# ON SOME METHODS OF CONSTRUCTION OF PARTIALLY BALANCED ARRAYS[1]

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**Summary.** Partially balanced arrays are generalizations of orthogonal arrays. Multifactorial designs derived from partially balanced arrays require a reduced number of assemblies in order to accommodate a given number of factors. For instance, an orthogonal array of strength two, six symbols and four constraints, would require at least $2.6^2 = 72$ assemblies. This is because there does not exist a pair of mutually orthogonal Latin Squares of order six. But for the same situation, a partially balanced array in 42 assemblies, is constructed in this paper. The method of construction is one of composition which utilizes the existence of a *pairwise partially balanced incomplete block design* and an orthogonal array.

**1. Introduction.** Suppose $A = ((a_{ij}))$ is a matrix, $i = 1, \cdots, m, j = 1, \cdots, N$ and the elements $a_{ij}$ of the matrix are symbols $0, 1, 2, \cdots, s - 1$. Consider the $s^t$ $1 \times t$ matrices $X' = (x_1, x_2, \cdots, x_t)$ that can be formed by giving different values to the $x_i$'s, $x_i = 0, 1, 2, \cdots, s - 1; i = 1, \cdots t$. Suppose associated with each $t \times 1$ matrix $X$ there is a positive integer $\lambda(x_1, x_2, \cdots, x_t)$ which is invariant under permutations of $(x_1, x_2, \cdots, x_t)$. If, for every $t$-rowed submatrix of $A$, the $s^t$ $t \times 1$ matrices $X$ occur as columns $\lambda(x_1, x_2, \cdots, x_t)$ times, then the matrix $A$ is called a *partially balanced array* of strength $t$ in $N$ assemblies, $m$ constraints (or factors), $s$ symbols (or levels) and the specified $\lambda(x_1, x_2, \cdots, x_t)$ parameters. When $\lambda(x_1, x_2, \cdots, x_t) = \lambda$ for all $(x_1, x_2, \cdots, x_t)$, the array is called an orthogonal array.

Orthogonal arrays were defined in [6] and [7] and construction of orthogonal arrays were considered in [1], [2], [3], [6] and [7]. Partially balanced arrays were defined in [5], where their use as multifactorial designs is also discussed.

In this paper, some methods of construction of partially balanced arrays are considered. One of the methods is applicable when $s = 2$ and derives partially balanced arrays from the well-known $\lambda - \mu - \nu$ configurations. The other method is an extension of the Bose-Shrikhande [2] method of construction of orthogonal arrays.

**2. An example of a partially balanced array.** Deleting the first three assemblies and the last row from the orthogonal array $A(18, 7, 3, 2)$ given in [1], one gets a

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partially balanced array of strength two, $s = 3$ symbols and $m = 6$ constraints in $N = 15$ assemblies. This array has the $\lambda(x_1, x_2)$ parameters

$$\lambda(x_1, x_2) = 2 \quad \text{if } x_1 \text{ and } x_2 \text{ are unlike,}$$
$$= 1 \quad \text{otherwise.}$$

The orthogonal array and the derived partially balanced array are given in Tables 1 and 2. The columns of the partially balanced array were a little re-arranged.

TABLE 1

*Orthogonal Array* $A(18, 7, 3, 2)$ *assemblies*

| Constraints | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 |
| 2 | 0 | 1 | 2 | 0 | 1 | 2 | 1 | 2 | 0 | 2 | 0 | 1 | 1 | 2 | 0 | 2 | 0 | 1 |
| 3 | 0 | 1 | 2 | 1 | 2 | 0 | 0 | 1 | 2 | 2 | 0 | 1 | 2 | 0 | 1 | 1 | 2 | 0 |
| 4 | 0 | 1 | 2 | 2 | 0 | 1 | 2 | 0 | 1 | 0 | 1 | 2 | 1 | 2 | 0 | 1 | 2 | 0 |
| 5 | 0 | 1 | 2 | 1 | 2 | 0 | 2 | 0 | 1 | 1 | 2 | 0 | 0 | 1 | 2 | 2 | 0 | 1 |
| 6 | 0 | 1 | 2 | 2 | 0 | 1 | 1 | 2 | 0 | 1 | 2 | 0 | 2 | 0 | 1 | 0 | 1 | 2 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 |

TABLE 2

*Partially balanced array* (15, 6, 3, 2) *assemblies*

| Constraints | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 |
| 2 | 0 | 2 | 1 | 1 | 2 | 0 | 0 | 1 | 2 | 2 | 0 | 0 | 2 | 1 | 1 |
| 3 | 1 | 0 | 2 | 2 | 1 | 0 | 2 | 0 | 1 | 2 | 1 | 2 | 0 | 0 | 1 |
| 4 | 1 | 1 | 0 | 2 | 2 | 2 | 0 | 2 | 0 | 1 | 2 | 1 | 0 | 1 | 0 |
| 5 | 2 | 2 | 1 | 0 | 1 | 1 | 2 | 2 | 0 | 0 | 0 | 1 | 1 | 0 | 2 |
| 6 | 2 | 1 | 2 | 1 | 0 | 2 | 1 | 0 | 2 | 0 | 1 | 0 | 1 | 2 | 0 |

Suppose, an orthogonal array $A(N, m, s, t)$ of index $\lambda$ is resolvable into two disjoint arrays. Further, let one of them be a partially balanced array or a *degenerate* partially balanced array (a degenerate array being one which has some but not all $\lambda(x_1, x_2, \cdots, x_t)$ equal to zero), with $\lambda(x_1, x_2, \cdots, x_t) < \lambda$ for all $t \times 1$ matrices $X$. Then the residual array is a partially balanced array with $\lambda$-parameters $\bar{\lambda}(x_1, x_2, \cdots, x_t) = \lambda - \lambda(x_1, x_2, \cdots, x_t)$. This provides a basis for the deletion process of deriving a partially balanced array from an orthogonal array.

## 3. Construction of partially balanced arrays for $s = 2$ from $\lambda - \mu - \nu$ configurations.

DEFINITION. A $\lambda - \mu - \nu$ configuration of $m$ elements is defined [4] as the configuration of $m$ elements taken $\nu$ at a time so that each set of $\mu$ elements shall occur together in just $\lambda$ of the sets.

Suppose there are $N_o$ sets of $\nu$ elements each in the configuration. Let $N_t$ denote the number of sets each containing a fixed subset of $t$ elements. Then it is easily seen that

$$N_t = \lambda \binom{m-t}{\mu-t} \Big/ \binom{\nu-t}{\mu-t} \qquad t = 0, 1, 2, \cdots, \mu. \tag{3.1}$$

Consider the matrix $A = ((a_{ij}))$ of order $m \times N_o$ derived from a $\lambda - \mu - \nu$ configuration of $m$ elements in $N_o$ sets in the following manner: Let $\alpha_1, \alpha_2, \cdots, \alpha_m$ denote the $m$ elements and $s_1, s_2, \cdots, s_{N_o}$ denote the $N_o$ sets of the configuration. Then let

$$\begin{aligned} a_{ij} &= 1 && \text{if } \alpha_i \text{ occurs in the set } s_j \\ &= 0 && \text{otherwise.} \end{aligned}$$

Consider a $\mu$-rowed submatrix of $A$ with elements $a_{ij}$ as defined above. Amongst the $N_o$ columns of the submatrix, a column matrix $X_{\mu,1}$ where its transpose $X'_{1,\mu} = (x_1, x_2, \cdots, x_\mu)$, $x_i = 0$ or $1$, $i = 1, \cdots, \mu$ occurs $\lambda(x_1, x_2, \cdots, x_\mu)$ times. Specifically, let $x_i = 1$ for $i = 1, \cdots, r$ and let $x_i = 0$ for $i = r+1, \cdots, \mu$ in $X$. Then it is easy to show that for such an $X$

$$\begin{aligned} \lambda(x_1, x_2, \cdots, x_\mu) &= N_r - \binom{\mu - r}{1} N_{r+1} + \binom{\mu - r}{2} N_{r+2} - \cdots \\ &= (-1)^{\mu-r} \Delta^{\mu-r} N_r \end{aligned} \tag{3.2}$$

where $\Delta$ stands for the symbol of finite difference,

$$\Delta N_r = N_{r+1} - N_r .$$

Value of $\lambda(x_1, x_2, \cdots, x_\mu)$ depends only on the count $r$ of unities in its argument and hence it is invariant under permutation of its arguments.

Now provided $\lambda(x_1, x_2, \cdots, x_\mu) > 0$ for all $s^\mu$ sets of $X$, we have

THEOREM 2.1. *The existence of a $\lambda - \mu - \nu$ of $m$ elements with $\lambda(x_1, x_2, \cdots, x_\mu)$ all positive, implies the existence of a partially balanced array of strength $\mu$ with parameters $s = 2$ and $\lambda(x_1, x_2, \cdots, x_\mu)$ as defined in* (3.2).

Well known examples of $\lambda - \mu - \nu$ configurations are the triple systems, quadruple systems, etc., which are defined in [4].

## 4. An extension of the Bose-Shrikhande method of construction of orthogonal arrays and its use in the construction of partially balanced arrays.

DEFINITION. A pairwise partially balanced design with parameters

$$(\nu, k_1, k_2, \cdots, k_m; b_1, b_2, \cdots, b_m; \lambda_1, \lambda_2, \cdots, \lambda_t; n_1, n_2, \cdots, n_t)$$

is defined as an arrangement of $\nu$ varieties in blocks of $m$ different sizes $k_1, k_2, \cdots, k_m$, there being $b_i$ blocks of size $k_i$, $\sum_{i=1}^m b_i = b$, satisfying the following conditions:

(i) No block contains a single variety more than once.

(ii) With respect to any variety, the remaining $\nu - 1$ varieties fall into $t$ categories, there being $n_i$ varieties in the $i$th category, called the $i$th associates of the variety; $\sum_{i=1}^t n_i = \nu - 1$.

(iii) Two varieties which are $i$th associates, occur together in $\lambda_i$ blocks, $i = 1, \cdots, t$.

Then the following relations among the parameters hold,

$$\sum_{i=1}^{m} b_i k_i(k_i - 1) = \sum_{i=1}^{t} n_i \nu \lambda_i = \nu \sum_{i=1}^{t} n_i \lambda_i . \tag{4.1}$$

Suppose there exist the orthogonal arrays

$$A_i\ (\lambda k_i^2, q_i, k_i, 2) \qquad i = 1, \cdots, m'$$

of strength two and index $\lambda$ and in $k_i$ symbols. Consider the pairwise partially balanced design defined earlier. There are $b_i$ blocks each of size $k_i$. These $b_i$ blocks provide $b_i$ sets of $k_i$ symbols each. Using each set of $k_i$ symbols once in the orthogonal array $A_i$, one gets $b_i$ such orthogonal arrays. If all such orthogonal arrays are arranged side by side, then one gets a matrix $A$ with number of columns $N = \lambda \sum_{i=1}^{m} b_i k_i^2$ and number of rows $q = \min(q_1, q_2, \cdots, q_m)$. In the columns of any two-rowed submatrix of matrix $A$, every ordered pair $(t_u, t_v)$ of two distinct symbols of varieties which are $i$th associates will occur $\lambda\lambda_i$ times and every ordered pair $(t_j, t_j)$ of two like symbols occur $\lambda r_j$ times, if the variety $t_j$ occurs in $r_j$ blocks of the pairwise partially balanced design. Hence we have.

THEOREM 4.1. *The existence of a pairwise partially balanced design with parameters* $(\nu; k_1, k_2, \cdots, k_m; b_1, b_2, \cdots, b_m; \lambda_1, \lambda_2, \cdots, \lambda_t; n_1, n_2, \cdots, n_t)$ *and of the orthogonal arrays* $A_i(\lambda k_i^2, q_i, k_i, 2)$ $i = 1, \cdots, m$, *imply the existence of the partially balanced array of strength two in* $\nu$ *symbols and* $q = \min(q_1, q_2, \cdots, q_m)$ *constraints and* $\lambda(x_1, x_2) = \lambda\lambda_i$, *where* $x_1, x_2$ *stand for two varieties which are* $i$th *associates and* $\lambda(x, x) = \lambda r_j$, *and where the variety* $x$ *occurs* $r_j$ *times in the pairwise partially balanced design.*

As an illustration, a partially balanced array which has been constructed using the method described above, is given below. This is a partially balanced array in $\nu = 6$ symbols, $N = 48$ assemblies, $m = 5$ constraints and

$$\begin{aligned} \lambda(x_1, x_2) &= 2 \quad \text{if } (x_1, x_2) \text{ are first associates} \\ &= 1 \quad \text{if } (x_1, x_2) \text{ are second associates} \\ &= 2 \quad \text{if } x_1 \text{ and } x_2 \text{ are like,} \end{aligned}$$

where $x_i$, $i = 1, \cdots, 6$ are the variety symbols. In constructing this array, the partially balanced design

$$(\nu = 6, r = 2, b = 3, k = 4, n_1 = 1, n_2 = 4, \lambda_1 = 2, \lambda_2 = 1)$$

in three blocks

$$\begin{matrix} x_1, & x_4, & x_2, & x_5 \\ x_2, & x_5, & x_3, & x_6 \\ x_3, & x_6, & x_1, & x_4 \end{matrix}$$

and the orthogonal array $A(16, 5, 4, 2)$ have been used.

TABLE 3

*Orthogonal Array* $A(16, 5, 4, 2)$

| $R$ | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | $t$ | $t$ | $t$ | $t$ | $t^2$ | $t^2$ | $t^2$ | $t^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $C$ | 0 | 1 | $t$ | $t^2$ | 0 | 1 | $t$ | $t^2$ | 0 | 1 | $t$ | $t^2$ | 0 | 1 | $t$ | $t^2$ |
| $L_1$ | 0 | 1 | $t$ | $t^2$ | 1 | 0 | $t^2$ | $t$ | $t$ | $t^2$ | 0 | 1 | $t^2$ | $t$ | 1 | 0 |
| $L_2$ | 0 | 1 | $t$ | $t^2$ | $t$ | $t^2$ | 0 | 1 | $t^2$ | $t$ | 1 | 0 | 1 | 0 | $t^2$ | $t$ |
| $L_3$ | 0 | 1 | $t$ | $t^2$ | $t^2$ | $t$ | 1 | 0 | 1 | 0 | $t^2$ | $t$ | $t$ | $t^2$ | 0 | 1 |

Making successively the identifications

| (1) | (2) | (3) |
|---|---|---|
| $1 = x_1$ | $1 = x_2$ | $1 = x_1$ |
| $t = x_2$ | $t = x_3$ | $t = x_3$ |
| $t^2 = x_4$ | $t^2 = x_5$ | $t^2 = x_4$ |
| $0 = x_5$ | $0 = x_6$ | $0 = x_6$ |

and using them on the above array in place of $(0, 1, t, t^2)$, one gets three arrays, $A_1$, $A_2$ and $A_3$. Then the array $A_0 = [A_1\, A_2\, A_3]$ is the desired partially balanced array in 6 symbols and 48 assemblies. Let $A^*$ denote the array derived from $A$ by truncating the first row and the first four columns (as indicated by the horizontal and vertical lines). Then the arrays $A_1^*$, $A_2^*$ and $A_3^*$ are obtained from $A^*$ using the three identifications of variety-symbols given above. Let $E$ denote the array

$$E: \begin{bmatrix} x_1 & x_2 & \cdots & \cdots & x_6 \\ x_1 & x_2 & \cdots & \cdots & x_6 \\ x_1 & x_2 & \cdots & \cdots & x_6 \\ x_1 & x_2 & \cdots & \cdots & x_6 \end{bmatrix}$$

Then the array $A_0^* = [E\, A_1^*\, A_2^*\, A_3^*]$ is a partially balanced array in $\nu = 6$ symbols, $N = 42$ assemblies, $m = 4$ constraints and $\lambda(x_i\, x) = 1$, $\lambda(x_i, x_j) = 2$ if $x_i$ and $x_j$ are first associates and $\lambda(x_i, x_j) = 1$ if they are second associates.

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