NOTES

A NEW RESULT ON THE DISTRIBUTION OF QUADRATIC FORMS

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1. Introduction and summary. The distribution of the non-homogeneous[2] quadratic form $Q = \sum_1^n a_i(x_i - b_i)^2$, where the x_i are independent standardized normal variables and the a_i and b_i are real constants with $a_i > 0$, has recently [1] been obtained as an infinite linear combination in scaled central and non-central χ^2 distribution functions in the form

$$(1.1) \qquad P[Q \leqq t] = \sum_0^\infty c_j(p)F_{n+2j}(t/p) = \sum_0^\infty d_j(p)G_{n+2j;\kappa}(t/p).$$

Here p is an arbitrary positive constant, $F_{n+2j}(\cdot)$ is the distribution function of χ^2 with $n + 2j$ degrees of freedom and $G_{n+2j;\kappa}(\cdot)$ is the distribution function of χ^2 with $n + 2j$ degrees of freedom and non-centrality parameter

$$\kappa = (\sum_1^n b_i^2)^{\frac{1}{2}}.$$

The main purpose of the present paper is to rederive the first of the two expansions in (1.1), for the special case $p \leqq \min_i a_i$ when the expansion is a proper mixture representation, by a simple conditional probability argument which may be of some general interest. At the same time the $c_j(p)$ will be expressed in simpler and more appealing form[3] than in [1]. In essence, the distribution of Q (including that of $\sum_1^n a_i x_i^2$) is found to be almost a direct consequence of the distribution of the *special* non-homogeneous form $\sum_1^n (x_i - b_i)^2$, that is, of non-central χ^2 with n degrees of freedom and non-centrality parameter

$$(\sum_1^n b_i^2)^{\frac{1}{2}}.$$

Specifically, the distribution of Q can be expressed as a weighted non-central chi-square in the sense that the non-centrality parameter is not fixed but is rather a random variable with a given distribution depending on the a_i and b_i.

2. The distribution of Q in terms of χ^2 distributions. On setting

$$x_i = (p/a_i)^{\frac{1}{2}}z_i - (1 - p/a_i)^{\frac{1}{2}}y_i, \qquad i = 1, 2, \cdots, n,$$

Received December 17, 1962; revised May 20, 1963.

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[2] The homogeneous quadratic form $\sum_1^n a_i x_i^2$ is, of course, a special case obtained by setting $b_i = 0$.

[3] It appears to be difficult to prove the equivalence of the two forms of $\{c_j(p)\}$ directly, though this follows from the equality of the generating functions of the two sets of coefficients.

where p is an arbitrary positive constant $\leqq \min_i a_i$ and the z_i and y_i are independent standardized normal variates, we obtain

$$P[Q \leqq t] \equiv P\left[\sum_1^n \{z_i - (a_i/p - 1)^{\frac{1}{2}} y_i - (a_i/p)^{\frac{1}{2}} b_i\}^2 \leqq t/p\right].$$

Now, for *fixed* $\mathbf{y}' = (y_1, \cdots, y_n)$, the variate

$$\sum_1^n \{z_i - (a_i/p - 1)^{\frac{1}{2}} y_i - (a_i/p)^{\frac{1}{2}} b_i\}^2$$

is a non-central χ^2 with n degrees of freedom and non-centrality parameter $\kappa = \kappa(\mathbf{y})$ defined by $\kappa^2 = \sum_1^n \{(a_i/p - 1)^{\frac{1}{2}} y_i + (a_i/p)^{\frac{1}{2}} b_i\}^2$. However, it is well-known that the distribution of non-central χ^2 can be expressed as a mixture of central χ^2 distributions in which the weights form a Poisson series (see, e.g., [2], p. 247). Thus $P[Q \leqq t \mid \mathbf{y}] = \sum_0^\infty e^{-\frac{1}{2}\kappa^2}[(\frac{1}{2}\kappa^2)^j/j!]F_{n+2j}(t/p)$, from which we obtain[4]

$$P[Q \leqq t] = \sum_0^\infty c_j(p) F_{n+2j}(t/p), \tag{2.1}$$

where

$$\begin{aligned} c_j(p) &= E_\kappa\left[e^{-\frac{1}{2}\kappa^2} \frac{(\frac{1}{2}\kappa^2)^j}{j!}\right] = \int_{-\infty}^\infty e^{-\frac{1}{2}\kappa^2(\mathbf{y})} \frac{[\frac{1}{2}\kappa^2(\mathbf{y})]^j}{j!} \cdot (2\pi)^{-\frac{1}{2}n} e^{-\frac{1}{2}\mathbf{y}'\mathbf{y}}\, d\mathbf{y} \\ &= \prod_1^n \left(\frac{p}{a_i}\right)^{\frac{1}{2}} \exp\left(-\tfrac{1}{2}\sum_1^n b_i^2\right) \cdot \frac{1}{j!} \int_{-\infty}^\infty [\tfrac{1}{2}Q^*(\mathbf{v})]^j \cdot (2\pi)^{-\frac{1}{2}n} e^{-\frac{1}{2}\mathbf{v}'\mathbf{v}}\, d\mathbf{v}, \end{aligned} \tag{2.2}$$

after substituting for $\kappa(\mathbf{y})$ and setting $v_i = (a_i/p)^{\frac{1}{2}} y_i + (a_i/p - 1)^{\frac{1}{2}} b_i$. Here $Q^*(\mathbf{v}) = \sum_1^n \{(1 - p/a_i)^{\frac{1}{2}} v_i + (p/a_i)^{\frac{1}{2}} b_i\}^2$, and the last line of (2.2) shows that $c_j(p)$ can be expressed in terms of the jth moment of a non-homogeneous quadratic form in n independent standardized normal variables. Thus

$$P[Q \leqq t] = \prod_1^n \left(\frac{p}{a_i}\right)^{\frac{1}{2}} \exp\left(-\tfrac{1}{2}\sum_1^n b_i^2\right) \cdot \sum_0^\infty \frac{E[(\frac{1}{2}Q^*(\mathbf{x}))^j]}{j!} F_{n+2j}(t/p), \tag{2.3}$$

where $0 < p \leqq \min_i a_i$,

$$Q^*(\mathbf{x}) = \sum_1^n \{(1 - p/a_i)^{\frac{1}{2}} x_i + (p/a_i)^{\frac{1}{2}} b_i\}^2, \tag{2.3a}$$

and the x_i are (as before) independent $N(0, 1)$ variables. In particular, we may choose $p = 1$ if it is assumed, without loss of generality, that $a_i \geqq 1$.

It should be noted, from the first line of (2.2), that

$$c_j(p) \geqq 0 \ (j = 0, 1, \cdots), \qquad \sum_0^\infty c_j(p) = 1 \tag{2.4}$$

[4] Justification of the interchange of the expectation and summation operators (i.e., term by term integration), implied in the derivation of (2.1) from the preceding equation, is trivial.

for $0 < p \leqq \min_i a_i$, i.e., the $c_j(p)$ form a probability series for each p in the stated range. (This confirms a result stated previously in (5.8) of [1].) Thus, (2.3) is a representation of the distribution of Q as a mixture of scaled χ^2 distributions. Furthermore, the series in (2.3) is uniformly convergent over the extended t-axis $-\infty \leqq t \leqq +\infty$.

In [1], the $c_j(p)$ were expressed, for all $p > 0$, as expectations of polynomials in linear and (homogeneous) quadratic functions of the x_i. The present form of the $c_j(p)$, for $0 < p \leqq \min_i a_i$, namely,

$$c_j(p) = \prod_1^n (p/a_i)^{\frac{1}{2}} \exp\left(-\tfrac{1}{2}\sum_1^n b_i^2\right)\cdot E[(\tfrac{1}{2}Q^*(\mathbf{x}))^j]/j!,$$

is simpler and more suggestive. A convenient recursion formula for the $c_j(p)$ is given in [1].

REFERENCES

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