THE RANDOM WALK BETWEEN A REFLECTING AND AN ABSORBING BARRIER

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1. Introduction. In this paper, the classical problem of random walk restricted between two barriers at 0 and b is discussed. A particle, starting from the initial position u on the x-axis ($0 < u \leqq b$ an integer) at $t = 0$, moves one unit to the left or right of its position at times $t = 1, 2, \cdots$. The probabilities for the moves are respectively q and $p(q + p = 1)$, the moves being independent. We assume that the barrier at 0 is absorbing and the one at b reflecting so that (i) when the particle reaches the barrier at 0, it is absorbed and the process terminates (ii) when at any integral time $\tau(\tau \geqq b - u)$, the particle is at the barrier at b, there is a probability p that it remains there at the next instant $(\tau + 1)$ and a probability q that it moves one unit to the left.

Random walk problems have been extensively studied (see Feller [1]), and their application to the theory of Brownian movement has been discussed by Kac [2] among others. With the assumption that there is one reflecting barrier at 0 and the other at ∞, Kac was able to derive an explicit expression for

$$P(n, m \mid s),$$

the probability that the particle starting from position n is at m after time s has elapsed. Other cases where both barriers are absorbing and where both barriers are reflecting have also been discussed by Feller [1]. We are concerned in this paper with the case where one barrier is absorbing and the other reflecting; we shall derive the expression for the generating function of the probabilities of absorption.

2. Generating function for the probabilities of absorption. Let $g(t \mid u)$ be the probability that the particle reaches the barrier at 0 for the first time (thus being absorbed) at time t starting from the initial position u at $t = 0$. The probability $g(t \mid u)$ satisfies the difference equation:

$$g(t \mid u) = g(t - 1 \mid u - 1)q + g(t - 1 \mid u + 1)p, \qquad (u = 1, 2, \cdots, b - 1; t = 1, 2, \cdots), \tag{1}$$

where $g(0 \mid 0) = 1$ and $g(t \mid u) = 0$ for $t < u$. For $u = b$, we have

$$g(t \mid b) = g(t - 1 \mid b - 1)q + g(t - 1 \mid b)p.$$

Let $P(u)$ be the $1 \times b$ row vector $(0 \cdots 0\, q\, 0\, p\, 0 \cdots 0)$ with q being the $(u - 1)$th component, and let $G(t - 1)$ be the $b \times 1$ column vector of elements $g(t - 1 \mid i)$, $(i = 1, 2, \cdots, b)$. Then equation (1) may be written in the matrix

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form as

$$g(t \mid u) = P(u)G(t-1). \tag{2}$$

A further application of the difference equation (1) immediately leads to

$$g(t \mid u) = P(u)\, Q\, G(t-2), \tag{3}$$

where Q is the $b \times b$ matrix defined by

$$Q = \begin{pmatrix} 0 & p & 0 \cdots\cdots\cdots & 0 \\ q & 0 & p & 0 \cdots\cdots 0 \\ 0 & q & 0 & p \cdots\cdots 0 \\ \vdots & & & \\ 0 & 0 \ldots\ldots\ldots 0 & q & p \end{pmatrix},$$

and where, as before, $G(t-2)$ is the column vector of elements $g(t-2 \mid i)$, $(i = 1, 2, \cdots, b)$. By successive applications of (1), it follows that

$$g(t \mid u) = P(u)Q^{t-2}G(1). \tag{4}$$

Let $\varphi(\theta \mid u) = \sum_{t=0}^{\infty} \theta^t g(t \mid u)$ be the generating function for the probabilities of absorption. We have from (4) that

$$\begin{aligned} \varphi(\theta \mid u) &= \theta^2 \sum_{i=0}^{\infty} P(u)(\theta Q)^i G(1) \\ &= \theta^2 P(u)(I - \theta Q)^{-1} G(1) \end{aligned} \tag{5}$$

provided θ lies in such a range that $\max [|\theta p|, |\theta q|] \leqq 1$.

We note that $G(1)$, being the column vector of elements $g(1 \mid i)$, has the first component q, all other elements being zero. It follows that the right hand side of (5) may be written as the ratio of two determinants, namely

$$\varphi(\theta \mid u) = \frac{q\theta^2 \begin{vmatrix} 0 \cdots\cdots\cdots\cdots 0 & q & 0 & p & 0 \cdots\cdots 0 \\ -\theta q & 1 & -\theta p & 0 \cdots\cdots\cdots & \\ 0 & -\theta q & 1 & -\theta p \cdots\cdots\cdots & \\ \vdots & & & & \\ 0 \cdots\cdots\cdots\cdots & & -\theta q & 1 & -\theta p \\ 0 \cdots\cdots\cdots\cdots & & 0 & -\theta q & 1 - \theta p \end{vmatrix}}{\begin{vmatrix} 1 & -\theta p & 0 \cdots\cdots\cdots\cdots & & 0 \\ -\theta q & 1 & -\theta p & 0 \cdots\cdots\cdots & 0 \\ 0 & -\theta q & 1 \cdots\cdots & & \\ \vdots & & & & \\ 0 \cdots\cdots\cdots -\theta q & & 1 & -\theta p & \\ 0 \cdots\cdots\cdots & 0 & -\theta q & 1 - \theta p & \end{vmatrix}} \tag{6}$$

$$= \frac{q\theta^2 \mid D \mid}{\mid I - \theta Q \mid}.$$

The determinant $|D|$ in the numerator is the same as $|I - \theta Q|$ except that the first row is replaced by $P(u)$. We first evaluate the determinant $|I - \theta Q|$ in the denominator. Consider an $n \times n$ determinant A_n of the form similar to $|I - \theta Q|$ except that the (n, n)th element is 1. Then for this determinant the following recurrence relation holds:

$$A_n = A_{n-1} - \theta^2 pq A_{n-2}, \qquad (n = 2, 3, \cdots,) \tag{7}$$

where $A_1 = 1$ and A_0 is defined to be 1 for convenience. Writing

$$\begin{pmatrix} A_n \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & -\theta^2 pq \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_{n-1} \\ A_{n-2} \end{pmatrix} = S \begin{pmatrix} A_{n-1} \\ A_{n-2} \end{pmatrix},$$

it follows from (7) immediately that

$$\begin{pmatrix} A_n \\ A_{n-1} \end{pmatrix} = S^{n-1} \begin{pmatrix} A_1 \\ A_0 \end{pmatrix}. \tag{8}$$

The two characteristic roots λ_1, λ_2 of the matrix S are found to have distinct values

$$\lambda_1 = \tfrac{1}{2}[1 + (1 - 4\theta^2 pq)^{\frac{1}{2}}], \qquad \lambda_2 = \tfrac{1}{2}[1 - (1 - 4\theta^2 pq)^{\frac{1}{2}}].$$

Writing S in the spectral form

$$S = B \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} B^{-1}$$

with $B = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$ and $B^{-1} = (\lambda_1 - \lambda_2)^{-1} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$, it follows from (8) that

$$\begin{pmatrix} A_n \\ A_{n-1} \end{pmatrix} = B \begin{pmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{pmatrix} B^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so that $A_n = (\lambda_1 - \lambda_2)^{-1}[\lambda_1^{n+1} - \lambda_2^{n+1}]$. Hence we have finally

$$\begin{aligned} |I - \theta Q| &= A_b - \theta p A_{b-1} \\ &= (\lambda_1 - \lambda_2)^{-1}[\lambda_1^{b+1} - \lambda_2^{b+1} - \theta p(\lambda_1^b - \lambda_2^b)]. \end{aligned} \tag{9}$$

To evaluate the determinant $|D|$ in the numerator, we first add to its first row the uth row multiplied by θ^{-1}, thus reducing it to $(0 \cdots \theta^{-1} 0 \cdots 0)$ where θ^{-1} is in the uth position. Expanding the determinant by the first row, we obtain

$$\begin{aligned} |D| &= \theta^{-1}(-1)^{u-1}(-\theta q)^{u-1}[A_{b-u} - \theta p A_{b-u-1}] \\ &= \theta^{u-2} q^{u-1} (\lambda_1 - \lambda_2)^{-1}[\lambda_1^{b-u+1} - \lambda_2^{b-u+1} - \theta p(\lambda_1^{b-u} - \lambda_2^{b-u})]. \end{aligned} \tag{10}$$

Hence from equations (9) and (10), we have

$$\varphi(\theta \mid u) = \frac{q\theta^2 \mid D \mid}{\mid I - \theta Q \mid} = \frac{\theta^u q^u[\lambda_1^{b-u+1} - \lambda_2^{b-u+1} - \theta p(\lambda_1^{b-u} - \lambda_2^{b-u})]}{[\lambda_1^{b+1} - \lambda_2^{b+1} - \theta p(\lambda_1^b - \lambda_2^b)]}. \tag{11}$$

From equation (11) we may draw the following conclusions:

(i) $$[\varphi(\theta \mid u)]_{\theta=1} = \frac{q^u[p^{b-u+1} - q^{b-u+1} - p(p^{b-u} - q^{b-u})]}{[p^{b+1} - q^{b+1} - p(p^b - q^b)]} = 1.$$

This is in agreement with the fact that eventual absorption is certain.

(ii) Rewriting

$$\varphi(\theta \mid u) = \frac{\theta^u q^u[\lambda_1^{-u+1} - \theta p\lambda_1^{-u}) - (\lambda_2/\lambda_1)^b(\lambda_2^{-u+1} - \theta p\lambda_2^{-u})]}{[(\lambda - \theta p) - (\lambda_2/\lambda_1)^b(\lambda_2 - \theta p)]},$$

since $\lambda_1 > \lambda_2$,

(12) $$\lim_{b\to\infty} \varphi(\theta \mid u) = \theta^u q^u \lambda_1^{-u} = (\lambda_2/\theta p)^u.$$

The above expression is the generating function for the probabilities of absorption when no reflecting barrier is present, and is identical to the result obtained by Feller [1].

(iii) The expected duration of time before absorption takes place may be obtained from

$$E(T) = [\partial(\varphi(\theta \mid u))/\partial\theta]_{\theta=1}$$

and is found to be

(13) $$E(T) = \frac{u}{q-p} + \frac{p^{b+1}}{q^b(q-p)^2}[1 - (q/p)^u] \quad \text{if} \quad p \neq q.$$

When $p = q = \frac{1}{2}$, $\lim_{\theta\to1} [\partial(\varphi(\theta \mid u))/\partial\theta]$ is evaluated using L'Hospital's rule, and in this case

(14) $$E(T) = u + u(2b - u).$$

3. Explicit expression for the probabilities of absorption. The form of equation (6) indicates that $\varphi(\theta \mid u)$ is simply a ratio of two polynomials in θ. Denote this by

(15) $$\varphi(\theta \mid u) = \frac{U(\theta)}{V(\theta)}.$$

Both the numerator and the denominator have degree b. If the roots of $V(\theta)$, $\theta_1, \theta_2, \cdots, \theta_b$ are distinct, equation (15) may be expanded into partial fractions

(16) $$\varphi(\theta \mid u) = \sum_{\nu=1}^{b} \frac{\rho_\nu}{(\theta_\nu - \theta)},$$

where ρ_ν are constants that can be determined by

(17) $$\rho_\nu = \frac{-U(\theta_\nu)}{[\partial(V(\theta))/\partial\theta]_{\theta=\theta_\nu}}.$$

We first find the roots of the denominator, making use of the variable α defined by

$$(\cos\alpha)^{-1} = 2(pq)^{\frac{1}{2}}\theta.$$

Then $\lambda_{1,2} = (2\cos\alpha)^{-1}[\cos\alpha \pm i\sin\alpha] = (2\cos\alpha)^{-1}e^{\pm i\alpha}$, and in terms of the new variable, $\varphi(\theta \mid u)$ may be written

$$(18) \qquad \varphi(\theta \mid u) = (q/p)^{u/2} \left[\frac{q^{\frac{1}{2}} \sin (b - u + 1)\alpha - p^{\frac{1}{2}} \sin (b - u)\alpha}{q^{\frac{1}{2}} \sin (b + 1)\alpha - p^{\frac{1}{2}} \sin b\alpha} \right].$$

The denominator of (18) is found to have b distinct roots α_ν ($\nu = 1, 2, \cdots, b$), which lie in the subintervals

$$\left(\frac{\nu\pi}{b - 1}, \frac{(\nu + 1)\pi}{b - 1} \right) \qquad (\nu = 1, 2, \cdots, b).$$

The roots of $V(\theta)$ are then

$$(19) \qquad \theta_\nu = (2(pq)^{\frac{1}{2}} \cos \alpha_\nu)^{-1}, \qquad (\nu = 1, 2, \cdots, b).$$

From equation (17), we obtain

$$(20) \qquad \begin{aligned} \rho_\nu &= -(q/p)^{u/2} \frac{[q^{\frac{1}{2}} \sin (b - u + 1)\alpha_\nu - p^{\frac{1}{2}} \sin (b - u)\alpha_\nu]}{[(b + 1)q^{\frac{1}{2}} \cos (b + 1)\alpha_\nu - bp^{\frac{1}{2}} \cos b\alpha_\nu] \left(\dfrac{\partial \alpha}{\partial \theta} \right)_{\alpha = \alpha_\nu}} \\ &= -(q/p)^{u/2} \frac{[q^{\frac{1}{2}} \sin (b - u + 1)\alpha_\nu - p^{\frac{1}{2}} \sin (b - u)\alpha_\nu] \sin \alpha_\nu}{2(pq)^{\frac{1}{2}}[(b + 1)q^{\frac{1}{2}} \cos (b + 1)\alpha_\nu - bp^{\frac{1}{2}} \cos \alpha_\nu] \cos^2 \alpha_\nu}. \end{aligned}$$

It remains now to expand each term in equation (16) into a geometric series. The coefficient, $g(t \mid u)$, of θ^t is found to be

$$g(t \mid u) = \sum_{\nu=1}^{b} \frac{\rho_\nu}{\theta^{t-1}},$$

this together with equations (19) and (20) yield finally

$$g(t \mid u) = -2^t p^{\frac{1}{2}(t-u)} q^{\frac{1}{2}(t+u)} \sum_{\nu=1}^{b} \cos^{t-1} \alpha_\nu \cdot \frac{[q^{\frac{1}{2}} \sin (b - u + 1)\alpha_\nu - p^{\frac{1}{2}} \sin (b - u)\alpha_\nu] \sin \alpha_\nu}{[(b + 1)q^{\frac{1}{2}} \cos (b + 1)\alpha_\nu - bp^{\frac{1}{2}} \cos b\alpha_\nu]}.$$

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