MARKOV RENEWAL PROCESSES WITH FINITELY MANY STATES[1]

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1. Summary. In this paper, Markov Renewal processes having a finite number of states are studied. Explicit expressions are derived for the distribution functions of first passage times, as well as for the marginal distribution function of the corresponding Semi-Markov process. Double generating functions are obtained for the distribution functions of the N_j-processes. The limiting behavior of a Markov Renewal process is discussed, the stationary probabilities being derived completely. General Markov Renewal processes are introduced, and a related stationary process is determined. Several examples are given.

2. Introduction. In [1], a class of stochastic processes, called Markov Renewal processes (M.R.P.), are defined and a preliminary investigation is made of their structure, and of the related Semi-Markov processes (S.-M.P.) introduced by Lévy [2], Smith [3] and Takács [4] independently in 1954. The reader is referred to [1] for the necessary definitions and notation. Roughly speaking, M.R.P.'s are generalizations both of continuous and discrete parameter Markov Chains which permit arbitrary distribution functions (d.f.), possibly depending both on the last state entered and on the next state to be entered, for the times between successive transitions.

In the present paper we restrict our attention to those M.R.P.'s determined by $(m, \mathbf{A}, \mathfrak{Q})$ with $m < \infty$. Recall, that because of Lemma I.4.1,[3] all such M.R.P.'s are regular, i.e., almost all sample functions are finite-valued step functions over $(-\infty, \infty)$. In Section 3, systems of integral equations are given for the functions $P_{ij}(t)$ and $G_{ij}(t)$, respectively, the d.f. of Z_t and the d.f. of the time until the first transition into state j, both given $Z_0 = i$. Equations relating these two functions are also given. Lemma 3.2 is due to Takács (equations (10) and (11) of [4]),[4] while (3.2), summed over i with respect to the initial probabilities, is essentially equation (8) in a paper by Weiss [5], who derived this and other integral relationships for the purpose of studying the asymptotic behavior of

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[2] This work was completed while the author was at Columbia University. The contents of the first 6 sections comprised an Invited paper given at the Annual meeting of the I.M.S., Sept. 1958, Cambridge, Massachusetts.

[3] All numbers which are prefixed by I, refer to the correspondingly numbered part of [1].

[4] In assumption 4 of [4], which postulates that the X-process is a sequence of independent random variables, the word independent should be interpreted as conditional independence given the successive states of the system.

the "Renewal functions" $M_{ij}(t)$, defined in (5.10). The relationships of Section 3 are solved in Section 4, with explicit expressions being given, respectively, in Theorems 4.1 and 4.2 for the matrix-valued functions $\mathcal{P}$ and $\mathcal{G}$ defined therein. In Section 5, the marginal d.f.'s for the N_j-processes are studied, explicit expressions being given for their double generating functions. As a consequence, the matrix-valued Renewal function $\mathfrak{M}$ is derived. The three most common subclasses of M.R.P.'s are briefly discussed in the following section; they are Markov Chains, continuous parameter Markov processes, and Renewal processes.

In Section 7, the problem of characterizing the limiting behavior of an M.R.P. is solved. The stationary distribution of the process is derived, thus extending a result of Smith [3]. The concept of a general Markov Renewal process (G.M.R.P.) is introduced, in which a different matrix of transition distributions may be used to determine the initial transition of the process then is used for all remaining transitions. By using the stationary probabilities obtained for a given M.R.P. it is possible to construct a related G.M.R.P. which is "stationary" in a sense made explicit in Theorem 7.2. Following this, some additional examples are briefly stated in the final section.

3. The probabilities P_{ij} and G_{ij} and related quantities. Of considerable importance in studying the behavior of M.R.P.'s are the times between successive occurrences of a given state. It is the purpose of this section to study the d.f.'s of these times as well as the marginal d.f.'s of the corresponding S.-M.P., the Z-process, which is to say, the probability of the process being in a given state at any instant of time. Several identities and relationships between these quantities will be given for them in terms of the basic $\mathcal{Q}$-matrix.

Let us assume that $\mathbf{A} > 0$ coordinate-wise for every M.R.P. under consideration, in order that all conditional probabilities will be well defined. The quantities in question are then defined for all i, j (cf., Section I.5) by

$$G_{ij}(t) = P[N_j(t) > 0 \mid Z_0 = i], \qquad P_{ij}(t) = P[Z_t = j \mid Z_0 = i] \tag{3.1}$$

for $t \geqq 0$, and denote their respective L.-S. transforms (when they exist) by g_{ij} and π_{ij}. Let b_{ij}, μ_{ij} and η_i be the first moments (possibly infinite) of the mass functions F_{ij}, G_{ij} and H_i respectively. Notice that for $i = j$, G_{ii} becomes the mass function (possibly of total variation less than one) representing the probabilistic behavior of the first passage time of the process from state i into state i, but that the process need not have left state i during this recurrence time.

The two quantities defined by (3.1) are closely related both to each other and to the elements of the basic $\mathcal{Q}$-matrix of the M.R.P. being studied. In the next three lemmas these relationships are explicitly stated. First of all, the relationship between the P_{ij}'s and the Q_{ij}'s is given in

LEMMA 3.1. *For all $t \geqq 0$, $s > 0$*

$$P_{ij}(t) = \delta_{ij} - \sum_{k=1}^{m} [\delta_{ij} - P_{kj}(t)] * Q_{ik}(t), \tag{3.2}$$

$$(3.3)\qquad \pi_{ij}(s) = \delta_{ij} - \sum_{k=1}^{m} [\delta_{ij} - \pi_{kj}(s)] q_{ik}(s).$$

[Throughout this paper, δ_{ij} is used to indicate both the Kronecker delta and the function $\delta_{ij}U_0(\cdot)$, where U_c is the d.f. with unit jump at c, whose domain (possibly restricted, as it is in (3.3)) is determined by the context. A similar remark applies to any real constant.]

Proof. Define for all $t \geqq 0$, $n \geqq 0$,

$$(3.4)\qquad P_{ij}(t; n) = P[Z_t = j, N(t) = n \mid Z_0 = i].$$

By Lemma I.4.1., $N(t) < \infty$ a.s. since $m < \infty$, and so $P_{ij}(t) = \sum_{n=0}^{\infty} P_{ij}(t;n)$. The quantities $P_{ij}(t; n)$ are straightforwardly shown to satisfy $P_{ij}(t; 0) = \delta_{ij}[1 - H_i(t)]$ and, for $n > 0$,

$$(3.5)\qquad P_{ij}(t; n) = \sum_{k=1}^{m} P_{kj}(t; n-1) * Q_{ik}(t).$$

Upon summing over n in these last two equations, (3.2) is obtained immediately. (3.3) then follows upon taking L.-S. transforms of the former.

The analogous relationship between the G_{ij}'s and the Q_{ij}'s is given without proof in

Lemma 3.2. *For* $t \geqq 0$, $s > 0$,

$$(3.6)\qquad G_{ij}(t) = \sum_{k=1}^{m} G_{kj}(t) * Q_{ik}(t) + [1 - G_{jj}(t)] * Q_{ij}(t)$$

$$(3.7)\qquad g_{ij}(s) = \sum_{k=1}^{m} g_{kj}(s) q_{ik}(s) + [1 - g_{jj}(s)] q_{ij}(s).$$

Between the G_{ij}'s and the P_{ij}'s, there is a particularly tractable relationship, as given in

Theorem 3.1. *For* $t \geqq 0$, $s > 0$,

$$(3.8)\qquad P_{ij}(t) = P_{jj}(t) * G_{ij}(t) + \delta_{ij}[1 - H_i(t)],$$

$$(3.9)\qquad \pi_{ij}(s) = \pi_{jj}(s) g_{ij}(s), \quad (i \neq j); \qquad \pi_{jj}(s) = \frac{1 - h_j(s)}{1 - g_{jj}(s)}.$$

Proof. It suffices to remark roughly that for $i \neq j$, $P_{ij}(t)$ is the probability of reaching state j for the first time before t (according to G_{ij}) and then, in the remaining time, ending up in state j (according to P_{jj}). In case $i = j$, one must add to the above the probability that no transition occurs in $(0, t]$, namely, $1 - H_i(t)$. (3.9) is an immediate consequence of (3.8).

Define

$$(3.10)\qquad D_i(t) = 1 - P[Z_u = i, 0 \leqq u < t \mid Z_0 = i].$$

Then D_i is a mass function representing the duration of time that the Z-process remains in state i. It is clear that $D_i = H_i$ in case $p_{ii} = 0$. In particular, $D_i = H_i$ for an S.-M.P. One may easily prove

LEMMA 3.3. *For $t \geqq 0$, $s > 0$,*

$$
\begin{aligned}
D_i(t) &= [H_i(t) - Q_{ii}(t)] * [1 - Q_{ii}(t)]^{(-1)} \\
d_i(s) &= [h_i(s) - q_{ii}(s)][1 - q_{ii}(s)]^{-1}.
\end{aligned}
\tag{3.11}
$$

Notice that D_i must have total variation equal to either zero or one, and that it equals zero if and only if $p_{ii} = 1$. In this case one would say that state i is an absorbing state.

The explicit evaluation of P_{jj} in terms of H_j and G_{jj} determined by (3.9) may be used to characterize a recurrent state in terms of the integrability of P_{jj}. The reader is referred to Section I.5 for the definition of a recurrent state in an M.R.P.

THEOREM 3.2. *If $\eta_j < \infty$, then state j is recurrent if and only if*

$$\int_0^\infty P_{jj}(t)\, dt = \infty. \tag{3.12}$$

PROOF. By its definition in (3.1) one has

$$\lim_{s \to 0} s^{-1} \pi_{jj}(s) = \lim_{s \to 0} \int_0^\infty e^{-st} P_{jj}(t)\, dt = \int_0^\infty P_{jj}(t)\, dt \tag{3.13}$$

as a consequence of the Lebesgue Monotone Convergence Theorem, whether or not the limit is finite. From Theorem 3.1, one obtains, when $\eta_j < \infty$,

$$\lim_{s \to 0} s^{-1} \pi_{jj}(s) = \lim_{s \to 0} \frac{1 - h_j(s)}{s} [1 - q_{jj}(s)]^{-1} = \eta_j [1 - G_{jj}(\infty)]^{-1}. \tag{3.14}$$

Therefore (3.13) and (3.14) together imply that state j is recurrent (i.e., $G_{jj}(\infty) = 1$) if and only if (3.12) holds, as required.

Note that because of Theorem I.5.1(a), it follows from the above theorem, that if $\eta_j < \infty$, then

$$\int_0^\infty P_{jj}(t)\, dt < \infty \quad \text{if and only if} \quad \sum_{n=0}^\infty p_{jj}^{(n)} < \infty$$

where $\mathbf{P}^n = (p_{ij}^{(n)})$. It should also be emphasized that in the proof of Theorem 3.2, the following stronger result is obtained, namely, that if $\eta_j < \infty$

$$G_{jj}(\infty) = 1 - \eta_j \left\{ \int_0^\infty P_{jj}(t)\, dt \right\}^{-1}.$$

4. The matrices $\mathcal{P}$ and $\mathcal{G}$. In this section, the relationships given in Lemmas 3.1 and 3.2 will be solved, with explicit expressions for the P_{ij} and the G_{ij} in terms of the Q_{ij} being given in matrix notation. Theoretically, therefore, for any M.R.P., the G_{ij} and the P_{ij} may be uniquely determined from the Q_{ij}.

Consider the matrix-valued functions defined by

$$\mathcal{P} = (P_{ij}), \qquad \pi = (\pi_{ij})$$
$$\mathcal{G} = (G_{ij}), \qquad g = (g_{ij})$$
$$\mathcal{H} = (\delta_{ij} H_i), \qquad h = (\delta_{ij} h_{ij}).$$

Define a convolution operation on matrix-valued functions (whenever the definition is valid) by $\mathcal{K} * \mathcal{L} = (\sum_{k=1}^{m} K_{ik} * L_{kj})$ which is formally the same as regular matrix multiplication except that the usual numerical product is replaced with convolution. Let $\mathbf{I}$ denote either the identity matrix (δ_{ij}) or the unit-step matrix-valued function $(\delta_{ij} U_0(\cdot))$. The domains of the various functions will be clear from their contexts. For an arbitrary matrix-valued function $\mathcal{K}$, set $\mathcal{K}^{(0)} = \mathbf{I}$ and define $\mathcal{K}^{(n)} = \mathcal{K}^{(n-1)} * \mathcal{K}$ for $n > 0$ and $(\mathbf{I} - \mathcal{K})^{(-1)} = \sum_{n=0}^{\infty} \mathcal{K}^{(n)}$ whenever the series converges. An explicit expression for $\mathcal{P}(\pi)$ in terms of $\mathcal{Q}$ and $\mathcal{H}(q$ and $h)$ is given in

THEOREM 4.1.

$$\mathcal{P} = (\mathbf{I} - \mathcal{Q})^{(-1)} * (\mathbf{I} - \mathcal{H}), \qquad \pi = (\mathbf{I} - q)^{-1}(\mathbf{I} - h), \tag{4.1}$$

the latter equation being defined over $(0, \infty)$, *while* $\pi(0) = 0$.

PROOF. It suffices to show that $(P_{ij}(\cdot\,; n) = \mathcal{Q}^{(n)} * (\mathbf{I} - \mathcal{H})$ where $P_{ij}(\cdot\,; n)$ is as defined by (3.4), since then in view of the fact that $N(t) < \infty$ a.s., (4.1) follows immediately upon summation over n. It is easily checked recursively that $P_{ij}(\cdot\,; n) * Q_{kl} = Q_{kl} * P_{ij}(\cdot\,; n)$. It therefore follows that $(P_{ij}(\cdot\,; 0)) = \mathbf{I} - \mathcal{H}$ and $(P_{ij}(\cdot\,; n)) = \mathcal{Q} * (P_{ij}(\cdot\,; n-1))$, $(n > 0)$. Consequently, one obtains

$$\mathcal{P} = \sum_{n=0}^{\infty} \mathcal{Q}^{(n)} * (\mathbf{I} - \mathcal{H}), \tag{4.2}$$

which immediately yields

$$\pi(s) = \sum_{n=0}^{\infty} [q(s)]^n [\mathbf{I} - h(s)], \tag{4.3}$$

for $s > 0$. To verify (4.1), let s be a fixed positive number and define $c_s = \max_{i,j} q_{ij}(s)$. Since by (i) of Definition I.3.1. each mass function Q_{ij} satisfies $Q_{ij}(0) = 0$, and since $m < \infty$, one has $c_s < 1$. Therefore $0 \leqq q^n(s) \leqq c_s^n \mathbf{1}$ where the inequality is termwise and where $\mathbf{1}$ denotes the $m \times m$ matrix in which each term equals 1. Consequently the series in (4.3) converges. Moreover $[\mathbf{I} - q(s)] \sum_{n=0}^{\infty} q^n(s) = \mathbf{I}$, and hence the inverse $(\mathbf{I} - q)^{-1} = \sum q^n$ is well defined for all $s > 0$. That $\pi(0) = \mathbf{I}$ follows directly from the definition, (3.1).

It should be remarked here that equation (3.3) of Lemma 3.1 may be rewritten in matrix notation as $\pi = \mathbf{I} - h + q\pi$. Equation (4.1) of the above theorem may then be considered as an immediate consequence of Lemma 3.1, once the non-singularity of $\mathbf{I} - q$ is demonstrated.

For any $m \times m$ matrix (or matrix-valued function) $\mathbf{A} = (a_{ij})$, define the diagonal and off-diagonal parts of $\mathbf{A}$ by

$$_d\mathbf{A} = (\delta_{ij}a_{ij}), \qquad _0\mathbf{A} = \mathbf{A} - {_d\mathbf{A}}.$$

With this notation one may rewrite (3.7) of Lemma 3.2 and (3.9) of Theorem 3.1, respectively, as

$$\mathcal{G} = \mathfrak{q}(\mathbf{I} + {_0\mathcal{G}}) \tag{4.4}$$

and

$$\boldsymbol{\pi} = (\mathbf{I} + {_0\mathcal{G}})(\mathbf{I} - {_d\mathcal{G}})^{-1}(\mathbf{I} - \mathfrak{h}). \tag{4.5}$$

Equivalently, because of (3.9), (4.5) may be rewritten as $\boldsymbol{\pi}({_d\boldsymbol{\pi}})^{-1} = \mathbf{I} + {_0\mathcal{G}}$ since for $s > 0$, $\pi_{jj}(s) > 0$ for all j. Consequently, substitution of this in (4.4) leads, as a result of Theorem 4.1, to the proof of

THEOREM 4.2. *As defined on* $(0, \infty)$

$$\mathcal{G} = \mathfrak{q}\boldsymbol{\pi}({_d\boldsymbol{\pi}})^{-1} = \mathfrak{q}(\mathbf{I} - \mathfrak{q})^{-1}\{_d[(\mathbf{I} - \mathfrak{q})^{-1}]\}^{-1} \tag{4.6}$$

An immediate consequence of Theorem 4.2 is that a formula can be given for the mean recurrence times μ_{ii} as defined after (3.1). Set $\boldsymbol{\mu} = (\mu_{ij})$. Clearly $\boldsymbol{\mu} = \lim_{s\to 0} s^{-1}(\mathbf{1} - \mathcal{G})$, and so from (4.6) it can be shown that

$$_d\boldsymbol{\mu} = \lim_{s\to 0}\{_d[s(\mathbf{I} - \mathfrak{q})^{-1}]\}^{-1} \tag{4.7}$$

where one must interpret $1/0 = \infty$.

5. The probability distribution of $N_j(t)$. It is possible to obtain a system of integral equations for the probability distributions of the r.v.'s $N_j(t)(1 \leqq j \leqq m)$. Explicit solutions of these equations are derived in terms of double generating functions of these probabilities. Theoretically, therefore, the probabilities are determined and, in particular, the moments of $N_j(t)$ may then be obtained in the usual way.

Define for all $1 \leqq i, j \leqq m$, $k \geqq 0$, $s, t \geqq 0$, and $|z| \leqq 1$,

$$\begin{aligned} v_{ij}(k; t) &= P[N_j(t) = k \mid Z_0 = i], \qquad \phi_{ij}(z; t) = \sum_{k=0}^{\infty} z^k v_{ij}(k; t), \\ \psi_{ij}(z; s) &= \int_{0-}^{\infty} e^{-st}\, d_t\, \phi_{ij}(z; t), \end{aligned} \tag{5.1}$$

and $\mathbf{V}_k = (v_{ij}(k; \cdot))$, $\mathbf{\Psi}_z = (\psi_{ij}(z; \cdot))$.

It is immediately seen that the v_{ij} can be expressed either in terms of the Q_{ij}'s or in terms of the G_{ij} by means of the following integro-difference equations. These expressions are self explanatory and require no proofs. For $t \geqq 0$

$$\begin{aligned} v_{ij}(k; t) &= \sum_{r\neq j} v_{rj}(k; t) * Q_{ir}(t) + v_{jj}(k - 1; t) * Q_{ij}(t), \quad (k > 0), \\ v_{ij}(0; t) &= \sum_{r\neq j} v_{rj}(0; t) * Q_{ir}(t) + 1 - H_i(t) \end{aligned} \tag{5.2}$$

and

$$v_{ij}(k; t) = v_{jj}(k-1, t) * G_{ij}(t), \quad v_{ij}(0; t) = 1 - G_{ij}(t), \quad (k > 0). \tag{5.3}$$

The solutions to (5.3) are easily seen to be

$$\begin{aligned} v_{ij}(k; t) &= G_{ij}(t) * G_{jj}^{(k-1)}(t) * [1 - G_{jj}(t)], \qquad (k > 0), \\ v_{ij}(0; t) &= 1 - G_{ij}(t). \end{aligned} \tag{5.4}$$

Actually (5.4) may be viewed as a known result in Renewal Theory. By (5.4), the v_{ij} are explicitly expressed in terms of the G_{ij}, which in turn have been expressed in Theorem 4.2 in terms of the basic Q_{ij}'s. This relationship may be more simply expressed by means of generating functions. Thus from (5.4), one obtains for $|z| \leqq 1$,

$$\phi_{ij}(z; t) = 1 - G_{ij}(t) + zG_{ij}(t) * [1 - zG_{jj}(t)]^{(-1)} * [1 - G_{jj}(t)],$$

and so for $s \geqq 0$

$$\psi_{ij}(z; s) = 1 - g_{ij}(s) + zg_{ij}(s)[1 - g_{jj}(s)][1 - zg_{jj}(s)]^{-1}.$$

From the last expression, one obtains

THEOREM 5.1. *As defined over* $(0, \infty)$ *for* $|z| \leqq 1$, *one has*

$$\boldsymbol{\Psi}_z = \mathbf{1} - (1 - z)\mathfrak{g}(\mathbf{I} - z\,{}_d\mathfrak{g})^{-1}. \tag{5.5}$$

Substitution for $\mathfrak{g}$ in terms of $\mathfrak{q}$ may be made in (5.5) by Theorem 4.2, thus giving an explicit expression for $\boldsymbol{\Psi}$ in terms of $\mathfrak{q}$ (cf., Corollary 5.1 below). Notice that $\boldsymbol{\Psi}_0 = \mathbf{1} - \mathfrak{g}$ and $\boldsymbol{\Psi}_1 = \mathbf{1}$, as required.

An alternative derivation of (5.5) is to bypass the explicit expression (5.4) and to derive the analogus equations to (5.3) for the generating functions. By so doing, one would obtain directly the matrix equation

$$\boldsymbol{\Psi}_z - z\mathfrak{g}\,{}_d\boldsymbol{\Psi}_z = \mathbf{1} - \mathfrak{g}. \tag{5.6}$$

Now for any square matrices $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ for which $c_{ii} \neq 1$ for every i, the solution for $\mathbf{B}$ in the equation $\mathbf{B} - \mathbf{C}\,{}_d\mathbf{B} = \mathbf{D}$ is easily checked to be

$$\mathbf{B} = \mathbf{D} + \mathbf{C}(\mathbf{I} - {}_d\mathbf{C})^{-1}\,{}_d\mathbf{D}, \tag{5.7}$$

since the given equation implies that $(\mathbf{I} - {}_d\mathbf{C})\,{}_d\mathbf{B} = {}_d\mathbf{D}$, and the assumption permits the taking of the inverse of $\mathbf{I} - {}_d\mathbf{C}$. Theorem 5.1 may then be obtained by applying (5.7) to (5.6).

In terms of generating functions the equation (5.2) becomes

$$\psi_{ij}(z; s) = \sum_{r=1}^{m} q_{ir}(s)\psi_{rj}(z; s) - (1 - z)q_{ij}(s)\psi_{jj}(z; s) + 1 - h_i(s),$$

or in matrix notation

$$(\mathbf{I} - \mathfrak{q})\boldsymbol{\Psi}_z + (1 - z)\mathfrak{q}\,{}_d\boldsymbol{\Psi}_z = (\mathbf{I} - \mathfrak{q})\mathbf{1}. \tag{5.8}$$

As in the proof of Theorem 4.1, $(\mathbf{I} - \mathfrak{q})$ is non-singular. Therefore, (5.8) may be rewritten as

$$\Psi_z + (1 - z)(\mathbf{I} - \mathfrak{q})^{-1}\mathfrak{q}\ {}_d\Psi_z = \mathbf{1},$$

whose solution, following (5.7), is given in

COROLLARY 5.1. *As defined over* $(0, \infty)$ *for* $|z| \leqq 1$, *one has*

$$\Psi_z = \mathbf{1} - (1 - z)(\mathbf{I} - \mathfrak{q})^{-1}\mathfrak{q}[z\mathbf{I} + (1 - z)\ {}_d\{(\mathbf{I} - \mathfrak{q})^{-1}\}]^{-1}. \tag{5.9}$$

This is indeed a corollary of Theorem 5.1, since, as mentioned earlier, it can be obtained from (5.5) by a substitution of (4.6). However, the above more direct approach is of interest in its own right.

Although the explicit result given in (5.9) is somewhat complex in appearance, it should be observed that it implies that when computing Ψ_z from $\mathfrak{q}$, only one major computation must be made, namely the inversion of $(\mathbf{I} - \mathfrak{q})$. This remark applies also to the results of the preceding section. The reader should also note that Theorem 4.2 may be viewed as a corollary to (5.9), since by definition $\mathbf{1} - \Psi_0 = \mathfrak{g}$.

Clearly the moments of $N_j(t)$, or more precisely, the L.-S. transforms of these moments, may be obtained by successive differentiations of Ψ_z. In particular, the L.-S. transform of the expectation of $N_j(t)$ is readily obtained from (5.9) because of the special form of Ψ_z. Define for $t \geqq 0$, $s > 0$

$$M_{ij}(t) = E[N_j(t) \mid Z_0 = i], \qquad m_{ij}(s) = \int_0^\infty e^{-st}\, dM_{ij}(t), \tag{5.10}$$

and set $\mathfrak{M} = (M_{ij}(\cdot))$, $\mathfrak{m} = (m_{ij}(\cdot))$. We shall call $\mathfrak{M}$ the *Renewal function* of the process. Clearly

$$\mathfrak{m} = (z - 1)^{-1}(\Psi_z - \mathbf{1})|_{z=1}.$$

From (5.5) and (5.9), one then obtains, respectively,

$$\mathfrak{m} = \mathfrak{g}(\mathbf{I} - {}_d\mathfrak{g})^{-1} = \mathfrak{q}(\mathbf{I} - \mathfrak{q})^{-1},$$

thus proving

THEOREM 5.2. *For a* M.R.P. *with* $m < \infty$, *the* (*conditional*) *expectations of* $\mathbf{N}(t)$ *satisfy*

$$\mathfrak{M} = \mathfrak{Q} * (\mathbf{I} - \mathfrak{Q})^{(-1)} = (\mathbf{I} - \mathfrak{Q})^{(-1)} - \mathbf{I} \qquad on\ [0, \infty) \tag{5.11}$$

and

$$\mathfrak{m} = \mathfrak{q}(\mathbf{I} - \mathfrak{q})^{-1} = (\mathbf{I} - \mathfrak{q})^{-1} - \mathbf{I} \qquad on\ (0, \infty). \tag{5.12}$$

This is, in several respects, a very important result, if not at first sight amazing. First of all, it implies that a knowledge of $\mathfrak{M}$ is equivalent to a knowledge of $\mathfrak{Q}$. That is, an M.R.P. is equally as well determined by $(m, \mathbf{A}, \mathfrak{M})$ as by $(m, \mathbf{A}, \mathfrak{Q})$. Because of known results in Renewal theory, the moments of the G_{ii} may be determined by a knowledge of the asymptotic behavior of the M_{ii}. In the next section we shall briefly consider some special cases of M.R.P.'s, one of which is the case of a Renewal process (the case of $m = 1$). The striking similarity between (5.11) and (5.12) and the corresponding known results for Renewal processes will then become apparent. It seems to this author that the suitability of the name Markov Renewal processes for the stochastic processes being studied in these papers, is best supported by this similarity, together with the ease with which Renewal theory yields limiting results for these processes, as is demonstrated in Section 7.

Theorem 3.2 together with (5.12) shows that one may write

$$\mathfrak{M} = \mathcal{G}({}_d\mathfrak{M} + \mathbf{I}), \qquad \mathcal{G} = \mathfrak{M}({}_d\mathfrak{M} + \mathbf{I})^{(-1)} \tag{5.13}$$

which implies $M_{ij} = G_{ij} * M_{jj} + G_{ij}$.

Because of the basic nature of Theorem 5.2, it is desirable to consider the following more direct proof which affords a much clearer insight into this relationship between $\mathfrak{M}$ and $\mathcal{Q}$, making it intuitive and natural, rather than "amazing". From the definition of an M.R.P. (in particular, I.(3.6)) it follows that

$$P[J_n = j, S_n \leqq t \mid Z_0 = i] = \sum_{\mathcal{S}_{n,i,j}} \mathop{*}_{k=0}^{n-1} Q_{\alpha_k \alpha_{k+1}}(t). \tag{5.14}$$

Hence, either by induction or by recalling the interpretation of elements of $\mathbf{P}^n$ in Markov Chain theory, one obtains

$$(P[J_n = j, S_n \leqq t \mid Z_0 = i]) = \mathcal{Q}^{(n)} \equiv (Q_{ij}^{(n)}). \tag{5.15}$$

Furthermore, define for each j, r.v.'s $\{U_{n,j}\,;\, n \geqq 1\}$ by $U_{n,j} = 1$ if $J_n = j$ and $S_n \leqq t$, and $= 0$ otherwise. Clearly $N_j(t) = \sum_{n=1}^{\infty} U_{n,j}$. Therefore, since $(E[U_{n,j} \mid Z_0 = i]) = \mathcal{Q}^{(n)}$, one obtains

$$\mathfrak{M} = (E[N_j(t) \mid Z_0 = i]) = \sum_{n=1}^{\infty} \mathcal{Q}^{(n)} = (\mathbf{I} - \mathcal{Q})^{(-1)} - \mathbf{I} \tag{5.16}$$

as desired.

6. Special cases of Markov Renewal processes.

(a) *Markov Chains.* As has been mentioned earlier, an M.R.P. becomes a Markov Chain whenever $F_{ij} = U_1(\cdot)$ for all i, j. Hence for a Markov Chain,

$$f_{ij}(s) = e^{-s}, \qquad q_{ij}(s) = p_{ij}e^{-s}, \qquad h_i(s) = e^{-s},$$
$$\mathfrak{f} = e^{-s}\mathbf{1}, \qquad \mathfrak{q} = e^{-s}\mathbf{P}, \qquad \mathfrak{h} = e^{-s}\mathbf{I}.$$

Consider first of all the relationship (3.12) which becomes

$$\pi_{jj}(s) = \frac{1 - h_j(s)}{1 - g_{jj}(s)} = \frac{1 - e^{-s}}{1 - g_{jj}(s)}. \tag{6.1}$$

Set $z = e^{-s}$. For purposes of this paragraph alone, introduce the notation

$$F_j(z) \equiv g_{jj}(-\log z) = \sum_{n=1}^{\infty} z^n f_j(n)$$

$$U_j(z) \equiv (1 - z)^{-1}\pi_{jj}(-\log z) = \sum_{n=0}^{\infty} z^n P_{jj}(n)$$

where $f_j(n) = G_{jj}(n) - G_{jj}(n-1)$ is the probability that state j is reentered for the first time at time n. Upon rewriting (6.1) with this notation, one obtains $U_j(z) = [1 - F_j(z)]^{-1}$, which is a very well known relationship for Markov Chains (for example, see Feller [6] pp. 285, 352).

Consider now equation (4.1). For a Markov Chain it becomes

$$\mathcal{P} = \sum_{k=0}^{\infty} U_k(\cdot)\mathbf{P}^k * [1 - U_1(\cdot)], \qquad \pi(s) = (\mathbf{I} - e^{-s}\mathbf{P})^{-1}(1 - e^{-s}).$$

In particular, one obtains the known result that

$$\mathcal{P}(n) = \sum_{k=0}^{\infty} [U_k(n) - U_{k+1}(n)]\mathbf{P}^k = \mathbf{P}^n.$$

Moreover, equation (4.6) of Theorem 4.2 becomes

$$\mathfrak{g}(s) = e^{-s}\mathbf{P}(\mathbf{I} - e^{-s}\mathbf{P})^{-1}\{_d[(\mathbf{I} - e^{-s}\mathbf{P})^{-1}]\}^{-1}$$

while (4.7) becomes

$$(6.2) \qquad {}_d\mathfrak{u} = \lim_{s\to 0}\{_d[s(\mathbf{I} - e^{-s}\mathbf{P})^{-1}]\}^{-1}.$$

By a straightforward generalization of well known Abelian and Tauberian theorems for series, one obtains for matrices that

$$\lim_{z\to 1}(1 - z)(\mathbf{I} - z\mathbf{P})^{-1} = \lim_{z\to 1}\frac{\sum_{k=0}^{\infty} z^k\mathbf{P}^k}{\sum_{k=0}^{\infty} z^k} = \mathbf{L}$$

if and only if

$$(6.3) \qquad \lim_{n\to\infty} n^{-1}(\mathbf{I} + \mathbf{P} + \cdots + \mathbf{P}^n) = \mathbf{L}.$$

Since ${}_d\mathfrak{u}$ obviously exists (possibly with some infinite entries) one deduces from (6.2) the well known ergodic result that ${}_d\mathfrak{u} = ({}_d\mathbf{L})^{-1}$.

From Theorem 5.2 one obtains for a Markov Chain, that for $n > 0$, $\mathfrak{M}(n) = \sum_{k=1}^{n} \mathbf{P}^k$, while, quite obviously, $\mathfrak{M}$ is constant over every interval of the form $[n, n+1)$.

(b) *Continuous parameter Markov processes with finitely many states.* Such a process is a special case of an M.R.P. for which $p_{ii} = 0$ and $F_{ij}(t) = 1 - e^{-\lambda_i t}$ for appropriate finite $\lambda_i > 0$. Clearly $F_i = H_i$. Setting $\mathfrak{y} = (\delta_{ij}\eta_i) = (\delta_{ij}\lambda_i^{-1})$, one may write

$$\mathfrak{q} = (\mathbf{I} + s\mathfrak{y})^{-1}\mathbf{P},$$

and hence from (4.1) and (5.12) one obtains

$$\pi(s) = [\mathbf{I} - (\mathbf{I} + s\mathfrak{y})^{-1}\mathbf{P}]^{-1}[\mathbf{I} - (\mathbf{I} + s\mathfrak{y})^{-1}]$$

$$\mathfrak{M}(s) + \mathbf{I} = [\mathbf{I} - (\mathbf{I} + s\mathfrak{y})\mathbf{P}]^{-1}.$$

The corresponding expression for $\mathcal{P}$ may also be written down, and from it, one would expect to be able to derive the semigroup property of $\mathcal{P}$. This does not seem to be, however, a very simple deduction.

(c) *Renewal processes.* A Renewal process (R.P.) is defined as a sequence of independent and identically distributed r.v.'s, say $\{X_n : n \geqq 1\}$. [The reader is referred to the survey paper by Smith [7] for details of Renewal theory, as well as for complete references to the proofs of the theorem stated below.] Equivalently, either the sequence of partial sums $\{S_n : n \geqq 1\}$, or the process $\{N(t) : t \geqq 0\}$ defined as before by $N(t) = \sup \{k : S_k \leqq t\}$, can be termed a Renewal process. With emphasis upon the latter description, it is easily seen that the family of all R.P.'s and the family of all M.R.P.'s having $m = 1$ are identical. Most of the above results become either vacuously true (e.g., the results of Sections 3 and 4), or obvious (e.g., the results of Section 5) for R.P.'s. Note that for a R.P., $Q_{11} = F_{11}$. The analogues of Theorems 5.1, 5.2, for example, become, dropping all redundant subscripts,

$$(6.4)\qquad \begin{aligned} v(k; \cdot) &= F^{(k)} - F^{(k+1)}, \qquad \psi(z; \cdot) = (1 - f)(1 - zf)^{-1}, \\ M &= (1 - F)^{(-1)} - 1, \qquad \text{and} \qquad m = f(1 - f)^{-1}, \end{aligned}$$

all of which can be derived directly with very little effort. It is the similarity between (5.12) and (6.4) that is referred to at the end of Section 5.

As may be seen in [7], Renewal theory goes very much deeper than these finite results, the basic emphasis being on the study of the limiting behavior of $M(t)$. In the following section, the limiting stationarity of an M.R.P. will be discussed as an application of the main limit theorem of Renewal theory, due to Blackwell and Smith, which states that if k is any non-negative, non-increasing, Lebesgue-integrable function defined on $[0, \infty)$, then

$$(6.5)\qquad k(t)*M(t) \to \begin{cases} \mu^{-1} \int_0^\infty k(x)\, dx & \text{if } F \text{ is non-lattice} \\ h\mu^{-1} \sum_{n=0}^{\infty} k(nh) & \text{if } F \text{ is lattice with span } h, \end{cases}$$

where for the lattice case, $t \to \infty$ over multiples of h.

7. Stationary probabilities. In this section we derive the limiting form of a certain d.f. pertaining to an M.R.P., and use this to select the appropriate initial distribution for making the corresponding S.-M.P. stationary. The method of derivation is to compute the pertinent d.f., a useful formula in itself, and to apply the Blackwell-Smith theorem (6.5) to it to obtain its limiting form.[5] In [3], Smith derived the asymptotic form of the probabilities $P_{ij}(t)$ of an M.R.P. For discrete or continuous parameter Markov processes it is known that such is sufficient for ascertaining the initial distribution which makes the process

[5] As was pointed out by the referee, it is also possible to apply the more general Renewal theorem of Smith (Corollary 2.1 of [3]) upon making a suitable redefinition of the state space.

stationary. However, the problem of obtaining the stationary probabilities of an M.R.P. is not solved by deriving the limits of the $P_{ij}(t)$. One must instead consider the problem of finding the limit, as $t \to \infty$, of the probability of being in state j at time t, of making the next transition sometime before $t + x$ and of this next transition being into state k.

First of all, define formally the probability just referred to, as

$$R_{jk}^{(i)}(x; t) = P[Z_t = j, J_{N(t)+1} = k, S_{N(t)+1} \leqq t + x \mid Z_0 = i].$$

Now by (5.14) and (5.15) one may straightforwardly show that

$$\begin{aligned} R_{jk}^{(i)}(x; t) &= \sum_{n=0}^{\infty} P[J_n = j, J_{n+1} = k, S_n \leqq t < S_{n+1} \leqq t + x \mid Z_0 = i] \\ &= \sum_{n=0}^{\infty} [Q_{jk}(t + x) - Q_{jk}(t)] * Q_{ij}^{(n)}(t). \end{aligned}$$

Hence by Theorem 5.2 and (5.13) one may write

$$\begin{aligned} R_{jk}^{(i)}(x; t) &= [Q_{jk}(t + x) - Q_{jk}(t)] * [M_{ij}(t) + \delta_{ij} U_0(t)] \\ &= [Q_{jk}(t + x) - Q_{jk}(t)] * G_{ij}(t) * [M_{jj}(t) + U_0(t)] \\ &\qquad + \delta_{ij}[Q_{jk}(t + x) - Q_{jk}(t)]. \end{aligned} \tag{7.1}$$

Upon applying the Renewal theorem, (6.5), to (7.1) with $M_{jj} = M$ and $k(t)$ equal first to $p_{jk} - Q_{jk}(t)$ and then to $p_{jk} - Q_{jk}(t + x)$,[6] one obtains that if G_{jj} is a non-lattice d.f. (which may be seen to be equivalent to assuming that j is recurrent and that not every non-zero Q_{ir} for $i, r \in C_j$ is a lattice mass function) and if $b_{jk} < \infty$

$$\begin{aligned} \lim_{t \to \infty} R_{jk}^{(i)}(x; t) &= G_{ij}(\infty) \mu_{jj}^{-1} \int_0^{\infty} [Q_{jk}(t + x) - Q_{jk}(t)]\, dt \\ &= G_{ij}(\infty) p_{jk}\, \mu_{jj}^{-1} \int_0^{x} [1 - F_{jk}(y)]\, dy \end{aligned} \tag{7.2}$$

If G_{jj} is a lattice d.f. of span h, (that is, j is recurrent and all non-zero Q_{ir} for $i, r \in C_j$ are lattice mass functions) and if $b_{jk} < \infty$, then

$$\lim_{n \to \infty} R_{jk}^{(i)}(x; nh) = G_{ij}(\infty) h \mu_{jj}^{-1} \sum_{n=0}^{[x/h]+1} [1 - F_{jk}(nh)] \tag{7.3}$$

where $[y]$ is the largest integer *less* than y.

Suppose G_{jj} has variation less than one, and so is not a d.f. Then, as seen, for example, from (5.13), $\lim_{t \to \infty} M_{jj}(t) = [1 - G_{jj}(\infty)]^{-1} < \infty$ and hence it follows directly from (7.1) that $\lim_{t \to \infty} R_{jk}^{(i)}(x; t) = 0$ for all i, k and all $x \geqq 0$. The above results are summarized in

THEOREM 7.1.

(i) *If state j is recurrent and $b_{jk} < \infty$, then*

[6] It may easily be demonstrated that the function G_{ij} causes no difficulty in applying (6.5), for which the family of k functions has been unnecessarily restricted.

$$\lim_{t\to\infty} R_{jk}^{(i)}(x;t) = G_{ij}(\infty)p_{jk}\mu_{jj}^{-1}\int_0^x [1 - F_{jk}(y)]\,dy, \tag{7.4}$$

where it is understood that if G_{jj} *is a lattice* d.f. *then both* t *and* x *may take on as values only multiples of its span.*

(ii) *If state* j *is transient, then* $\lim_{t\to\infty} R_{jk}^{(i)}(x;t) = 0$ *for all* i, k *and all* $x \geqq 0$.

Since $P_{ij}(t) = \sum_{k=1}^{m} R_{jk}^{(i)}(\infty;t)$ and $m < \infty$, one obtains as a consequence of this theorem, the following result of Smith ([3], Theorem 5).

Corollary 7.1. *For an* M.R.P. *for which* $\eta_j < \infty$,

$$\lim_{t\to\infty} P_{ij}(t) = G_{ij}(\infty)\eta_j/\mu_{jj}, \tag{7.5}$$

with the understanding that t *takes on only multiples of the span if* G_{ij} *is a lattice* d.f.

Actually, in [3] the right hand side of (7.5) is given as

$$G_{ij}(\infty)\int_0^\infty x dD_j(x) \Big/ \int_0^\infty x\, dK_{ij}(x)$$

where D_j is as defined in (3.10) and where

$$K_{ij}(t) = P[Z_u \neq i, Z_v = j \quad \text{for some} \quad u < v \leqq t \mid Z_0 = i].$$

That is, K_{jj} is the first passage time d.f. of state i in the corresponding S.M.P. determined by $(m, \mathbf{A}, \mathfrak{Q}^*)$, (cf., Section 3 of [1]), in which a transition into state i from itself is not observed. The equivalence of the two limits follows from (3.11) and the relationships for $i \neq j$;

$$P_{ij} = K_{ij} * (1 - K_{jj})^{(-1)} * (1 - D_j), \qquad \pi_{ij} = k_{ij}(1 - d_j)(1 - k_{jj})^{-1}, \tag{7.6}$$

$$P_{jj} = (1 - K_{jj})^{(-1)} * (1 - D_j), \qquad \pi_{jj} = (1 - d_j)(1 - k_j)^{-1}, \tag{7.7}$$

$$K_{jj} = (1 - Q_{jj})^{(-1)} * (G_{jj} - Q_{jj}), \qquad k_{jj} = (g_{jj} - q_{jj})(1 - q_{jj})^{-1}, \tag{7.8}$$

which are obtained using the methods and results of Section 3.

For a Renewal process $(m = 1)$. Theorem 7.1. reduces to a result due essentially to Doob [8] (cf., also Smith [9]). Actually if there were to exist only one recurrent state (and hence one for which $p_{jj} = 1$) then Theorem 7.1 is essentially this result.

Assume throughout the remainder of this section that the M.R.P. under consideration has only one recurrent class, C say, and that it is a positive class. By Theorem I.5.1.(c) this means that $\eta_j < \infty$ for all $j \in C$. Assume also, for simplicity of notation, that each G_{jj} is non-lattice for every recurrent state j. Under these assumptions it follows that for every state i, $G_{ij}(\infty) = 1$ and $\mu_{jj} < \infty$ whenever $j \in C$. Consequently,

$$\lim_{t\to\infty} R_{jk}^{(i)}(x;t) = \begin{cases} p_{jk}\,\mu_{jj}^{-1}\int_0^x [1 - F_{jk}(y)]\,dy & \text{if } j \in C \\ 0 & \text{if } j \notin C \end{cases} \tag{7.9}$$

which limits are independent of i.

Define now a slightly more general process than an M.R.P., namely one in which the first transition time and state, (X_1, J_1), has a d.f. determined by an auxillary matrix $\tilde{\mathfrak{Q}}$ of transition distributions. That is, it may be considered as an M.R.P. with a random origin determined by $\tilde{\mathfrak{Q}}$. In keeping with existing terminology in Renewal theory, define a *general Markov Renewal process* (G.M.R.P.) determined by $(m, \mathbf{A}, \tilde{\mathfrak{Q}}, \mathfrak{Q})$ as a functional of the process $\{(\tilde{J}_n, \tilde{X}_n): n \geqq 0\}$ in exactly the same way as is an M.R.P. (cf., Section I.3.), with the following different probabilistic structure on the $(\tilde{J}, \tilde{X})$-process. Let

$$\begin{aligned} &\tilde{X}_0 = 0 \text{ a.s.}, \qquad P[\tilde{J}_0 = k] = a_k \\ (7.10) \quad &P[\tilde{J}_1 = k, \tilde{X}_1 \leqq x \mid J_0] \overset{\text{a.s}}{=} \tilde{Q}_{J_0,k}(x) \\ &P[\tilde{J}_n = k, \tilde{X}_n \leqq x \mid \tilde{J}_0, \tilde{J}_1, \tilde{X}_1, \cdots, \tilde{J}_{n-1}, \tilde{X}_{n-1}] \overset{\text{a.s.}}{=} Q_{J_{n-1},k}(x) \end{aligned}$$

for $n > 1$. Compare this description particularly with that of Definition I.3.3. A similar definition may be made for a G.S.-M.P.

Define $\tilde{\mathbf{A}} = (\tilde{a}_1, \cdots, \tilde{a}_m)$ with $\tilde{a}_j = \eta_j \mu_{jj}^{-1}$, and $\tilde{\mathfrak{Q}} = (\tilde{Q}_{ij})$ with

$$(7.11) \qquad \tilde{Q}_{ij}(t) = p_{ij}\eta_i^{-1} \int_0^t [1 - F_{ij}(y)]\, dy.$$

We wish to show that the G.S.-M.P. determined by $(m, \tilde{\mathbf{A}}, \tilde{\mathfrak{Q}}, \mathfrak{Q})$, which shall be denoted as the $\tilde{Z}$-process, is a stationary process.

From the definition of the functions $R_{jk}^{(i)}(x; t)$ it may easily be seen that they satisfy the following recursion relationship for all $t, s \geqq 0$.

$$(7.12) \qquad \begin{aligned} R_{jk}^{(i)}(x; t+s) = \sum_{r,u=1}^{m} \int_0^s R_{jk}^{(u)}(x; s-y)\, d_y\, R_{ru}^{(i)}(y; t) \\ + R_{jk}^{(i)}(s+x; t) - R_{jk}^{(i)}(s; t). \end{aligned}$$

Upon defining

$$(7.13) \qquad R_{jk}(x) = \lim_{t\to\infty} R_{jk}^{(i)}(x; t) = \tilde{a}_j \tilde{Q}_{jk}(x)$$

by (7.9) and (7.11), and letting $t \to \infty$ in (7.12) one obtains

$$(7.14) \qquad R_{jk}(x) = \sum_{r,u=1}^{m} \int_0^s R_{jk}^{(u)}(x; s-y)\, dR_{ru}(y) + R_{jk}(s+x) - R_{jk}(s)$$

for all $x, s \geqq 0$ and $1 \leqq j, k \leqq m$. For all $x, t \geqq 0$, $1 \leqq i, j, k \leqq m$, define

$$\tilde{R}_{jk}^{(i)}(x; t) = P[\tilde{Z}_t = j, \tilde{J}_{\tilde{N}(t)+1} = k, \tilde{S}_{\tilde{N}(t)+1} \leqq t + x \mid \tilde{Z}_0 = i],$$

which is to say that $\tilde{R}_{jk}^{(i)}(x; t)$ is the counterpart of $R_{jk}^{(i)}(x; t)$ as defined for a G.S.-M.P. For a G.S.-M.P. it may be seen that

$$(7.15) \qquad \tilde{R}_{jk}^{(i)}(x; t) = \sum_{u=1}^{m} \int_0^t R_{jk}^{(u)}(x; t-y)\, d\tilde{Q}_{iu}(y) + \delta_{ij}\, [\tilde{Q}_{ik}(t+x) - \tilde{Q}_{ik}(t)].$$

In words this equation states that in order to be in state j at time t and make the next transition into state k before time $t + x$, when the process starts in state

i, one must either make a first transition at some time $y \leqq t$ into some arbitrary state u in accordance with $\tilde{Q}_{iu}$ and then in the remaining time $t - y$ end up in state j and make the next transition into state k before time $t + x$, or, in case $i = j$, make the first transition during the time interval $(t, t + x]$, and make it into state k. The integrands in (7.15) are without $\sim$'s because after the initial transition has occurred the process behaves like an ordinary S.-M.P. From (7.15) it follows that

$$\sum_{i=1}^{m} a_i R_{jk}^{(i)}(x;t) = \sum_{i,u=1}^{m} \int_0^t R_{jk}^{(u)}(x;t-y)\, da_i\tilde{Q}_{iu}(y) + a_j\tilde{Q}_{jk}(t+x) - a_j\tilde{Q}_{jk}(t). \tag{7.16}$$

For the particular G.S.-M.P. determined by $(m, \tilde{\mathbf{A}}, \tilde{\mathfrak{Q}}, \mathfrak{Q})$ with $\tilde{\mathbf{A}}$, $\tilde{\mathfrak{Q}}$ as defined in the paragraph containing (7.11), (7.16) states, in view of (7.13), that

$$\sum_{i=1}^{m} \tilde{a}_i \tilde{R}_{jk}^{(i)}(x;t) = \sum_{i,u=1}^{m} \int_0^t R_{jk}^{(u)}(x;t-y)\, dR_{iu}(y) + R_{jk}(t+x) - R_{jk}(t).$$

As a consequence of (7.14), as well as of the definition of $\tilde{R}_{jk}^{(i)}(x;t)$, it therefore follows that

$$P[\tilde{Z}_t = j, \tilde{J}_{\tilde{N}(t)+1} = k, \tilde{S}_{\tilde{N}(t)+1} \leqq t + x] = R_{jk}(x) = p_{jk}\,\mu_{jj}^{-1} \int_0^x [1 - F_{jk}(y)]\, dy$$

which is independent of t. In particular $P[\tilde{Z}_t = j] = \eta_j\mu_{jj}^{-1} = \tilde{a}_j$.

For each t, define the three-dimensional r.v. $W_t = (\tilde{J}_{\tilde{N}(t)}, \tilde{J}_{\tilde{N}(t)+1}, \tilde{S}_{\tilde{N}(t)+1} - t)$ whose coordinates respectively record for a G.S.-M.P. the state it is in at time t, the state into which the next transition will be made, and the remaining time until the next transition will occur. It should be clear that the W-process is essentially equivalent to the $\tilde{Z}$-process in that almost all sample functions of the one can be determined from the other and vice versa. We now summarize the results of the preceding paragraphs in

THEOREM 7.2. *Consider a given* S.-M.P. *determined by* $(m, \mathbf{A}, \mathfrak{Q})$ *for which there is only one positive class. Define* $\tilde{\mathbf{A}} = (\tilde{a}_1, \cdots, \tilde{a}_m)$ *with* $\tilde{a}_i = \eta_i\mu_{ii}^{-1}$ *and* $\tilde{Q} = (\tilde{Q}_{ij})$ *with*

$$\tilde{Q}_{ij}(t) = p_{ij}\,\eta_i^{-1} \int_0^t [1 - F_{ij}(y)]\, dy,$$

the limiting transition distribution given in Theorem 7.1. *Then the* W*-process which corresponds to the* G.S.-M.P. *determined by* $(m, \tilde{\mathbf{A}}, \tilde{\mathfrak{Q}}, \mathfrak{Q})$ *is a stationary process whose marginal* d.f. *is given by*

$$P[W_t \leqq (j, k, x)] = \sum_{\substack{j \leqq i \\ k \leqq r}} p_{jk}\,\mu_{jj}^{-1} \int_0^x [1 - F_{jk}(y)]\, dy.$$

It should be clear that a result corresponding to that given in Theorem 7.2 is possible for a G.M.R.P. Indeed, if one represents the M.R.P. by the associated S.-M.P. as defined in Section I.3, then the result for the G.M.R.P. could be viewed as a special case of Theorem 7.2, generalized to cover the case of infinite m. Since this paper has dealt only with the case of $m < \infty$, we shall leave the analogue of Theorem 7.2 for M.R.P.'s until later.

8. Examples. In conclusion, we list briefly some specific examples of M.R.P.'s to indicate the broad scope of applications for this family of stochastic processes. First of all, the special case in which $Q_{ij} = p_i F_j$ for each i and j, the p_i's being real numbers and the F_j's being d.f.'s, arises in electronic counter theory and is studied in detail in [10]. This special case, by a slight reinterpretation of the sample functions, is seen to be essentially equivalent to the "zero order" M.R.P. in which $Q_{ij} = Q_i$ for each i and j.

A second very important special case of an M.R.P., although in one sense somewhat degenerate, is a zero-one process. That is, $m = 2$ and $p_{11} = p_{22} = 0$. For example, in a queueing model, the server is either in a busy state or in an idle state, and these states are entered alternately. In a counter problem, the counter is either dead or free, and it alternates between these two states. The only "parameters" of such zero-one processes are the two d.f.'s of the duration times of the two states. Of course, one still assumes independence between the successive time periods. For such a process it is clear that $Q_{12} = F_{12} = G_{12} = H_1 = D_1$, $Q_{21} = F_{21} = G_{21} = H_2 = D_2$, and $G_{11} = F_{12} * F_{21} = G_{22}$. Set $F_{12} = F_1$, and $F_{21} = F_2$. It therefore follows straightforwardly from Theorem 4.1 that, for $i = 1, 2$,

$$P_{ii}(t) = [1 - F_i(t)][1 - F_1(t) * F_2(t)]^{(-1)}$$

$$P_{3-i,i}(t) = [1 - F_i(t)] * F_{3-i}(t) * [1 - F_1(t) * F_2(t)]^{(-1)}$$

which checks with Theorem 1 of [11]. Zero-one processes arise in many problems, and hence have been studied by various authors with various emphases. Let it suffice here to mention the several papers by Takács (cf. [12] and references contained therein) in which the total time spent in one of the states during a given interval of time is particularly studied. The limiting normality of these "sojourn" times, under general conditions, derived by Takács, will be a corollary of a general Central limit theorem given in [13].

In an M.R.P., the probability distribution of the next state depends only on the present state of the process. An important generalization of an M.R.P. arises if one allows this distribution to depend also upon the time it took for the last transition. A special class of such processes is seen to be the class of 2-dependent stationary processes with non-negative r.v.'s, which fact indicates the different approach which would have to be used in studying the more general processes. The following example of an M.R.P. arises as a "first approximation" to this generalization. Let F_1 and F_2 be d.f.'s. Define, for $c > 0$, $G_c(x, y) = F_1(x)$ if $y < c$ and $= F_2(x)$ if $y \geqq c$. For the process $\{X_n ; n \geqq 1\}$, set

$$P[X_n \leqq x \mid X_{n-1}, \cdots, X_1] = G_c(x, X_{n-1}).$$

This process is clearly equivalent to an M.R.P. with two states, where the state indicates the subscript of the d.f., F_1 or F_2, which was used last. Therefore, for $i = 1, 2$.

$$Q_{i1}(x) = F_i(\min[x, c]), \qquad Q_{i2}(x) = F_i(\max[x, c]) - F_i(c).$$

Such a model and its analogue for arbitrary m may be applicable in life-testing or behavioural problems. One particular application to an inventory problem is the following. Suppose that state 2 indicates that a new efficient water pumping station is in use while state 1 indicates that an old inefficient auxillary station is also in use. Suppose that the transition times represent the successive times that it takes for the capacity of a reservoir to fall below a fixed level. We assume here that the reservoir is instantly refilled at these times. For such a model, the above example of an M.R.P. could be used in studying the cost of the pumping system as well as the proportion of times the auxillary pumping station is used.

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