# THE DISTRIBUTION OF NONCENTRAL MEANS WITH KNOWN COVARIANCE

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**1. Summary.** The noncentral means distribution for infinite error degrees of freedom and the noncentral Wishart distribution are derived as expansions in zonal polynomials. A method of calculating the zonal polynomials is outlined, and orthogonality properties of their coefficients stated.

**2. Introduction.** Let the $k \times n$ matrix variate $X$, with $k \leqq n$, be distributed as

$$dF(X; M, \Sigma) = (2\pi)^{-\frac{1}{2}kn}|\Sigma|^{-\frac{1}{2}n} \operatorname{etr}\{-\tfrac{1}{2}[\Sigma^{-1}(X - M)(X - M)']\} \prod dx_{i\nu} \quad (1)$$

where $\operatorname{etr}(A) = \exp(\operatorname{tr}(A))$ and thus

$$E[X] = M,$$
$$\operatorname{Cov}(x_{i\mu}, x_{j\nu}) = \delta_{\mu\nu}\sigma_{ij}, \; \mu, \nu, = 1, \cdots, n; \qquad i, j = 1, \cdots, k,$$
$$\Sigma = (\sigma_{ij}).$$

We shall find zonal function expansions of:

1. the noncentral Wishart distribution which is the distribution of

$$S = n^{-1}(XX');$$

2. the noncentral means distribution when the covariance matrix is known, i.e., the distribution of the latent roots $l_1, \cdots, l_k$ of the determinantal equation

$$|S - l\Sigma| = 0. \quad (2)$$

This is the limiting case of the general distribution as the error degrees of freedom tend to infinity. It should not be confused with the asymptotic results of Hsu [10, 11], who considered the asymptotic distribution, as the covariance matrix of $X$ tends to zero. The distribution considered in this paper is a generalization of the noncentral $\chi^2$ distribution. Hsu's results are a generalization of the normal approximation to the noncentral $\chi^2$ distribution.

If $Y = \Sigma^{-\frac{1}{2}}X$, then the density of $Y$ is $dF(Y; \Sigma^{-\frac{1}{2}}M, I_k)$ and $l_1, \cdots, l_k$ are the latent roots of $n^{-1}YY'$.

The central distributions ($M = 0$) were found for cases

1. by Wishart [23] as

$$\frac{n^{\frac{1}{2}nk}}{2^{\frac{1}{2}nk}\pi^{\frac{1}{4}k(k-1)}|\Sigma|^{\frac{1}{2}n}\prod_{i=1}^{k}\Gamma\left(\frac{1}{2}(n+1-i)\right)} |S|^{\frac{1}{2}(n-k-1)} \operatorname{etr}\left(-\frac{1}{2}n\Sigma^{-1}S\right)\prod_{i\leqq j}^{k} ds_{ij}; \quad (3)$$

Received February 15, 1960; revised December 28, 1960.

2. by Fisher [5], Hsu [9], and Roy [19] as

$$(4)\quad \frac{n^{\frac{1}{2}nk}\pi^{\frac{1}{2}k}}{2^{\frac{1}{2}nk}\prod_{i=1}^{k}\Gamma\left(\frac{1}{2}(n+1-i)\right)\prod_{i=1}^{k}\Gamma\left(\frac{1}{2}i\right)} \exp\left(-\frac{1}{2}n\sum_{i=1} l_i\right)\prod l_i^{\frac{1}{2}(n-k-1)}\prod_{j<j}(l_i-l_j)\prod dl_i .$$

The noncentral Wishart distribution has been studied by T. W. Anderson and Girshick [2] and subsequently by Anderson [1], who obtained it for the case in which the rank of $M \leqq 2$. M. Weibull [24] derived the noncentral Wishart distribution for rank 3. In previous papers [13, 14] I obtained a power series expansion for the distribution, but for some purposes, notably for deriving the noncentral means distribution, a zonal function expansion is preferable. Herz [6] has shown that the function entering into the distribution is a Bessel function of matrix argument, which he has studied in relation to hypergeometric functions of similar type.

Roy [20, 21] has investigated the noncentral means distribution, for the general case of finite error degrees of freedom, and obtained the distribution for the case of one nonzero parameter of noncentrality, i.e., with $M$ of rank 1. Bartlett [3] showed that the general distribution could be expanded in a power series, of which he calculated the coefficients up to 3rd order. Constantine and James [4] supplied improved methods of calculating the coefficients and tabulated the coefficients of 4th order; but the series is still far too complicated.

**3. The distributions expressed as multiple integrals.** By taking out the factors involving $M$, (1) can be rewritten as

$$(5)\quad dF(X; M, \Sigma) = \operatorname{etr}(-\tfrac{1}{2}\Sigma^{-1}MM')\operatorname{etr}(M'\Sigma^{-1}X)\, dF(X; 0, \Sigma),$$

or

$$(6)\quad dF(Y; \Sigma^{-\frac{1}{2}}M, I_k) = \operatorname{etr}(-\tfrac{1}{2}\Sigma^{-1}MM')\operatorname{etr}((\Sigma^{-\frac{1}{2}}M)'Y)\, dF(Y; 0, I_k).$$

(Note that, although $\Sigma^{-1}XM'$ is a $k \times k$ matrix and $M'\Sigma^{-1}X$ is a $n \times n$ matrix, $\operatorname{tr}(\Sigma^{-1}XM') = \operatorname{tr}(M'\Sigma^{-1}X)$).

The distributions $dF(X; 0, \Sigma)$ and $dF(Y; 0, I_k)$, for which $M = 0$, give rise to the *central* distribution (3) of $S$ and the null distribution (4) of $l_1, \cdots, l_k$ respectively. By arguments presented in James [13], one readily sees the following

THEOREM I.

1. *The noncentral Wishart distribution is the central distribution* (3) *multiplied by the function*

$$(7)\quad \operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)\int_{O(n)}\operatorname{etr}(M'\Sigma^{-1}XH)\, dH$$

*where $dH$ stands for the invariant measure on the group $O(n)$ of $n \times n$ orthogonal matrices $H$ normalized so that $\int dH = 1$.*

2. *The noncentral distribution of $l_1, \cdots, l_k$ is the central distribution* (4) *multiplied by the factor*

$$\text{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)\int_{O(k)}\int_{O(n)} \text{etr}\left((\Sigma^{-\frac{1}{2}}M)'H_1YH_2\right) dH_1 dH_2. \tag{8}$$

(8) *is a symmetric function of the latent roots $nl_1, \cdots, nl_k$ of $YY'$ and of the latent roots of $M\Sigma^{-1}M'$ which are the parameters of noncentrality.*

## 4. Results from zonal function theory.

Theorem II. *If $W$ is a $k \times n$ matrix, $k \leqq n$, then*

$$\int_{O(n)} (\text{tr}\,(WH))^{2f}\, dH = \sum_{p \varepsilon P(f,k)} \frac{\chi_p(1)}{Z_p(I_n)} Z_p(WW'), \tag{9}$$

$$(\text{tr}(A))^f = \frac{2^f f!}{(2f)!} \sum_{p\varepsilon P(f,k)} \chi_p(1) Z_p(A), \tag{10}$$

$$\int_{O(k)} Z_p(AHBH')\, dH = \frac{Z_p(A)Z_p(B)}{Z_p(I_k)}, \tag{11}$$

where $P(f, k)$ is the set of partitions $p = (f_1, f_2, \cdots, f_k)$ of the positive integer $f$ into not more than $k$ parts, $\chi_p(1)$ is the dimension of the representation $[2f_1, 2f_2, \cdots, 2f_k]$ of the symmetric group which is given in equation (16), or can be found from its character tables; $A$ and $B$ are symmetric $k \times k$ matrices and $Z_p(A)$ is the zonal polynomial corresponding to the partition $p$.

That is, $Z_p(A)$ is a symmetric homogeneous polynomial in the latent roots of $A$ which, under the transformations

$$Z_p(A) \to Z_p(L^{-1}AL^{-1'}),$$

by nonsingular $k \times k$ matrices $L$ of the linear group, generates a representation space of the irreducible representation $\{2f_1, 2f_2, \cdots, 2f_k\}$ of the linear group. $Z_p(A)$ has been normalized, i.e., multiplied by a constant, so that equation (10) holds.

Equation (11) was established by James [15]. (9) and (10) will be derived in another paper (James [17]).

If $W$ or $A$ has rank $r < k$, the zonal polynomials $Z_p$, corresponding to partitions $p$ of $f$ into more than $r$ parts, vanish (see the remark at the end of Section 8). Since the nonzero latent roots of the $k \times k$ matrix $W'W$ are the same as those of the $n \times n$ matrix $WW'$, $Z_p(W'W) = Z_p(WW')$.

Lemma:

$$\int_{O(n)} (\text{tr}\,(WH))^{2f+1}\, dH = 0.$$

Proof: Substitute $(-I)H$ for $H$. Since $dH$ is the invariant measure on $O(n)$, $d((-I)H) = dH$. Then

$$\int_{O(n)} (\operatorname{tr}(WH))^{2f+1} dH = \int_{O(n)} (\operatorname{tr}(W(-I)H))^{2f+1} d(-I)H$$
$$= -\int_{O(n)} (\operatorname{tr}(WH))^{2f+1} dH$$

THEOREM III. *The function* (7), *which multiplies the probability density of the central Wishart distribution to give the noncentral distribution, is*

$$\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)\int_{O(n)} \operatorname{etr}(M'\Sigma^{-1}XH)\, dH$$
$$= \operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)\sum_{f=0}^{\infty}\frac{1}{(2f)!}\sum_{p\varepsilon P(f,k)}\frac{\chi_p(1)}{Z_p(I_n)} Z_p(\Sigma^{-1}MM'\Sigma^{-1}nS).$$

$Z_p(\Sigma^{-1}MM'\Sigma^{-1}nS)$ *is to be understood as the zonal function of the latent roots of* $\Sigma^{-1}MM'\Sigma^{-1}nS$.

PROOF: Expanding the exponential under the integral sign in (12) in a power series, we have

$$\int_{O(n)} \operatorname{etr}(M'\Sigma^{-1}XH)\, dH = \sum_{g=0}^{\infty}\frac{1}{g!}\int_{O(n)} (\operatorname{tr}(M'\Sigma^{-1}XH))^{g}\, dH.$$

The integrals of the odd powers, $g = 2f + 1$, vanish, and those of the even powers, $g = 2f$, are given by (9) with $W = M'\Sigma^{-1}X$, an $n \times n$ matrix of rank $\leq k$. Hence

$$\int_{O(n)} = \sum_{f=0}^{\infty}\frac{1}{(2f)!}\sum_{p\varepsilon P(f,k)}\frac{\chi_p(1)}{Z_p(I_n)} Z_p(M'\Sigma^{-1}XX'\Sigma^{-1}M).$$

Since $XX' = nS$, and the nonzero latent roots of $M'\Sigma^{-1}XX'\Sigma^{-1}M$ agree with those of $\Sigma^{-1}MM'\Sigma^{-1}XX'$, (12) follows.

**5. Noncentral means distribution for known covariance.** Using equations (9) and (11) of Theorem II to evaluate the integral in (8), we have, putting $W = (\Sigma^{-\frac{1}{2}}M)'H_1Y$,

$$\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)\int_{O(k)} dH_1\int_{O(n)} \operatorname{etr}((\Sigma^{-\frac{1}{2}}M)'H_1YH_2)\, dH_2$$
$$= \operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)\sum_{f=0}^{\infty}\frac{1}{(2f)!}\sum_{p\varepsilon P(f,k)}\frac{c(p)}{Z_p(I_n)}$$
$$\cdot\int_{O(k)} Z_p(\Sigma^{-\frac{1}{2}}MM'\Sigma^{-\frac{1}{2}}H_1YY'H_1')\, dH_1 \tag{13}$$
$$= \operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)\sum_{f=0}^{\infty}\frac{1}{(2f)!}\sum_{p\varepsilon P(f,k)}$$
$$\cdot\frac{\chi_p(1)}{Z_p(I_n)Z_p(I_k)} Z_p(\Sigma^{-1}MM')Z_p(nl_1, \cdots, nl_k),$$

where $Z_p(nl_1, \cdots, nl_k) = Z_p(YY')$. Hence

THEOREM IV. *The distribution of the latent roots of the determinantal equation*

$$|S - l\Sigma| = 0,$$

*where* $S = n^{-1}XX'$ *and* $X$ *is distributed as in* (1) *with* $M \neq 0$, *is the density* (4) *multiplied by the function given on the right hand side of equation* (13).

The zonal polynomials up to 4th order are tabulated in James [15], and those of 5th order are appended.

**6. Calculation of the zonal polynomials.** The coefficients of the zonal polynomials can be calculated by considering a certain representation of the symmetric group. The mathematical theory, upon which this calculation is based, is rather extensive and will be postponed to another paper (James [17]).

Consider the set $D$ of *distributions* corresponding to the partition $(2^f)$ of the integer $2f$ into $f$ parts each of size 2. Such a distribution, or *doublet* as we shall prefer to call it, can be considered as a pairing $\{i_1i_2\}\ \{i_3i_4\} \cdots \{i_{2f-1}i_{2f}\}$ of the ordinal numbers $1, 2, 3, \cdots, 2f$, not taking account of the order of the pairs or the order of the numbers within a pair, e.g., $\{4\ 3\}\ \{1\ 2\} = \{1\ 2\}\ \{3\ 4\}$. There are clearly $N = (2f)!/(2^f f!)$ doublets in $D$.

The symmetric group $S_{2f}$ of permutations of the ordinals $1, 2, \cdots, 2f$,

$$1, 2, \cdots, 2f \rightarrow \sigma 1, \sigma 2, \cdots, \sigma(2f), \qquad \sigma \in S_{2f},$$

induces a transitive group of permutations

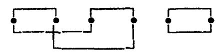
$$\{i_1i_2\}\{i_3i_4\} \cdots \{i_{2f-1}i_{2f}\} \rightarrow \{\sigma i_1 \sigma i_2\}\{\sigma i_3 \sigma i_4\} \cdots \{\sigma i_{2f-1} \sigma i_{2f}\}$$

of the doublets. Choose a doublet, e.g., $\{1\ 2\}\{3\ 4\} \cdots \{2f - 1\ 2f\}$, as origin in $D$ and let $T$ be the isotropy subgroup of $S_{2f}$, i.e., the subgroup which leaves the origin fixed. $T$ is clearly of order $2^f f!$. The isotropy group $T$ divides $D$ into equivalence classes or *orbits*, two doublets being in the same orbit if and only if there is an element of $T$ which transforms one into the other. (The orbit determines an *invariant relation* in the sense of James [16].)

The orbits are characterized by partitions $(1^{\nu_1}, 2^{\nu_2}, \cdots)$ of $f$; $\nu_1 + 2\nu_2 + \cdots = f$. Namely, one compares any doublet of the orbit with the doublet taken as origin and counts the lengths of the cycles linking them. For example, in the case $f = 3$, to compare the doublet $\{13\}\{24\}\{56\}$ with the origin $\{12\}\{34\}\{56\}$, write a row of $2f = 6$ dots and join them with bars above according to the

pairings of the doublet $\{12\}\ \{34\}\ \{56\}$ and below according to
pairings of the doublet $\{13\}\ \{24\}\ \{56\}$, as follows

$\{12\}\ \{34\}\ \{56\}$



$\{13\}\ \{24\}\ \{56\}$

The $2f$ dots are divided into cycles of even length, viz. 4 and 2. Dividing the lengths of the cycles by two, we have the partition (21) of $f = 3$ as the orbit-partition which characterizes the orbit to which the doublet $\{13\}\{24\}\{56\}$ belongs.

To the orbit-partition $\nu = (1^{\nu_1}, 2^{\nu_2}, \cdots)$ of $f$ is assigned the monomial $s_1^{\nu_1}s_2^{\nu_2} \cdots$ in the sums $s_i$ of the $i$th powers of the latent roots of the argument matrix $A$ of the zonal polynomial,

$$Z_p(A) = \sum_{\nu} z_{p\nu} s_1^{\nu_1} s_2^{\nu_2} \cdots .$$

The coefficient $z_{p\nu}$ in the zonal polynomial $Z_p(A)$ corresponding to the partition $p = (f_1 f_2 \cdots)$, is proportional to the sum of the values of the character $\chi_p(\sigma)$ of the representation $[2f_1, 2f_2, \cdots]$ of $S_{2f}$ over those elements $\sigma \varepsilon S_{2f}$ which map the origin $\{12\}\{34\} \cdots \{2f - 1\, 2f\}$ into the orbit determined by the partition $\nu = (1^{\nu_1}, 2^{\nu_2}, \cdots)$ of $f$,

$$z_{p\nu} = k \sum_{\sigma \varepsilon T\tau T} \chi_p(\sigma). \tag{14}$$

$\tau$ is any element of $S_{2f}$ which maps the origin into the given orbit. The set of all elements of $S_{2f}$ which do so is then clearly the double coset $T\tau T$ of the isotropy group $T$. We can choose $k$ to make the coefficient of $s_1^f$ unity.

Instead of characters, one may use coefficients of primitive idempotents of the symmetric group algebra such as those of Alfred Young. (See Littlewood [18], Section 5.4 or Rutherford [22], Chapter II and also James [16].)

We now give an example of the calculation of $Z_p$ for the partition $p = (f_1 f_2) = (21)$ of $f = 3$. Arrange the ordinals $1, 2, \cdots, 6$ in two rows of a Young symmetry diagram corresponding to the partition $(2f_1, 2f_2) = (42)$.

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & & \end{matrix}$$

Let us write a function on the doublets $D$ as a formal linear combination of doublets with its values for coefficients. In particular, the function which has value unity at the origin $\{12\}\{34\}\{56\}$ and zero on all other doublets will again be denoted by the symbol $\{12\}\{34\}\{56\}$.

Apply to this function, the Young symmetrizer. This is the element $s$ of the group algebra of $S_{2f}$ which is the sum of the permutations within the rows of the symmetry diagram. To the resulting function, we apply the alternator $a$ which is the linear combination of permutations within the columns of the symmetry diagram whose coefficients are $\pm 1$ according as the permutation is even or odd. In the symmetrizer $s$, it is only necessary to include one element from each coset $\sigma T'$, where $T'$ is the subgroup of $T$ whose elements permute within the rows of the symmetry diagram. Likewise, one need only include in the alternator $a$, one element from each coset $T''\sigma$, where $T''$ is the subgroup of $T$ which permutes within the columns. Thus we put

$$s = () + (13) + (14)$$

$$a = () - (15)$$

and calculate

$$s\{12\}\{34\}\{56\} = \{12\}\{34\}\{56\} + \{14\}\{23\}\{56\} + \{13\}\{24\}\{56\}$$

$$as\ \{12\}\{34\}\{56\} = \{12\}\{34\}\{56\} + \{14\}\{23\}\{56\} + \{13\}\{24\}\{56\} - \{16\}\{34\}\{25\} - \{16\}\{23\}\{45\} - \{16\}\{24\}\{35\}.$$

Since these doublets are in orbits determined by the orbit-partitions

$$\begin{aligned} &(1^3) + (12) + (12) \\ &- (12) - (3) - (3) \\ &= (1^3) + (12) - 2(3) \end{aligned}$$

the zonal polynomial for $p = (21)$ is

$$Z_p(A) = s_1^3 + s_1 s_2 - 2s_3 .$$

**7. Orthogonality of coefficients of zonal polynomials.** The zonal polynomials are really idempotents of the tensor representation of the linear group which correspond to idempotents of the representation of the symmetric group in the space of functions on the doublets. Since the coefficients of idempotents belonging to inequivalent representations must be orthogonal, it follows that the coefficients of the zonal polynomials must satisfy orthogonality relations which can be shown to be

$$(15) \qquad \sum_{\nu} \frac{z_{p\nu}\, z_{q\nu}}{z_{(f)\nu}} = \delta_{pq} \frac{N}{\chi_p(1)},$$

where $p$ and $q$ are partitions of the integer $f$ and the summation with respect to $\nu$ is taken over all such partitions. $\chi_p(1)$, denoted by $c(p)$ in James [15], is the dimension of the representation $[2f_1, 2f_2, \cdots, 2f_s]$ of $S_{2f}$, which is known to be

$$(16) \qquad \chi_p(1) = (2f)!\, \frac{\sum_{i<j}(l_i - l_j)}{l_1!\, l_2! \cdots l_p!},$$

where $l_1 = 2f_1 + s - 1$, $l_2 = 2f_2 + s - 2, \cdots, l_s = 2f_s$. $(f)$ is the partition of $f$ into a single part. The coefficient $z_{(f)\nu}$ in the zonal polynomial of $(f)$ is, in fact, the number of doublets in the orbit of $\nu$.

Hua [7, 8] has shown that the zonal polynomial $Z_p(A)$ can be expressed as a linear combination of characters of the linear group (Schur functions) corresponding to symmetry diagrams of order $\leqq p$,

$$Z_p(A) = \sum_{q \leqq p} \lambda_q \chi_q(A),$$

where $q$ runs over those partitions of $f$ which are not above $p$.

In particular, the zonal function corresponding to the lowest partition $p = (1^f)$ is the $f$th elementary symmetric function of the latent roots of $A$, and if all but $r$ roots are zero, the zonal polynomials corresponding to partitions $p = (f_1, f_2, \cdots, f_{r+1}, \cdots)$ of $f$ into more than $r$ parts vanish.

**Acknowledgment.** The author gratefully acknowledges the help of Mr. A. G. Constantine who pointed out an error in his initial conjecture of the noncentral means distribution. He would also like to thank Professor D. B. Lowdenslager of Princeton University for lending him a microfilm of an English translation of the papers of Hua.

# APPENDIX

*Zonal Polynomials of 5th Degree*

of the representation of the linear group in the space of polynomials in the elements of a positive definite real symmetric matrix. The zonal polynomials of lower degree are tabulated in James [15]. $s_i$ is the sum of the $i$th powers of the latent roots of the matrix.

| Partition $p$ | Zonal polynomial $Z_p(A)$ | Constants $\chi_p(1)$ | $Z_p\ (I_k)$ |
|---|---|---|---|
| (5) | $s_1^5 + 20s_1^3s_2 + 60s_1s_2^2 + 80s_1^2s_3 + 160s_2s_3 + 240s_1s_4 + 384s_5$ | 1 | $k(k+2)(k+4)(k+6)(k+8)$ |
| (41) | $s_1^5 + 11s_1^3s_2 + 6s_1s_2^2 + 26s_1^2s_3 - 20s_2s_3 + 24s_1s_4 - 48s_5$ | 35 | $k(k+2)(k+4)(k+6)(k-1)$ |
| (32) | $s_1^5 + 6s_1^3s_2 + 11s_1s_2^2 - 4s_1^2s_3 + 20s_2s_3 - 26s_1s_4 - 8s_5$ | 90 | $k(k+2)(k+4)(k-1)(k+1)$ |
| $(31^2)$ | $s_1^5 + 3s_1^3s_2 - 10s_1s_2^2 + 2s_1^2s_3 - 4s_2s_3 - 8s_1s_4 + 16s_5$ | 225 | $k(k+2)(k+4)(k-1)(k-2)$ |
| $(2^21)$ | $s_1^5 + 5s_1s_2^2 - 10s_1^2s_3 - 10s_2s_3 + 10s_1s_4 + 4s_5$ | 252 | $k(k+2)(k-1)(k-2)(k+1)$ |
| $(21^3)$ | $s_1^5 - 4s_1^3s_2 - 3s_1s_2^2 + 2s_1^2s_3 + 10s_2s_3 + 6s_1s_4 - 12s_5$ | 300 | $k(k+2)(k-1)(k-2)(k-3)$ |
| $(1^5)$ | $s_1^5 - 10s_1^3s_2 + 15s_1s_2^2 + 20s_1^2s_3 - 20s_2s_3 - 30s_1s_4 + 24s_5$ | 42 | $k(k-1)(k-2)(k-3)(k-4)$ |

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