ON THE LINE—GRAPH OF THE COMPLETE BIGRAPH

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1. Introduction and summary. The line-graph, $L(G)$, of any given simple graph, G, is that graph whose points can be put in a one-to-one correspondence with the edges of G in such a way that two points in $L(G)$ are adjacent if, and only if, their corresponding edges in G are adjacent. The complete m by n bigraph, B_{mn}, contains mn edges which join each point in one set of m points to each point in a second set, disjoint from the first, containing n points. As general references on terminology see, e.g., Harary [4] and Ore [8].

If we suppose that $m \geqq n \geqq 1$ it can be seen that $L(B_{mn})$ has the following three properties:

(1) The graph has mn points each of which is adjacent to $m + n - 2$ other points.

(2) $n \binom{m}{2}$ of the pairs of adjacent joints are mutually adjacent to $m - 2$ other points and the remaining $m \binom{n}{2}$ pairs of adjacent points are mutually adjacent to $n - 2$ other points.

(3) Any two distinct nonadjacent points are mutually adjacent to two points.

The object of this note is to show that if any graph satisfies these three conditions then it is isomorphic to $L(B_{mn})$ except possibly when $(m, n) = (4, 4)$, $(4, 3)$ or $(5, 4)$. This will generalize a result of Shrikhande [10] (see also Mesner [7]) who, using different terminology, has already shown this for the case that $m = n$. The corresponding problem for the line-graph of the ordinary complete graph of n points has been treated by Connor [3], Shrikhande [9], Hoffman [5], and Chang [1], [2].

2. A lemma.

Lemma. *Let there be given a graph G satisfying Conditions* (1), (2), *and* (3), *where $m \geqq n \geqq 1$, but $(m, n) \neq (4, 4)$ $(5, 4)$ or $(4, 3)$. Let p_{11} and p_{12} be two adjacent points of G which are mutually adjacent to each of the $m - 2$ points in $A = \{p_{13}, \cdots, p_{1m}\}$. Let $C_1 = \{p_{21}, \cdots, p_{n1}\}$ be the set of $n - 1$ points which are adjacent to p_{11} but not to p_{i2}. Furthermore let there be at least $m - 2$ points in $A \cup C_1$ such that each of these points and p_{11} are mutually adjacent to $m - 2$ other points. Then $A \cup p_{11} \cup p_{12}$ and $C_1 \cup p_{11}$ are the vertex sets of complete graphs of m and n points, respectively, and no point of A is adjacent to any point of C_1.*

Proof. We consider first the case in which $m \geqq n \geqq 5$. No point in C_1 can be adjacent to more than one point in A without violating Condition (3) with respect to the point p_{12}. Therefore, each point in C_1 is adjacent to at least $n - 3$

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of the remaining $n - 2$ points in C_1 in order to satisfy Condition (2) with respect to the point p_{11}. Then, if there exist two nonadjacent points in C_1 they must be mutually adjacent to the remaining $n - 3$ points in C_1 as well as to p_{11}. This contradicts Condition (3) since $n - 2 \geqq 3$ in the case being considered. Hence every point in C_1 is adjacent to every other point in C_1. If some point in C_1, p_{21} say, is adjacent to some point in A then there are $n - 1$ points adjacent to both p_{21} and p_{11}. Thus $n - 1$ must equal $m - 2$, by (2), or $m = n + 1$.

If $m = n + 1$ suppose that some point in A, p_{13} say, is adjacent to some points of C_1. It is easily seen that p_{13} cannot be adjacent to more than one point of C_1 but not to all points of C_1 without violating Condition (3). But if p_{13} is adjacent to all points of C_1 then the number of points which are adjacent to both p_{11} and p_{13} is, including p_{12}, at least n which contradicts Condition (2). Hence p_{13} can be adjacent to at most one point in C_1. From Condition (2) it follows that there is at least one other point in A, p_{14} say, which is not adjacent to p_{13}. But p_{13} and p_{14} are each adjacent to at least $m - 5$ of the remaining $m - 4$ points of A and if $m > 6$ there will be at least one of these points which is adjacent to both p_{13} and p_{14}. This, however, contradicts Condition (3) since p_{13} and p_{14} are both adjacent to p_{11} and p_{12}.

The only alternative remaining to be treated, under the assumption that $m = n + 1$ and that some edges join points in A to points in C_1, is when $m = n + 1 = 6$. In this case it is not difficult to see that the only configuration which can satisfy Condition (2) without implying a contradiction of the type just described is one in which each point of A is adjacent to a different point in C_1 and p_{13} is adjacent to p_{15}, say, and p_{14} is adjacent to p_{16}. Suppose that p_{21} and p_{31} are the different points in C_1 which are adjacent to p_{13} and p_{15}, respectively. Then p_{21} and p_{15} are not adjacent to each other but are mutually adjacent to p_{31}, p_{13}, and p_{11}, contradicting Condition (3). Hence no point of A is adjacent to any point of C_1 under the given assumptions. This and the fourth sentence of the hypothesis of the lemma implies that each point in A is adjacent to every other point in A which suffices to complete the proof of the lemma when $m \geqq n \geqq 5$.

Next consider the case in which $n = 4$ and $m \geqq 6$. No point in C_1 can be adjacent to $m - 2$ other points of $A \cup C_1$ by an earlier remark and the fact that $m - 2 \geqq 4$. Hence, from the hypothesis, each of the $m - 2$ points of A must be adjacent to $m - 3$ other points of $A \cup C_1$. Using again the fact that no point in C_1 can be adjacent to more than one point in A it follows that there is at least one point of A which is not adjacent to any point in C_1 and hence is adjacent to each of the remaining $m - 3$ points of A. To avoid contradicting Condition (3) it must be that $A \cup p_{11} \cup p_{12}$ is the vertex set of a complete graph of m points. Condition (2) now implies that no point of A is adjacent to any point of C_1 and that $C_1 \cup p_{11}$ is the vertex set of a complete graph of 4 points, which completes the proof of the lemma for this case.

An entirely analogous argument proves the lemma when $n = 3$ and $m \geqq 5$.

Its validity when $n = m = 3$ follows from the results of Shrikhande [10] and the remaining cases, when $n = 1$ or 2, are also easily established.

3. The main theorem.

THEOREM. *Let there be given a graph G satisfying Conditions* (1), (2) *and* (3), *where* $m \geqq n \geqq 1$ *but* $(m, n) \neq (4, 4), (5, 4)$, *or* $(4, 3)$. *Then G is isomorphic to* $L(B_{mn})$.

PROOF. Condition (2) implies that there are $2n\binom{m}{2}$ points p in G, counting multiplicities, for which there exists another point q of G such that p is adjacent to q, and p and q are mutually adjacent to $m - 2$ other points of G. Since

$$2n\binom{m}{2}/mn = m - 1$$

it follows that there exist two points, p_{11} and p_{12} say, which satisfy the hypothesis of the lemma. Retaining the notation of the lemma let $C_2 = \{p_{22}, \cdots, p_{n2}\}$ be the set of $n - 1$ points which are adjacent to p_{12} but not to p_{11}; by symmetry it follows that $C_2 \cup p_{12}$ is the vertex set of a complete graph of n points and no point in C_2 is adjacent to any point in A. By applying Condition (3) to the points of C_1 with respect to p_{12} and to the points of C_2 with respect to p_{11} we see that each point of C_1 is adjacent to one, and only one, point of C_2 and vice versa. We may assume that the points are labelled in such a way that p_{j1} is adjacent to p_{j2}, for $j = 2, \cdots, n$.

The hypotheses of the lemma are now satisfied with any pair of distinct points, p_{1i} and p_{1j}, playing the roles of the points earlier labelled as p_{11} and p_{12}. Hence we may assert that for each point p_{1j}, $j = 1, \cdots, m$, there exists a set of $n - 1$ points, $C_j = \{p_{2j}, \cdots, p_{nj}\}$, such that $C_j \cup p_{1j}$ is the vertex set of a complete graph of n points and no point of C_j is adjacent to any point p_{1i}, where $i \neq j$. Also there are no points common to C_i and C_j and each point of C_i is adjacent to one and only one point of C_j, for $i, j = 1, \cdots, m$, $i \neq j$. This exhausts the points and edges of G.

Let p_{23} be that point of C_3 which is adjacent to p_{22}. If p_{23} is not adjacent to p_{21} suppose that p_{33} is the point of C_3 which is adjacent to p_{21} and that p_{31} is the point of C_1 which is adjacent to p_{23}. Then the nonadjacent points p_{21} and p_{23} are mutually adjacent to the distinct points p_{22}, p_{33}, and p_{31} which contradicts Condition (3). Hence p_{23} and p_{21} are adjacent. Letting p_{2j} be that point of C_j which is adjacent to $p_{2,j-1}$, for $j = 4, \cdots, m$, and repeating this argument it can be seen that the points p_{2j}, $j = 1, \cdots, m$, form the vertex set of a complete graph of m points.

Next let p_{33} be that point of C_3 which is adjacent to p_{32} and repeat the above argument. Carrying through this procedure $n - 1$ times it is seen that the mn points of G may be labelled p_{ij}, $i = 1, \cdots, n$, $j = 1, \cdots, m$, in such a way that p_{ij} is adjacent to p_{rs} if, and only if, $i = r$ or $j = s$, but not both. This shows that G is isomorphic to $L(B_{mn})$ under the given conditions.

4. Special cases. The above theorem does not hold when $(m, n) = (4, 4)$ and Shrikhande [10] has shown that there is just one counter-example. The determination of whether the theorem holds if $(m, n) = (4, 3)$ or $(5, 4)$ seems to be somewhat involved. Hoffman [6] has described a method for enumerating the counter-examples for the corresponding problem on the line-graph of the ordinary complete graph. Even if his method could be adapted to the present situation the effort required would likely be considerable.

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