Lastly, a nonstationary example is provided by the Brownian motion kernel. For

$$K(s, t) = \min (s, t), \qquad 0 \leqq s, t \leqq 1,$$

the unit sphere of $H(K)$ consists of absolutely continuous functions m for which $m(0) = 0$, and

$$\int_0^1 |m'(t)|^2 \, dt \leqq 1.$$

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THE OPINION POOL[1]

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1. Introduction and summary. When a group of k individuals is required to make a joint decision, it occasionally happens that there is agreement on a utility function for the problem but that opinions differ on the probabilities of the relevant states of nature. When the latter are indexed by a parameter θ, to which probability density functions on some measure $\mu(\theta)$ may be attributed, suppose the k opinions are given by probability density functions $p_{s1}(\theta), \cdots, p_{sk}(\theta)$. Suppose that D is the set of available decisions d and that the utility of d, when the state of nature is θ, is $u(d, \theta)$.

For a probability density function $p(\theta)$, write

$$u[d \mid p(\theta)] = \int u(d, \theta) p(\theta) \, d\mu(\theta).$$

The Group Minimax Rule of Savage [1] would have the group select that d minimising

$$\max_{i=1,\cdots,k} \{\max_{d' \varepsilon D} u[d' \mid p_{si}(\theta)] - u[d \mid p_{si}(\theta)]\}.$$

As Savage remarks ([1], p. 175), this rule is undemocratic in that it depends only on the *different* distributions for θ represented in those put forward by the

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group and not on the number of members of the group supporting each different representative.

An alternative rule for choosing d may be stated as follows: "Choose weights $\lambda_1, \cdots, \lambda_k$ ($\lambda_i \geqq 0, i = 1, \cdots, k$ and $\sum_1^k \lambda_i = 1$); construct the pooled density function

$$p_{s\lambda}(\theta) = \sum_1^k \lambda_i p_{si}(\theta);$$

choose the d, say $d_{s\lambda}$, maximising $u[d \mid p_{s\lambda}(\theta)]$." This rule, which may be called the Opinion Pool, can be made democratic by setting $\lambda_1 = \cdots = \lambda_k = 1/k$.

Where it is reasonable to suppose that there is an actual, operative probability distribution, represented by an 'unknown' density function $p_a(\theta)$, it is clear that the group is then acting as if $p_a(\theta)$ were known to be $p_{s\lambda}(\theta)$. If $p_a(\theta)$ were known, it would be possible to calculate $u[d_{s\lambda} \mid p_a(\theta)]$ and $u[d_{si} \mid p_a(\theta)]$, where d_{si} is the d maximising $u[d \mid p_{si}(\theta)]$, $i = 1, \cdots, k$ and then to use these quantities to assess the effect of adopting the Opinion Pool for any given choice of $\lambda_1, \cdots, \lambda_k$.

It is of general theoretical interest to examine the conditions under which

$$u[d_{s\lambda} \mid p_a(\theta)] \geqq \min_{i=1,\cdots,k} u[d_{si} \mid p_a(\theta)]. \quad (1.1)$$

Theorems 2.1 and 3.1 provide different sets of sufficient conditions for (1.1) to hold. Theorem 2.1 requires $k = 2$ and places a restriction on $p_a(\theta)$ (or, equivalently, on $p_{s1}(\theta)$ and $p_{s2}(\theta)$); Theorem 3.1 puts conditions on D and $u(d, \theta)$ instead.

2. The case of $k = 2$. The following example shows that conditions *are* needed for (1.1) to hold. With $k = 2$, suppose that $p_{s1}(\theta)$, $p_{s2}(\theta)$, $p_a(\theta)$ are given by atoms of probability one on $\theta_1, \theta_2, \theta_a$ respectively, where $\theta_1, \theta_2, \theta_a$ are different; also suppose that D has only three elements d_1, d_2, d_3 and that

$$u(d_1, \theta_1) = 1, \quad u(d_2, \theta_1) = 0, \quad u(d_3, \theta_1) = \tfrac{3}{4},$$

$$u(d_1, \theta_2) = 0, \quad u(d_2, \theta_2) = 1, \quad u(d_3, \theta_2) = \tfrac{3}{4},$$

$$u(d_1, \theta_a) = \tfrac{1}{2}, \quad u(d_2, \theta_a) = \tfrac{1}{2}, \quad u(d_3, \theta_a) = 0.$$

Then $d_{s1} = d_1$, $d_{s2} = d_2$ and, for $\lambda_1 = \lambda_2 = \frac{1}{2}$, $d_{s\lambda} = d_3$ and (1.1) does not obtain.

However, the following theorem may be stated:

THEOREM 2.1. *If, for some* μ_1, μ_2, $p_a(\theta) = \mu_1 p_{s1}(\theta) + \mu_2 p_{s2}(\theta)$, *then* (1.1) *holds for any weights* λ_1, λ_2. (As heretofore explicit, the assumption is made that $d_{s1}, d_{s2}, d_{s\lambda}$ exist.)

PROOF. d_{si} maximises $u[d \mid p_{si}(\theta)]$, $i = 1, 2$, and $d_{s\lambda}$ maximises $u[d \mid p_{s\lambda}(\theta)]$ or $\lambda_1 u[d \mid p_{s1}(\theta)] + \lambda_2 u[d \mid p_{s2}(\theta)]$. Writing b_{ij} for $u[d_{si} \mid p_{sj}(\theta)] - u[d_{s\lambda} \mid p_{sj}(\theta)]$, it follows that

(2.1) $$b_{11} \geqq 0,$$

(2.2) $$b_{22} \geqq 0,$$

(2.3) $$\lambda_1 b_{11} + \lambda_2 b_{12} \leqq 0,$$

(2.4) $$\lambda_1 b_{21} + \lambda_2 b_{22} \leqq 0.$$

For (1.1) to hold, it is necessary that either

(2.5) $$\mu_1 b_{11} + \mu_2 b_{12} \leqq 0$$ or

(2.6) $$\mu_1 b_{21} + \mu_2 b_{22} \leqq 0.$$

Now it is necessary that $\mu_1 + \mu_2 = 1$ so that, if $\mu_1 \leqq \lambda_1$, (2.1) and (2.3) imply (2.5); while, if $\mu_1 > \lambda_1$, (2.2) and (2.4) imply (2.6). Therefore (1.1) holds and the theorem is established.

EXAMPLE. If each of $p_a(\theta)$, $p_{s1}(\theta)$, $p_{s2}(\theta)$ is atomic on two θ-points and if $p_{s1}(\theta)$, $p_{s2}(\theta)$ are not identical, $p_a(\theta)$ may be written as $\mu_1 p_{s1}(\theta) + \mu_2 p_{s2}(\theta)$ and (1.1) obtains. If $p_{s1}(\theta) = p_{s2}(\theta)$, (1.1) clearly obtains.

3. The general case. That the condition $p_a(\theta) = \mu_1 p_{s1}(\theta) + \cdots + \mu_k p_{sk}(\theta)$ is not sufficient for (1.1), when $k > 2$, follows from the following example: Suppose that $k = 3$ and that $p_{si}(\theta)$ is given by an atom of probability one at $\theta = \theta_i$ for $i = 1, 2, 3$ where $\theta_1, \theta_2, \theta_3$ are different; also suppose that D has only four elements d_0, d_1, d_2, d_3 for which

$$\begin{array}{llll} u(d_0, \theta_1) = \tfrac{3}{2}, & u(d_1, \theta_1) = 2\tfrac{1}{2}, & u(d_2, \theta_1) = \tfrac{1}{4}, & u(d_3, \theta_1) = \tfrac{1}{4}, \\ u(d_0, \theta_2) = \tfrac{3}{2}, & u(d_1, \theta_2) = \tfrac{1}{4}, & u(d_2, \theta_2) = 2\tfrac{1}{2}, & u(d_3, \theta_2) = \tfrac{1}{4}, \\ u(d_0, \theta_3) = 0, & u(d_1, \theta_3) = \tfrac{1}{4}, & u(d_2, \theta_3) = \tfrac{1}{4}, & u(d_3, \theta_3) = 2\tfrac{1}{2}. \end{array}$$

Choose a small positive number ϵ. Suppose $[\mu_1, \mu_2, \mu_3]$ is such that $p_a(\theta)$ is atomic on $[\theta_1, \theta_2, \theta_3]$ with

$$[p_a(\theta_1), p_a(\theta_2), p_a(\theta_3)] = [\tfrac{1}{3}(1 - \tfrac{1}{2}\epsilon), \tfrac{1}{3}(1 - \tfrac{1}{2}\epsilon), \tfrac{1}{3}(1 + \epsilon)].$$

Take $[\lambda_1, \lambda_2, \lambda_3]$ so that $p_{s\lambda}(\theta)$ is atomic on $[\theta_1, \theta_2, \theta_3]$ with

$$[p_{s\lambda}(\theta_1), p_{s\lambda}(\theta_2), p_{s\lambda}(\theta_3)] = [\tfrac{1}{3}(1 + \tfrac{1}{2}\epsilon), \tfrac{1}{3}(1 + \tfrac{1}{2}\epsilon), \tfrac{1}{3}(1 - \epsilon)].$$

Then $u[d_0 \mid p_{s\lambda}(\theta)] = 1 + \tfrac{1}{2}\epsilon$, $u[d_1 \mid p_{s\lambda}(\theta)] = u[d_2 \mid p_{s\lambda}(\theta)] = 1 + 9\epsilon/24$, $u[d_3 \mid p_{s\lambda}(\theta)] = 1 - 3\epsilon/4$; whence $d_{s\lambda} = d_0$. Also, by symmetry, $u[d_0 \mid p_a(\theta)] = 1 - \tfrac{1}{2}\epsilon$, $u[d_1 \mid p_a(\theta)] = u[d_2 \mid p_a(\theta)] = 1 - 9\epsilon/24$, $u[d_3 \mid p_a(\theta)] = 1 + 3\epsilon/4$; whence

$$u[d_{s\lambda} \mid p_a(\theta)] = u[d_0 \mid p_a(\theta)] < \min \{u[d_{si} \mid p_a(\theta)] \mid i = 1, 2, 3\}$$

so that (1.1) does not hold.

Theorem 2.1 gives conditions on k and $p_a(\theta)$ for (1.1) to obtain. The following theorem gives conditions on only D and $u(d, \theta)$ for (1.1) to obtain.

THEOREM 3.1. *If* (i) *D is an interval of real numbers* (ii) *$-u(d, \theta)$ is, for each θ, a strictly convex function of d then* (1.1) *holds for all weights $\lambda_1, \cdots, \lambda_k$*. (The assumption is made that $d_{s1}, \cdots, d_{sk}, d_{s\lambda}$ exist.)

PROOF. Consider any three different elements d_1, d_2, d_3 of D such that $d_1 = \rho d_2 + (1 - \rho)d_3$, $0 < \rho < 1$. Then, for all θ, $u(d_1, \theta) > \rho u(d_2, \theta) + (1 - \rho)u(d_3, \theta)$ and hence $u[d_1 \mid p(\theta)] > \rho u[d_2 \mid p(\theta)] + (1 - \rho)u[d_3 \mid p(\theta)]$. Therefore $-u[d \mid p_a(\theta)]$, $-u[d \mid p_{si}(\theta)]$, $i = 1, \cdots, k$, are strictly convex in d. Let $d_m = \min \{d_{s1}, \cdots, d_{sk}\}$ and $d_M = \max \{d_{s1}, \cdots, d_{sk}\}$. For $d_m \leqq d \leqq d_M$, by the convexity of $-u[d \mid p_a(\theta)]$,

$$u[d \mid p_a(\theta)] \geqq \min \{u[d_m \mid p_a(\theta)], u[d_M \mid p_a(\theta)]\}. \tag{3.1}$$

Hence

$$\min_{i=1,\cdots,k} u[d_{si} \mid p_a(\theta)] = \min \{u[d_m \mid p_a(\theta)], u[d_M \mid p_a(\theta)]\}. \tag{3.2}$$

For weights $\lambda_1, \cdots, \lambda_k$, if $d_m \leqq d_{s\lambda} \leqq d_M$, (3.1) and (3.2) together imply (1.1). However, if $d_{s\lambda} < d_m$, there exists a $d^* \epsilon D$ and ρ_i^*, $0 < \rho_i^* < 1$, $i = 1, \cdots, k$, such that $d_{s\lambda} < d^* < d_m$ and $d^* = \rho_i^* d_{s\lambda} + (1 - \rho_i^*)d_{si}$, $i = 1, \cdots, k$. By the established strict convexities,

$$\begin{aligned} u[d^* \mid p_{si}(\theta)] &> \rho_i^* u[d_{s\lambda} \mid p_{si}(\theta)] + (1 - \rho_i^*)u[d_{si} \mid p_{si}(\theta)] \\ &\geqq \rho_i^* u[d_{s\lambda} \mid p_{si}(\theta)] + (1 - \rho_i^*)u[d_{s\lambda} \mid p_{si}(\theta)] \\ &= u[d_{s\lambda} \mid p_{si}(\theta)], \qquad i = 1, \cdots, k; \end{aligned}$$

whence $\sum_1^k \lambda_i u[d^* \mid p_{si}(\theta)] > \sum_1^k \lambda_i u[d_{s\lambda} \mid p_{si}(\theta)]$ or

$$u[d^* \mid p_{s\lambda}(\theta)] > u[d_{s\lambda} \mid p_{s\lambda}(\theta)],$$

a contradiction. Hence $d_{s\lambda} < d_m$ is impossible; and so is $d_M < d_{s\lambda}$. Therefore the theorem is established.

EXAMPLE. D is an interval, θ is a real parameter and $u(d, \theta) = -(d - \theta)^2$. Because $(d - \theta)^2$ is strictly convex in d for each θ, (1.1) obtains.

In conclusion, it may be noted that it is quite possible to have

$$u[d_{s\lambda} \mid p_a(\theta)] > \max \{u[d_{si} \mid p_a(\theta)] \mid i = 1, \cdots, k\}.$$

For example, this will occur (for all but degenerate cases) when

$$p_a(\theta) = \sum_1^k \mu_i p_{si}(\theta)$$

and $\lambda_i = \mu_i$, $i = 1, \cdots, k$.

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