by analytic continuation [see MacRobert [2], page 122], the equality in (14) holds for $-\infty < \lambda < \infty$ and namely for the $n + k/2$ under consideration in expression (6).

Expression (6) can now be written as

$$f(\rho^2) = \left(\frac{1}{2\sigma^2}\right)^{k/2} (\rho^2)^{\frac{1}{2}(k-2)} \exp\left[-\frac{1}{2\sigma^2}(\rho^2 + r^2)\right] \sum_{n=0}^{\infty} \frac{(\rho^2 r^2/2^2\sigma^4)^n}{\Gamma(n+1)\Gamma(n+\frac{1}{2}k)} \quad (k \text{ odd}). \tag{15}$$

But (8), with k even, can be written in exactly the same form as (15). Thus, (15) is the density function for ρ^2, with k even or odd.

Letting $\gamma = r^2/2\sigma^2$ and $\chi'^2 = \rho^2/\sigma^2$, (15) can also be written as

$$f\left(\frac{\rho^2}{\sigma^2}\right) d\rho^2 = f(\chi'^2)\sigma^2\, d\chi'^2 = \frac{1}{2}\left(\frac{\chi'^2}{2}\right)^{\frac{1}{2}(k-2)} \exp(-\gamma) \exp\left(-\frac{\chi'^2}{2}\right) \cdot \sum_{n=0}^{\infty} \frac{(\frac{1}{2}\chi'^2\gamma)^n\, d\chi'^2}{n!\Gamma(n+\frac{1}{2}k)}. \tag{16}$$

Equation (16) is the non-central chi-square distribution, and the Fourier integral derivation of Equation (16) is different from that usually found in the literature—see Mann [3], pages 65–68 and Anderson [1], pages 112 and 113.

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A CHARACTERIZATION OF THE INVERSE GAUSSIAN DISTRIBUTION

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1. Introduction and summary. M. C. K. Tweedie [2] defined the inverse Gaussian distributions via the density functions

$$\begin{aligned} f(x; m, \lambda) &= [\lambda/(2\pi x^3)]^{\frac{1}{2}} \exp[-\lambda(x-m)^2/(2m^2x)] &\quad \text{for } x > 0 \\ &= 0 &\quad \text{for } x \leqq 0. \end{aligned} \tag{1}$$

The parameters λ and m are positive. The corresponding densities reflected about the origin, and with λ and m negative, may also be considered as in the Inverse Gaussian family. The characteristic function of the Inverse Gaussian

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distribution with parameters λ, m is

$$\phi(t) = \exp [\lambda\{1 - (1 - 2im^2t\lambda^{-1})^{\frac{1}{2}}\}/m], \qquad i = \sqrt{-1}, \tag{2}$$

for all real values of t. If $x_1, x_2, \cdots, x_n$ are n independent observations from (1), then $y = \sum_{j=1}^n x_j$ and $z = \sum_{j=1}^n x_j^{-1} - n^2y^{-1}$ are independently distributed. The distribution of y is $f(y, nm, n^2\lambda)$ and that of λz is Chi-Square with $(n - 1)$ degrees of freedom.

In this note, we prove that, if $x_1, x_2, \cdots, x_n$ are independently and identically distributed variates, with the existence of certain moments (different from zero), and if y and z are independently distributed, then the distribution of x_j is Inverse Gaussian.

2. A characterization of the Inverse Gaussian distributions.

THEOREM. *Let $x_1, x_2, \cdots, x_n$ be independently and identically distributed variates and let the expected values of x, x^2, x^{-1} and $(\sum_{j=1}^n x_j)^{-1}$ exist and be different from zero. Then the necessary and sufficient condition that the variates follow Inverse Gaussian distributions is that $y = \sum_{j=1}^n x_j$ and $z = \sum_{j=1}^n x_j^{-1} - n^2y^{-1}$ are independently distributed.*

PROOF. Let $\phi(t)$ and $F(x)$ be respectively the characteristic function and the distribution function of x. Then, since $|\exp(ipx)| = 1$ for any finite p, for any real value of t lying between $(-T, \infty)$, $T \neq \infty$, we have on account of the result (7.3.3) of Cramér ([1], p. 68),

$$\int dF(x) \int_{-T}^{t} \exp (ipx) \, dp = \int_{-T}^{t} dp \int \exp (ipx) \, dF(x) = \int_{-T}^{t} \phi(p) \, dp,$$

and, since $E(x^{-1})$ exists, we have

$$\int dF(x) \int_{-T}^{t} \exp (ipx) \, dp = \int (ix)^{-1}\{\exp (itx) - \exp (-iTx)\} \, dF(x).$$

Hence, we have

$$\int x^{-1} \exp (itx) \, dF(x) = i \int_{-T}^{t} \phi(p) \, dp + \int x^{-1} \exp (-iTx) \, dF(x). \tag{3}$$

Similarly, remembering x_i, $i = 1, 2, 3, \cdots, n$, as independent and identical variates and the existence of the expected value of y^{-1}, we have

$$\begin{aligned} \int \cdots \int y^{-1} \exp (ity) \prod_{j=1}^{n} dF(x_j) &= i \int_{-T}^{t} [\phi(p)]^n \, dp \\ &\quad + \int \cdots \int y^{-1} \exp (-iTy) \prod_{j=1}^{n} dF(x_j). \end{aligned} \tag{4}$$

Now since y and z are independently distributed, and $E(x^{-1})$ and $E(y^{-1})$ exist, i.e., $E(z)$ exists and, say, $E(z) = (n - 1)q$, we have

$$E[z \exp (ity)] = (n - 1)q[\phi(t)]^n. \tag{5}$$

Also, on substituting the values of z and y in the definition of expection, we have

$$E[z \exp(ity)] = \int \cdots \int \sum_{j=1}^{n} x_j^{-1} \prod_{k=1}^{n} \exp(itx_k)\, dF(x_k) \tag{6}$$
$$- n^2 \int \cdots \int y^{-1} \exp(ity) \prod_{k=1}^{n} dF(x_k).$$

Here we may note, on the right of (6), the signs of summation and integration can be interchanged. Then using (3) and (4) in (6) and noting (5), we have

$$n[\phi(t)]^{n-1}\left[\int_{-T}^{t} \phi(p)\, dp - i\int x^{-1} \exp(-iTx)\, dF(x)\right]$$
$$- n^2\left[\int_{-T}^{t} \{\phi(p)\}^n\, dp - i\int \cdots \int y^{-1} \exp(-iTy) \prod_{j=1}^{n} dF(x_j)\right] \tag{7}$$
$$+ i(n-1)q[\phi(t)]^n = 0.$$

Differentiating (7) with respect to (w.r.t.) t and then noting that $\phi(t) \neq 0$ in the neighborhood of $t = 0$, we have

$$\phi'(t)\left[\int_{-T}^{t} \phi(p)\, dp - i\int x^{-1} \exp(-iTx)\, dF(x)\right] = [\phi(t)]^2 - iq\phi'(t)\phi(t). \tag{8}$$

According to the assumptions of the theorem, $E(x)$ and $E(x^2)$ exist and are different from zero. Hence $\phi'(0) \neq 0$, $\phi''(0)$ exists and the differentiability of $\phi'(t)$ implies its continuity in the neighborhood of $t = 0$. Then, dividing (8) by $\phi'(t)$ and differentiating it w.r.t. t, we have

$$\phi(t) = iq\phi'(t) + \{[\phi(t)]^2\phi''(t)/[\phi'(t)]^2\}, \tag{9}$$

to be true in the neighborhood of $t = 0$, or to be true for all real values of t for which $\phi(t) \neq 0$ and $\phi'(t) \neq 0$ or $\phi'(t)/\phi(t) \neq 0$ and $\neq \infty$.

Let $h(t) = \log \phi(t)$. Then $h'(t) = \phi'(t)/\phi(t) \neq 0$ and $\neq \infty$ in the neighborhood of $t = 0$ and $h''(t) = \{\phi''(t)/\phi(t)\} - \{\phi'(t)/\phi(t)\}^2$. On substituting these values in (9), we rewrite it as

$$h''(t)[h'(t)]^{-3} = -iq. \tag{10}$$

Integrating (10) w.r.t. t and then using $h'(0) = im$, (say), we have

$$[h'(t)]^{-2} = -(1 - 2iqm^2t)/m^2.$$

That is, we can write

$$h'(t) = \pm i(1 - 2iqm^2t)^{-\frac{1}{2}}m \tag{11}$$

to be true in the neighborhood of $t = 0$ or to be true for all real values of t for which $h'(t) \neq 0$ and $\neq \infty$. Now integrating (11) w.r.t. t and noting $h(0) = 0$, we have

$$h(t) = \pm\{1 - (1 - 2iqm^2t)^{\frac{1}{2}}\}/(qm).$$

That is, we have

$$\phi(t) = \exp [\pm\{1 - (1 - 2iqm^2t)^{\frac{1}{2}}\}/(qm)] \tag{12}$$

to be true for all real values of t for which $\phi(t) \neq 0$ and $\phi'(t) \neq 0$. It was already demonstrated that (12) is valid in an interval around the origin in which $\phi(t) \neq 0$. Since $\phi(t)$ is continuous, one has $\lim_{h \downarrow 0} \phi(t_0 - h) = \phi(t_0) = 0$, however, it follows from (12) that this limit is not zero and this contradiction proves that $\phi(t) \neq 0$ for all t, so that (12) is valid for all t. On account of the uniqueness theorem on the characteristic function ([1], (10.3.1) p. 93), we have the distribution of x as Inverse Gaussian, with the parameters q^{-1} and $(\pm m)$.

Lastly, the converse result, namely if $x_1, x_2, \cdots, x_n$ are independently and identically distributed as Inverse Gaussian then y and z are independently distributed, is proved by M. C. K. Tweedie [2].

Thus, the theorem is completely proved.

3. Acknowledgment. The author is indebted to the referee for his guidance in the preparation of the paper.

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NULL DISTRIBUTION AND BAHADUR EFFICIENCY OF THE HODGES BIVARIATE SIGN TEST

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1. Summary. This note presents a simplified expression for the exact null distribution of the Hodges bivariate sign test. The form is suitable for computing both for small and large sample size and gives the limiting distribution easily. A small table is given for the limiting distribution. The Bahadur limiting efficiency of the test is computed relative to the bivariate Hotelling T^2 test for normal alternatives. The value $2/\pi$ is obtained, which is the same as for the one dimensional sign test relative to the t-test.

2. Introduction. Let $V_i = (X_i - X_i', Y_i - Y_i'), i = 1, 2, \cdots, n$, be a sample of n bivariate observations from a continuous distribution. The bivariate sign test was proposed by Hodges [5] in 1955 to test the hypothesis that the median is zero for the joint distribution. The test is based upon the statistic M, which

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