Once again we note that (5) is a *universal* relation valid for any sequence of skew vectors.

EXAMPLE 3.

Expectation of L_n. (Spitzer and Widom [3])[3]. It is easy to see that

$$n!E\{L_n\} = E\{\textstyle\sum_{(\sigma)} L_n(\sigma)\}. \tag{7}$$

By an argument similar to that leading to (5), we find

$$\textstyle\sum_{(\sigma)} L_n(\sigma) = \sum_A 2(m-1)!(n-m)!|\vec{Z}_A|. \tag{8}$$

Thus,

$$\begin{aligned} E\{L_n\} &= \sum_A 2(m-1)!(n-m)!E\{|\vec{Z}_A|\}/n! \\ &= \sum_{m=1}^{n} 2(m-1)!(n-m)!\binom{n}{m} E\{|S_m|\}/n! \\ &= \sum_{m=1}^{n} E\{|S_m|\}/m. \end{aligned}$$

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[1] ERIK SPARRE ANDERSEN, "On the fluctuations of sums of random variables," *Math. Scand.*, Vol. 1 (1953), pp. 263–285.

[2] FRANK SPITZER, "A combinatorial lemma and its application to probability theory," *Trans. Amer. Math. Soc.*, Vol. 82 (1956), pp. 323–339.

[3] F. SPITZER AND H. WIDOM, "The circumference of a convex polygon," *Proc. Amer. Math. Soc.*, Vol. 12 (1961), pp. 506–509.

[3] By a limiting argument which we could also employ in this example Spitzer and Widom remove the condition that $Z_k = X_k + iY_k$ have a density.

A COMBINATORIAL DERIVATION OF THE DISTRIBUTION OF THE TRUNCATED POISSON SUFFICIENT STATISTIC[1]

BY T. CACOULLOS

Columbia University

Let $X_1, \cdots, X_n$ be independently distributed with the Poisson distribution truncated away from zero, i.e.,

$$P(x) = \frac{e^{-\lambda}}{1-e^{-\lambda}} \frac{\lambda^x}{x!}, \qquad x = 1, 2, \cdots\cdots. \tag{1}$$

Tate and Goen showed [2] that $T = \sum_{i=1}^{n} X_i$ has the distribution

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$$h(t) = \Pr[T = t] = \frac{\lambda^t n!}{(e^\lambda - 1)^n t!} \mathfrak{S}_t^n, \tag{2}$$

where $\mathfrak{S}_t^n$ denotes the Stirling number of the second kind defined by

$$\begin{aligned} \mathfrak{S}_t^n &= \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} k^t, \qquad t = n, n+1, \cdots, \\ \mathfrak{S}_t^n &= 0, \qquad t < n. \end{aligned} \tag{3}$$

Their proof was based on characteristic functions, but a much simpler approach is available as follows:

We have

$$h(t) = \Pr\left[\sum_{i=1}^{n} X_i = t\right] = \sum_{(x_1, \cdots, x_n)} \Pr[X_1 = x_1, \cdots, X_n = x_n],$$

where the summation is over all ordered n-tuples $(x_1, \cdots, x_n)$ of integers such that $x_i \geqq 1$ and $\sum_{i=1}^{n} x_i = t$. Hence, by (1), we get

$$h(t) = \sum_{(x_1, \cdots, x_n)} \frac{e^{-n\lambda}\lambda^t}{(1 - e^{-\lambda})^n} \frac{1}{\prod_{i=1}^{n} x_i!} = \frac{e^{-n\lambda}\lambda^t}{(1 - e^{-\lambda})^n t!} \sum_{(x_1, \cdots, x_n)} \frac{t!}{\prod_{i=1}^{n} x_i!}, \tag{4}$$

where the summation must be explained as above. We observe however, that

$$t!/\prod_{i=1}^{n} x_i!$$

is the number of partitions of a population of t elements into an ordered n-tuple of subpopulations of size $x_1, \cdots, x_n$, respectively. Therefore, we conclude that

$$\sum_{(x_1, \cdots, x_n)} t!/\prod_{i=1}^{n} x_i! \tag{5}$$

equals the number of possible ways in which t (distinguishable) balls can be placed in n cells (x_i being the number of balls in the i-th cell) so that no cell remains empty. Hence, we find that (5) (see, for example, p. 92 of [1]) is equal to

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} k^t.$$

Therefore, and by virtue of (4) and (3), (2) follows.

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