 1, 2, 3, 4)$.

u	q	s	t	u	q	s	t
$25 \leqq u \leqq 26$	$u - 25$	5	5	$85 \leqq u \leqq 96$	$u - 80$	5	16
$u = 30$	5	5	5	$97 \leqq u \leqq 102$	$u - 85$	5	17
$35 \leqq u \leqq 36$	$u - 35$	5	7	$103 \leqq u \leqq 114$	$u - 95$	5	19
$40 \leqq u \leqq 42$	$u - 35$	5	7	$115 \leqq u \leqq 119$	$u - 102$	6	17
$u = 43$	1	6	7	$120 \leqq u \leqq 138$	$u - 115$	5	23
$45 \leqq u \leqq 48$	$u - 40$	5	8	$139 \leqq u \leqq 150$	$u - 125$	5	25
$50 \leqq u \leqq 54$	$u - 45$	5	9	$151 \leqq u \leqq 174$	$u - 145$	5	29
$55 \leqq u \leqq 56$	$u - 55$	5	11	$175 \leqq u \leqq 192$	$u - 160$	5	32
$u = 57$	1	7	8	$193 \leqq u \leqq 222$	$u - 185$	5	37
$u = 59$	5	6	9	$223 \leqq u \leqq 258$	$u - 215$	5	43
$60 \leqq u \leqq 66$	$u - 55$	5	11	$259 \leqq u \leqq 294$	$u - 245$	5	49
$u = 67$	1	6	11	$295 \leqq u \leqq 336$	$u - 280$	5	56
$68 \leqq u \leqq 69$	$u - 63$	7	9	$337 \leqq u \leqq 390$	$u - 325$	5	65
$70 \leqq u \leqq 78$	$u - 65$	5	13	$391 \leqq u \leqq 462$	$u - 385$	5	77
$79 \leqq u \leqq 81$	$u - 72$	8	9	$463 \leqq u \leqq 546$	$u - 455$	5	91
$82 \leqq u \leqq 84$	$u - 77$	7	11	$547 \leqq u \leqq 579$	$u - 535$	5	107

Blocks: $\{(0, j), (1, j), (2, j), (3, j); (4, j)\}$,
(these blocks show that $25 \varepsilon B_5'$).
$\{(i, 0), (i, 1), (i, 2), (i, 3), (i, 4)\}$,
$\{(i, 0), (i + 2^\beta, 1), (i + 2^{\beta+1}, 2), (i + 2^{\beta+2}, 3), (i + 2^{\beta+3}, 4)\}$,
$\beta = 0, 1, 2, 3$.

(7.10.2)* $61 \varepsilon B(5, 1)$, (see [1]).
Elements: (i), $(i = 0, 1, \cdots, 60)$.
Blocks: $\{(i + 2^{2\beta}), (i + 2^{2\beta+12}), (i + 2^{2\beta+24}), (i + 2^{2\beta+36}), (i + 2^{2\beta+48})\}$
$\beta = 0, 1, 2$.

(7.10.3)* $65 \varepsilon B(5, 1)$, (see [1]).
Elements: (i, j), $(i = 0, 1, \cdots, 12; j = 0, 1, 2, 3, 4)$.
Blocks: $\{(i, 0), (i, 1), (i, 2), (i, 3), (i, 4)\}$,
$\{(i + 2^\beta, j), (i + 2^{\beta+6}, j), (i + 2^{\beta+3}, j + 1), (i + 2^{\beta+9}, j + 1),$
$(i, j + 3)\}$, $\beta = 0, 1, 2$.

(7.10.4) $81 \varepsilon B(5, 1)$.
Elements: (g), $(g = \sum_{i=0}^{3} a_i x^i; a_i = 0, 1, 2; i = 0, 1, 2, 3)$;
$x^4 = 2x^3 + 2x^2 + x + 1$.
Blocks: $\{(g + x^{4\beta+\gamma}), (g + x^{4\beta+\gamma+16}), (g + x^{4\beta+\gamma+32}), (g + x^{4\beta+\gamma+48}),$
$(g + x^{4\beta+\gamma+64})\}$, $\beta = 0, 1; \gamma = 0, 1$.

(7.10.5) $141 \varepsilon B(5, 1)$?
So far no proof is available. On the other hand we remark that in the proof of $u \varepsilon B(K(5, \{0, 1\}), 1)$ for $u > 35$ (in Section (7.8)), we made no use of $35 \varepsilon K(5, \{0, 1\})$ and therefore the omission of proof of (7.10.5) does not impair the validity of proposition (7.10) for other values of v.

(7.10.6) $145 \varepsilon B(5, 1)$.
Elements: (i, j), $(i = 0, 1, \cdots, 28; j = 0, 1, 2, 3, 4)$.

Blocks: $\{(i, 0), (i, 1), (i, 2), (i, 3), (i, 4)\}$,
$\{(i + 2^{\beta}, j), (i + 2^{\beta+14}, j), (i + 2^{\beta+7}, j + 1), (i + 2^{\beta+21}, j + 1),$
$(i, j + 3)\}, \quad \beta = 0, 1, \cdots, 6.$

(7.10.7) $161 \varepsilon B(5, 1)$.
Elements: $(i, j), (i = 0, 1, \cdots, 22; j = 0, 1, \cdots, 6)$.
Blocks: $\{(i, j), (i + 5^{11\beta}, j + 3^{2\gamma}), (i + 5^{11\beta+4}, j + 3^{2\gamma+1}),$
$(i + 5^{11\beta+8}, j + 3^{2\gamma+4}), (i + 5^{11\beta+12}, j + 3^{2\gamma+3})\},$
$\beta = 0, 1; \gamma = 0, 1, 2,$
$\{(i, j), (i + 5^{2}, j), (i + 5^{6}, j), (i + 5^{10}, j), (i + 5^{14}, j)\},$
$\{(i + 5^{3}, j), (i + 5^{14}, j), (i, j + 3^{0}), (i, j + 3^{2}), (i, j + 3^{4})\}.$

(7.10.8) $281 \varepsilon B(5, 1)$.
Elements: $(i), (i = 0, 1, \cdots, 280)$.
Blocks: $\{(i + 3^{2\beta}), (i + 3^{2\beta+56}), (i + 3^{2\beta+112}), (i + 3^{2\beta+168}),$
$(i + 3^{2\beta+224})\}, \quad \beta = 0, 1, \cdots, 13.$

(7.10.9) $285 \varepsilon B(5, 1)$.
Elements: $(i, j), (i = 0, 1, \cdots, 55; j = 0, 1, 2, 3, 4)$ and $A, h)$,
$(h = 0, 1, 2, 3, 4)$.
Blocks: For every j, $(j = 0, 1, 2, 3, 4)$ take the 61 elements $(i, j), (i = 0, 1, \cdots, 55)$ and $(A, h), (h = 0, 1, 2, 3, 4)$ and form a design $B[5, 1, 61]$ as in (7.10.2) such that
$\{(A, 0), (A, 1), (A, 2), (A, 3), (A, 4)\}$
is one of the blocks. The union of the systems $B[5, 1, 61]$ for $j = 0, 1, 2, 3, 4$ and of the system $T_{56}[5, 56]$ with
$\tau_j = j) \{(i, : i = 0, 1, \cdots, 55\}, j = 0, 1, 2, 3, 4,$
gives the required design.

(7.10.10) $301 \varepsilon B(5, 1)$.
Elements: $(i, j), (i = 1, 2, \cdots, 60; j = 0, 1, 2, 3, 4)$ and (A).
Blocks: Consider the system $B[5, 1, 61]$ constructed in (7.10.2). For every quintuple $\{(0), (b_1), (b_2), (b_3), (b_4)\}$ of this system containing the element (0) take the set of 21 elements $(b_1, j), (b_2, j), (b_3, j), (b_4, j), (j = 0, 1, 2, 3, 4)$ and (A) and form out of them the system $B[5, 1, 21]$ as in (7.8.1).
For every quintuple $\{(a_0), (a_1), (a_2), (a_3), (a_4)\}$ of $B[5, 1, 61]$ which does not contain the element (0) form the blocks
$\{(a_0, j), (a_1, j + \alpha), (a_2, j + 2\alpha), (a_3, j + 3\alpha), (a_4, j + 4\alpha)\},$
$(j = 0, 1, 2, 3, 4; \alpha = 0, 1, 2, 3, 4)$. All the blocks so constructed together with the a. m. systems $B[5, 1, 21]$ form the required design.

(7.10.11) $305 \varepsilon B(5, 1)$.
Elements: $(i, j), (i = 0, 1, \cdots, 60; j = 0, 1, 2, 3, 4)$.
Blocks: $\{(i, 0), (i, 1), (i, 2), (i, 3), (i, 4)\}$,
$\{(i + 2^{\beta}, j), (i + 2^{\beta+30}, j), (i + 2^{\beta+15}, j + 1), (i + 2^{\beta+45}, j + 1),$
$(i, j + 3)\}, \quad \beta = 0, 1, \cdots, 14.$

(7.11) *If* $v \equiv 0$ *or* $1 \pmod 5$, *then* $v \varepsilon B(5, 4)$.

By (7.3) with $k = 5$, $R = \{0, 1\}$, $\lambda = 4$ we have to prove that $v \varepsilon B(5, 4)$ for $v \varepsilon K(5, \{0, 1\})$, (see (7.8)). For $v = 5$ this is trivial and for $v = 11$ it follows from (6.8.1). For other values of v we prove:

(7.11.1) $6 \varepsilon B(5, 4)$.

Elements: (i, j), $(i = 0, 1, 2; j = 0, 1)$.

Blocks: $\{(i, j), (i + 2^0, j), (i + 2^1, j), (i + 2^0, j + 1), (i + 2^1, j + 1)\}$.

(7.11.2) $10 \varepsilon B(5, 4)$.

Elements: (i, j), $(i = 0, 1, 2; j = 0, 1, 2)$ and (A).

Blocks: $\{(A), (i, j), (i + 1, j), (i, j + 1), (i + 1, j + 2)\}$,
$\{(0, j), (1, j), (2, j), (i, j + 1), (i + 1, j + 2)\}$.

(7.11.3) $15 \varepsilon B(5, 4)$, (for nonexistence of $B[5, 2, 15]$ see [14, 4, 7]).

Elements: (i, j), $(i = 0, 1, 2, 3, 4; j = 0, 1, 2)$.

Blocks: $\{(i, j), (i + 2, j), (i + 3, j), (i, j + 1), (i + 4, j + 2)\}$,
$\{(i, j), (i + 1, j), (i, j + 1), (i + 2, j + 1), (i, j + 2)\}$,
$\{(i, 0), (i + 2, 1), (i + 3, 1), (i + 4, 1), (i + 1, 2)\}$,
$\{(i, 0), (i + 1, 0), (i + 2, 1), (i + 3, 2), (i + 4, 2)\}$,
$\{(0, \alpha), (1, \alpha), (2, \alpha), (3, \alpha), (4, \alpha)\}$, $\alpha = 0, 2$.

(7.11.4) $16 \varepsilon B(5, 4)$, (compare (4.3)).

Elements: (g), $(g = \sum_{i=0}^{3} a_i x^i; a_i = 0, 1; i = 0, 1, 2, 3)$; $x^4 = x + 1$.

Blocks: $\{(g + x^{\beta}), (g + x^{\beta+3}), (g + x^{\beta+6}), (g + x^{\beta+9}), (g + x^{\beta+12})\}$, $\beta = 0, 1, 2$.

(7.11.5) $20 \varepsilon B(5, 4)$.

Elements: (i, j), $(i = 0, 1, 2, 3, 4; j = 0, 1, 2, 3)$.

Blocks: $\{(i, j), (i + 4, j), (i, j + 1), (i + 2, j + 1), (i, j + 2)\}$,
$\{(i, j), (i + 1, j), (i, j + 1), (i + 3, j + 1), (i + 1, j + 3)\}$,
$\{(i, j), (i + 4, j), (i + 1, j + 1), (i, j + 2), (i + 2, j + 2)\}$,
$\{(i, \alpha), (i + 1, \alpha), (i + 2, \alpha + 1), (i + 4, \alpha + 1), (i + 3, 3)\}$,
$\alpha = 0, 1, 2$; for $\alpha = 2$, take $\alpha + 1 = 0$.
$\{(0, 3), (1, 3), (2, 3), (3, 3), (4, 3)\}$.

(7.11.6) $35 \varepsilon B(5, 2)$.

Elements: (i, j), $(i = 0, 1, \cdots, 6; j = 0, 1, 2, 3, 4)$.

Blocks: $\{(i, 0), (i, 1), (i, 2), (i, 3), (i, 4)\}$, twice,
$\{(i + 3^{\beta}, j), (i + 3^{\beta+3}, j), (i + 3^{\beta+1}, j + 1), (i + 3^{\beta+4}, j + 1)$
$(i, j + 3)\}$, $\beta = 0, 1, 2$.

(7.11.7) $36 \varepsilon B(5, 4)$.

Elements: (g, j), $(g = a_0 + a_1 x; a_1 = 0, 1, 2; i = 0, 1; j = 0, 1, 2, 3)$;
$x^2 = 2x + 1$.

Blocks: $\{(g + x^{2\beta}, j), (g + x^{2\beta+2}, j), (g + x^{2\beta+4}, j + 1), (g, j + 2)$,
$(g + x^{2\beta+6}, j + 3)\}$, $\beta = 0, 1, 2, 3$,
$\{(g + x^{2\gamma+1}, j), (g + x^{2\gamma+3}, j), (g, j + 1), (g, j + 2), (g, j + 3)\}$,
$\gamma = 0, 1$,
$\{(g, j), (g + x^0, j), (g + x^2, j), (g + x^4, j), (g + x^6, j)\}$.

(7.11.8) $40 \varepsilon B(5, 4)$.

Elements: $(g, j), (g = a_0 + a_1x + a_2x^2; a_1 = 0, 1; i = 0, 1, 2;$
$j = 0, 1, 2, 3, 4); x^3 = x + 1$.

Blocks: $\{(g, 0), (g, 1), (g, 2), (g, 3), (g, 4)\}$, 4 times,
$\{(g + x^\beta, j), (g + x^{\beta+1}, j), (g + x^{\beta+2}, j + 1), (g + x^{\beta+3}, j + 1),$
$(g, j + 3)\}, \quad \beta = 0, 1, \cdots, 6$.

(7.11.9) $70 \varepsilon B(5, 4)$.

Elements: $(i, j, h), (i = 0, 1, 2, 3, 4; j = 0, 1; h = 0, 1, \cdots, 6)$.

Blocks: For every h, $(h = 0, 1, \cdots, 6)$ form the blocks
$\{(a_0, h), (a_1, h), (a_2, h), (a_3, h), (a_4, h)\}$,
where $\{(a_0), (a_1), (a_2), (a_3), (a_4)\}$ are blocks of the design $B[5, 4, 10]$ formed out of the elements (i, j), $(i = 0, 1, 2, 3, 4; j = 0, 1)$, (see (7.11.2)). Further form the blocks:
$\{(i, j, h), (i + 2^{2\gamma+1}, j + \delta, h + 3^{2\beta}), (i + 2^{2\gamma+1}, j + \delta, h + 3^{2\beta+3})$
$(i + 2^{2\delta}, j + \gamma, h + 3^{2\beta+1}), (i + 2^{2\delta}, j + \gamma + 1, h + 3^{2\beta+4})\}$
$\beta = 0, 1, 2; \gamma = 0, 1; \delta = 0, 1$.

(7.11.10) $71 \varepsilon B(5, 2)$, (compare (4.4)).

Elements: $(i), (i = 0, 1, \cdots, 70)$.

Blocks: $\{(i + 7^\beta), (i + 7^{\beta+14}), (i + 7^{\beta+28}), (i + 7^{\beta+42}), (i + 7^{\beta+56})\}$,
$\beta = 0, 1, \cdots, 6$.

(7.11.11) $75 \varepsilon B_5'(5, 4)$.

Put in (3.11): $m = 5$, $u = 15$, $K' = K = \{5\}$, $\lambda' = 4$, $\lambda'' = 1$ and apply to (7.11.3) and (7.10.1).

(7.11.12) $76 \varepsilon B(5, 4)$.

Elements: $(i, j), (i = 0, 1, \cdots, 14; j = 0, 1, 2, 3, 4)$ and (A).

Blocks: Apply the design (7.11.11) to the elements (i, j), $(i = 0, 1, \cdots, 14; j = 0, 1, 2, 3, 4)$. The design may be arranged in such a way that among the blocks should appear the quintuples
$\{(i, 0), (i, 1), (i, 2), (i, 3), (i, 4)\}, \quad i = 0, 1, \cdots, 14,$
four times each. Leave all other blocks of (7.11.11) without change and instead of the block $\{(i, 0), (i, 1), (i, 2), (i, 3), (i, 4)\}$ taken 4 times take the design (7.11.1) on the elements:
$(A), (i, 0), (i, 1), (i, 2), (i, 3), (i, 4), \quad i = 0, 1, \cdots, 14.$

(7.12) *For every* v, $v \varepsilon B(5, 20)$ *holds.*

By (7.3) with $k = 5$, $R = \{0, 1, 2, 3, 4\}$, $\lambda = 20$ we have to prove that $v \varepsilon B(5, 20)$ for $v \varepsilon K(5, \{0, 1, 2, 3, 4\})$, (see (7.9)). For $v = 5$ this is trivial and for $v = 6, 10, 11, 15, 16$ and 20 this follows from (7.11.1), (7.11.2), (6.8.1), (7.11.3), (7.11.4) and (7.11.5) respectively. For other values of v we have:

(7.12.1) $7 \varepsilon B(5, 10)$, (compare (4.5)).

Elements: $(i), (i = 0, 1, \cdots, 6)$.

Blocks: $\{(i), (i + 3^\beta), (i + 3^{\beta+1}), (i + 3^{\beta+3}), (i + 3^{\beta+4})\}, \quad \beta = 0, 1, 2$.

(7.12.2) $8 \varepsilon B(5, 20)$, (compare (4.2)).

Elements: $(g), (g = a_0 + a_1x + a_2x^2; a_i = 0, 1; i = 0, 1, 2); x^3 = x + 1$.

Blocks: $\{(g + x^\beta), (g + x^{\beta+1}), (g + x^{\beta+2}), (g + x^{\beta+3}), (g + x^{\beta+4})\}$,
$\beta = 0, 1, \cdots, 6$.

(7.12.3) $9 \varepsilon B(5, 5)$, (compare (4.5)).

Elements: (g), $(g = a_0 + a_1x;\ a_i = 0, 1, 2;\ i = 0, 1)$; $x^2 = 2x + 1$.

Blocks: $\{(g), (g + x^\beta), (g + x^{\beta+2}), (g + x^{\beta+4}), (g + x^{\beta+6})\}$, $\beta = 0, 1$.

(7.12.4) $12 \varepsilon B(5, 20)$.

Elements: (g, j), $(g = a_0 + a_1x;\ a_i = 0, 1;\ i = 0, 1;\ j = 0, 1, 2)$; $x^2 = x + 1$.

Blocks: $\{(g + x^\beta, j), (g + x^{\beta+1}, j), (g + x^{\beta+2}, j), (g + x^\beta, j + 1), (g + x^{\beta+1}, j + 2)\}$, $\beta = 0, 1\ 2$, twice,

$\{(g + x^\beta, j), (g + x^{\beta+1}, j), (g + x^{\beta+1}, j + 1), (g + x^{\beta+2}, j + 1), (g + x^{\beta+2}, j + 2)\}$, $\beta = 0, 1, 2$,

$\{(g + x^0, j), (g + x^1, j), (g + x^2, j), (g, j + 1), (g, j + 2)\}$, twice.

(7.12.5) $13 \varepsilon B(5, 5)$, (compare (4.5)).

Elements: (i), $(i = 0, 1, \cdots, 12)$.

Blocks: $\{(i), (i + 2^\beta), (i + 2^{\beta+3}), (i + 2^{\beta+6}), (i + 2^{\beta+9})\}$, $\beta = 0, 1, 2$.

(7.12.6) $14 \varepsilon B(5, 20)$.

Elements: (i, j), $(i = 0, 1, \cdots, 6;\ j = 0, 1)$.

Blocks: $\{(i, j), (i + 3^\beta, j), (i + 3^{\beta+1}, j), (i + 3^{\beta+3}, j + 1), (i + 3^{\beta+4}, j + 1)\}$, $\beta = 0, 1, 2$, twice,

$\{(i, j), (i + 3^\beta, j), (i + 3^{\beta+3}, j), (i + 3^\beta, j + 1), (i + 3^{\beta+3}, j + 1)\}$, $\beta = 0, 1, 2$,

$\{(i, j), (i + 3^\gamma, j), (i + 3^{\gamma+2}, j), (i + 3^{\gamma+4}, j), (i, j + 1)\}$, $\gamma = 0, 1$, twice.

(7.12.7) $17 \varepsilon B(5, 5)$, (compare (4.5)).

Elements: (i), $(i = 0, 1, \cdots, 16)$.

Blocks: $\{(i), (i + 3^\beta), (i + 3^{\beta+4}), (i + 3^{\beta+8}), (i + 3^{\beta+12})\}$, $\beta = 0, 1, 2, 3$.

(7.12.8) $18 \varepsilon B(5, 20)$.

Elements: (g, j), $(g = a_0 + a_1x;\ a_i = 0, 1, 2;\ i = 0, 1;\ j = 0, 1)$; $x^2 = 2x + 1$.

Blocks: $\{(g, j), (g + x^\beta, j), (g + x^{\beta+1}, j), (g + x^{\beta+4}, j + 1), (g + x^{\beta+5}, j + 1)\}$, $\beta = 0, 1, 2, 3$, twice,

$\{(g, j), (g + x^\beta, j), (g + x^{\beta+4}, j), (g + x^\beta, j + 1), (g + x^{\beta+4}, j + 1)\}$, $\beta = 0, 1, 2, 3$,

$\{(g, j), (g + x^\gamma, j), (g + x^{\gamma+2}, j), (g + x^{\gamma+4-2\delta}, j + \delta), (g + x^{\gamma+6-2\delta}, j + \delta)\}$, $\gamma = 0, 1;\ \delta = 0, 1$,

$\{(g, j), (g + x^0, j), (g + x^2, j), (g + x^5, j + 1), (g + x^6, j + 1)\}$.

(7.12.9) $19 \varepsilon B(5, 10)$, (compare (4.5)).

Elements: (i), $(i = 0, 1, \cdots, 18)$.

Blocks: $\{(i), (i + 2^\beta), (i + 2^{\beta+1}), (i + 2^{\beta+9}), (i + 2^{\beta+10})\}$, $\beta = 0, 1, \cdots, 8$.

(7.12.10) $22 \varepsilon B(5, 20)$.

Elements: (i, j), $(i = 0, 1, \cdots, 10;\ j = 0, 1)$.

Blocks: $\{(i, j), (i + 2^{\beta} j,), (i + 2^{\beta+1}, j), (i + 2^{\beta+5}, j + 1),$
$(i + 2^{\beta+6}, j + 1)\}$, $\beta = 0, 1, 2, 3, 4$, twice,
$\{(i,j), (i + 2^{\beta}, j), (i + 2^{\beta+5}, j), (i + 2^{\beta}, j + 1), (i + 2^{\beta+5}, j + 1)\}$,
$\beta = 0, 1, 2, 3, 4$,
$\{(i + 2^{\beta}, j), (i + 2^{\beta+1}, j), (i + 2^{\beta+5}, j), (i + 2^{\beta+6}, j), (i, j + 1)\}$,
$\beta = 0, 1, 2, 3, 4$,
$\{(i + 2^{0}, j), (i + 2^{2}, j), (i + 2^{4}, j), (i + 2^{6}, j), (i + 2^{8}, j)\}$.

(7.12.11) $23 \in B(5, 10)$, (compare (4.5)).
Elements: (i), $(i = 0, 1, \cdots, 22)$.
Blocks: $\{(i), (i + 5^{\beta}), (i + 5^{\beta+1}), (i + 5^{\beta+11}), (i + 5^{\beta+12})\}$,
$\beta = 0, 1, \cdots, 10$.

(7.12.12) $24 \varepsilon B(5, 20)$.
Elements: (g, j), $(g = a_0 + a_1x + a_2x^2; a_i = 0, 1; i = 0, 1, 2; j = 0, 1, 2)$;
$x^3 = x + 1$.
Blocks: $\{(g + x^{\beta}, j), (g + x^{\beta+1}, j), (g + x^{\beta+2}, j + 1), (g + x^{\beta+3}, j + 1),$
$(g, j + 2)\}$, $\beta = 0, 1, \cdots, 6$,
$\{(g, j), (g + x^{\beta}, j), (g + x^{\beta+1}, j), (g, j + 1), (g + x^{\beta}, j + 2)\}$,
$\beta = 0, 1, \cdots, 6$,
$\{(g + x^{\gamma}, j), (g + x^{\gamma+1}, j), (g + x^{\gamma+2}, j), (g + x^{\gamma+1}, j + 1),$
$(g + x^{\gamma+6}, j + 1)\}$, $\gamma = 0, 1, \cdots, 5$,
$\{(g + x^{\delta}, j), (g + x^{\delta+5}, j), (g + x^{\delta+6}, j), (g + x^{2\delta+2}, j + 1),$
$(g + x^{2\delta+3}, j + 1)\}$, $\delta = 0, 1$,
$\{(g + x^{0}, j), (g + x^{1}, j), (g + x^{2}, j), (g, j + 1), (g + x^{5}, j + 2)\}$.

(7.12.13) $27 \varepsilon B(5, 10)$, (compare (4.5)).
Elements: (g), $(g = a_0 + a_1x + a_2x^2; a_i = 0, 1, 2; i = 0, 1, 2); x^3 = x + 2$.
Blocks: $\{(g), (g + x^{\beta}), (g + x^{\beta+1}), (g + x^{\beta+13}), (g + x^{\beta+14})\}$,
$\beta = 0, 1, \cdots, 12$.

(7.12.14) $28 \varepsilon B(5, 20)$.
Elements: (i, j), $(i = 0, 1, \cdots, 6;\ j = 0, 1, 2, 3)$.
Blocks: $\{(i + 3^{2\beta}, j), (i + 3^{2\beta+1}, j), (i + 3^{2\beta+3}, j + 1), (i, j + 2),$
$(i + 3^{2\beta+4}, j + 3)\}$, $\beta = 0, 1, 2$, taken 3 times,
$\{(i + 3^{\beta}, j), (i + 3^{\beta+3}, j), (i, j + 1), (i, j + 2), (i, j + 3)\}$,
$\beta = 0, 1, 2$, twice,
$\{(i + 3^{\gamma}, j), (i + 3^{\gamma+2}, j), (i + 3^{\gamma+4}, j), (i, j + 1), (i, j + 2)$,
$\gamma = 0, 1$, taken 3 times,
$\{(i + 3^{0}, j), (i + 3^{2}, j), (i + 3^{4}, j), (i, j + 1), (i, j + 3)\}$
3 times,
$\{(i, j), (i + 3^{0}, j), (i + 3^{2}, j), (i + 3^{4}, j), (i, j + \delta)\}$,
$\delta = 1, 2, 3$.

(7.12.15) $29 \varepsilon B(5, 5)$, (compare (4.5)).
Elements: (i), $(i = 0, 1, \cdots, 28)$.
Blocks: $(i), (i + 2^{\beta}), (i + 2^{\beta+7}), (i + 2^{\beta+14}), (i + 2^{\beta+21})\}$,
$\beta = 0, 1, \cdots, 6$.

(7.12.16) $32 \varepsilon B(5, 20)$, (compare (4.2)).
Elements: (g), $(g = \sum_{i=0}^{4} a_i x^i; a_i = 0, 1; i = 0, 1, 2, 3, 4); x^5 = x^2 + 1$.

Blocks: $\{(g + x^{\beta}), (g + x^{\beta+1}), (g + x^{\beta+2}), (g + x^{\beta+3}), (g + x^{\beta+4})\}$, $\beta = 0, 1, \cdots, 30$.

(7.12.17) $33 \varepsilon B(5, 5)$.
Elements: $(i, j), (i = 0, 1, \cdots, 10;\ j = 0, 1, 2)$.
Blocks: $\{(i + 2^{\beta}, j), (i + 2^{\beta+5}, j), (i + 2^{\beta+1}, j + 1), (i + 2^{\beta+6}, j + 1), (i, j + 2)\}$, $\beta = 0, 1, 2, 3, 4$,
$\{(i + 2^0, j), (i + 2^2, j), (i + 2^4, j), (i + 2^6, j), (i + 2^7, j)\}$,
$\{(i, j), (i + 2^0, j), (i, j + 1), (i + 2^0, j + 1), (i + 2^2, j + 2)\}$,
$\{(i, j), (i + 2^1, j), (i + 2^9, j), (i, j + 1), (i, j + 2)\}$.

(7.12.18) $34 \varepsilon B(5, 20)$.
Elements: $(i, j), (i = 0, 1, \cdots, 16;\ j = 0, 1)$.
Blocks: $\{(i, j), (i + 3^{\beta}, j), (i + 3^{\beta+8}, j), (i + 3^{\beta+4}, j + 1), (i + 3^{\beta+12}, j + 1)\}$ $\beta = 0, 1, \cdots, 7$,
$\{(i, j), (i + 3^{\beta}, j), (i + 3^{\beta+8}, j), (i, j + 1), (i + 3^{\beta+4}, j + 1)\}$, $\beta = 0, 1, \cdots, 7$,
$\{(i + 3^{\gamma}, j), (i + 3^{\gamma+4}, j), (i + 3^{\gamma+8}, j), (i + 3^{\gamma+12}, j), (i, j + 1)\}$, $\gamma = 0, 1, 2, 3$, twice,
$\{(i, j), (i + 3^{\nu}, j), (i + 3^{\nu+4}, j), (i + 3^{\nu+9}, j + 1), (i + 3^{\nu+15}, j + 1)\}$, $\nu = 1, 2, 3, 5, 6$,
$\{(i, j), (i + 3^{\mu}, j), (i + 3^{\mu+4}, j), (i + 3^{\mu+15}, j + 1), (i, j + 1)\}$, $\mu = 0, 4$,
$\{(i, j), (i + 3^5, j), (i + 3^6, j), (i + 3^{14}, j), (i + 3^{15}, j)\}$,
$\{(i, j), (i + 3^1, j), (i + 3^4, j), (i + 3^5, j), (i + 3^9, j)\}$.

(7.12.19) $38 \varepsilon B(5, 20)$.
Elements: $(i, j), (i = 0, 1, \cdots, 18; j = 0, 1)$.
Blocks: $\{(i, j), (i + 2^{\beta}, j), (i + 2^{\beta+1}, j), (i + 2^{\beta+9}, j + 1), (i + 2^{\beta+10}, j + 1)\}$, $\beta = 0, 1, \cdots, 17$,
$\{(i, j), (i + 2^{\gamma}, j), (i + 2^{\gamma+1}, j), (i + 2^{\gamma+1}, j + 1), (i + 2^{\gamma+2}, j + 1)\}$, $\gamma = 0, 1, \cdots, 8$,
$\{(i + 2^{3\delta+\epsilon}, j), (i + 2^{3\delta+\epsilon+1}, j), (i + 2^{3\delta+\epsilon+2}, j), (i + 2^{3\delta+\epsilon+3}, j), (i, j + 1)\}$, $\delta = 0, 1, 2; \epsilon = 0, 1$,
$\{(i, j), (i + 2^{3\delta}, j), (i + 2^{3\delta+1}, j), (i + 2^{3\delta+2}, j), (i + 2^{3\delta+3}, j)\}$, $\delta = 0, 1, 2$,
$\{(i, j), (i + 2^2, j), (i + 2^8, j), (i + 2^{14}, j), (i, j + 1)\}$.

(7.12.20) $39 \varepsilon B(5, 10)$.
Elements: $(i, j), (i = 0, 1, \cdots, 12;\ j = 0, 1, 2)$.
Blocks: $\{(i + 2^{\beta}, j), (i + 2^{\beta+6}, j), (i + 2^{\beta+3}, j + 1), (i + 2^{\beta+9}, j + 1), (i, j + 2)\}$, $\beta = 0, 1, 2$, twice,
$\{(i + 2^{\beta}, j), (i + 2^{\beta+3}, j), (i + 2^{\beta+6}, j), (i + 2^{\beta+9}, j), (i, j + 1)\}$, $\beta = 0, 1, 2$,
$\{(i + 2^{\gamma}, j), (i + 2^{\gamma+4}, j), (i + 2^{\gamma+8}, j), (i, j + 1), (i, j + 2)\}$, $\gamma = 0, 1$, taken 5 times.

(7.13) *If* $v \equiv 1 (\mathrm{mod}\ 4)$, *then* $v \varepsilon B(5, 5)$, *provided that* $v \varepsilon B_5(5, 5)$ *for* $v = 4u + 1$, $u \varepsilon K(5, \{0, 1, 2, 3, 4\})$.

For $v = 5$ the proposition is trivial, for $v = 9, 13$ and 17 see (7.12.3), (7.12.5) and (7.12.7) respectively and for $v \geqq 21$ apply (7.4) with $k = 5$, $R = \{0, 1, 2, 3, 4\}$, $\lambda = 5$.

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