EXISTENCE, UNIQUENESS AND MONOTONICITY OF SEQUENTIAL PROBABILITY RATIO TESTS[1]

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Summary. Existence and uniqueness questions concerning SPRT's for testing one distribution against another in the case of independent and identically distributed random variables are considered under general conditions. Tests that need not take an observation ($n \geqq 0$), as well as tests that are required to take at least one observation ($n \geqq 1$) are being considered. SPRT's are allowed to have arbitrary stopping rule at the stopping bounds. The error point α of a test is the vector of error probabilities, and $A(u, v)$ is the set of all error points of SPRT's with stopping bounds $u, v (u \leqq v)$. For given $u < v$ there are four nonrandomized SPRT's (three if $u = v$), where u may be a stopping point or a continuation point, and similarly v. It is shown in Section 4, Theorem 6, that the error points of these four tests are the extreme points of the convex set $A(u, v)$. A special mixture of these four tests is denoted $R(s, t)$, its error point $\alpha(s, t)$, where $s = (u, \lambda)$ and $t = (v, \mu)$ are the randomized stopping bounds, and s, t may be considered points in an ordered topological space Z. Let $D = \{\alpha : \alpha_i \leqq \alpha_i(s, s) \text{ for some } s \,\varepsilon\, Z \text{ and } i = 1, 2\}$.

Using the characterization of SPRT's as Bayes tests, it is shown in Theorem 2 that there is a SPRT with given error point α^* if and only if $\alpha_1^* + \alpha_2^* \leqq 1$ in the case $n \geqq 0$, and $\alpha^* \,\varepsilon\, D$ in the case $n \geqq 1$. In Theorem 1 a somewhat stronger result is claimed, namely that the SPRT in Theorem 2 can be taken to be a test of the form $R(s, t)$. The main tool in dealing with the uniqueness question is the monotonicity Theorem 3, stating that if of a test $R(s, t)$ s is decreased or t increased, and the new s, t have positions u, v, then the change $\Delta\alpha$ in the error point satisfies $u\Delta\alpha_1 + \Delta\alpha_2 \leqq 0$ and $\Delta\alpha_1 + \Delta\alpha_2/v \leqq 0$, with at least one strict inequality unless the new test is equivalent to the old one. Theorem 4 says that a test of the form $R(s, t)$ with given error point is unique up to an equivalence. On the other hand, a SPRT is in general not unique. Theorem 5 claims only uniqueness of the stopping bounds u, v, up to an equivalence, in the case of a SPRT with given error point.

1. Introduction. Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables, having joint distribution either P_1 (hypothesis H_1) or P_2 (hypothesis H_2). For any test T of H_1 against H_2 let $\alpha_i(T) = P_i(T \text{ rejects } H_i)$ be the error probabilities, $i = 1, 2$. The vector $\alpha(T) = (\alpha_1(T), \alpha_2(T))$ will be called the *error point* of T. For $n = 1, 2, \cdots$ let $Y_n = p_{2n}/p_{1n}$ be the probability ratio at the nth stage of sampling, where p_{in} is the joint density of

Received April 2, 1962; revised May 27, 1963.

[1] Work supported by the National Science Foundation, grants G-11382 and G-21507.

$X_1, \cdots, X_n$ under H_i with respect to a measure dominating both P_i. At the zero-th stage of sampling we shall define $Y_0 = 1$. In this paper a sequential probability ratio test (SPRT), with lower stopping bound u and upper stopping bound v, will be understood to be as it was defined by Wald in [7], except that the stopping rule at the stopping bounds is arbitrary. That is, whenever $Y_n = u$ or $= v$ we have the option of stopping or continuing sampling, with the (possibly randomized) decision depending in a measurable way on $X_1, \cdots, X_n$. Thus, the test does not necessarily strictly depend only on the sequence $\{Y_n\}$. Let $C(u, v)$ be the class of all SPRT's with stopping bounds u, v, and $A(u, v)$ the set of its error points. We shall consider two kinds of SPRT's: those that start at sampling stage $n = 0$, and those that start at $n = 1$. The first kind is permitted to take no observation (this will happen if $u > 1$ or $v < 1$, since $Y_0 = 1$), and the second kind is forced to take at least one observation, no matter what u and v are. If necessary, the tests will be distinguished by the specification $n \geqq 0$, $n \geqq 1$, respectively. The notation $C(u, v)$, $A(u, v)$, will be used for both kinds of tests.

There are four members of $C(u, v)$ that are of special interest. We shall denote them $T(u, v)$, $T(u-, v)$, $T(u, v+)$, and $T(u-, v+)$. The notation u or v means that the test stops whenever $Y_n = u$ or $= v$, whereas $u-$ or $v+$ means that the test continues whenever $Y_n = u$ or $= v$. Thus, the SPRT defined by Wald in [7] is of the form $T(u, v)$. If $u = v$, $T(u, v)$ is not defined, but the remaining three are. The error points of the four tests will be denoted $\alpha(u, v)$, $\alpha(u-, v)$, etc. Any mixture of these four tests is a member of $C(u, v)$. Of particular interest is the type of mixture that can be described as follows: if $u < v$ choose u to be a stopping point with probability λ, a continuation point with probability $1 - \lambda$, and, independently, v to be a continuation point with probability μ, a stopping point with probability $1 - \mu$. This suggests putting $s = (u, \lambda)$, $t = (v, \mu)$, and calling s, t the *randomized stopping bounds* of a test which we shall denote $R(s, t)$, and its error point by $\alpha(s, t)$. A modification of this definition of $R(s, t)$ is necessary if $u = v$, $\lambda \leqq \mu$: choose $T(u-, u)$, $T(u, u+)$, and $T(u-, u+)$ with probabilities $1 - \mu$, λ, and $\mu - \lambda$, respectively.

In this paper the following questions will be considered: Given a point α^* in the error plane, is there a SPRT with $n \geqq 0$ (or with $n \geqq 1$, respectively) that has α^* as its error point? Is this test unique in some sense? The same questions can be asked for SPRT's of the special form $R(s, t)$. We shall first summarize some known results in this respect. For tests of the type $T(u, v)$ and $n \geqq 1$ (i.e. Wald's original SPRT) the uniqueness question has been answered in the affirmative by Weiss [10] if the probability ratio is continuously distributed, by Anderson and Friedman [1] assuming $u \leqq 1 \leqq v$, and in [13] without restrictions. For those same tests and continuously distributed probability ratio the existence question has been answered in [11]: let D be the set in the α-plane bounded by the coordinate axes and the locus of points $\alpha(u, u)$, $0 < u < \infty$, then there is a test $T(u, v)$ with $n \geqq 1$ and given error point α^* if and only if

$\alpha^* \varepsilon D$. If $n \geqq 1$ is replaced by $n \geqq 0$, then $\alpha^* \varepsilon D$ should be replaced by $\alpha_1^* + \alpha_2^* \leqq 1$.

If the distributions P_i are arbitrary, no simple results to the existence question can be expected unless randomization is allowed. Consequently, we consider all SPRT's that constitute the classes $C(u, v)$, as defined in the beginning of this section. Alternatively, it suffices to look only at tests of the form $R(s, t)$. There are mainly two methods of attacking the existence question. The first is by making use of the characterization of SPRT's as Bayes tests. This was done by Ghosh [4], and we shall return to this type of proof in Section 2. This method furnishes a shorter proof, but also somewhat weaker result than the second method does. The latter was followed in [12]. It considers only tests of the form $R(s, t)$ and studies the properties of $\alpha(s, t)$ as a function of s and t. To facilitate this study and obtain results that are formally the same as in the case of non-randomized stopping bounds and continuously distributed probability ratio, s and t are considered points in a certain ordered topological space Z. Here Z is a space of points $z = (x, y)$, $0 < x < \infty$, $0 \leqq y \leqq 1$, with the points $(0, 1)$ and $(\infty, 0)$ added. The latter two points will be denoted 0 and ∞, respectively. The ordering is lexicographical, and the topology is generated by the intervals defined by the ordering. It can then be verified that the $\alpha_i(s, t)$ are, in each variable separately, continuous and monotonic on Z. Let $D = \{\alpha : \alpha_i \leqq \alpha_i(s, s)$ for some $s \varepsilon Z$ and both $i\}$, i.e. D is the closed set bounded by the coordinate axes and the locus of error points $\alpha(s, s)$, $s \varepsilon Z$ (in the continuous case this is the same D as defined before). The type of existence proof in [11] for the continuous case can be carried over to the general case by letting Z play the role that the real line did before. For future reference we state the result as

THEOREM 1. *There is a test $R(s, t)$ with $n \geqq 0$ and error point α^* if and only if $\alpha_1^* + \alpha_2^* \leqq 1$, and there is a test $R(s, t)$ with $n \geqq 1$ and error point α^* if and only if $\alpha^* \varepsilon D$.*

Uniqueness results in the case of arbitrary P_i are somewhat less simple. To begin with, in the case of discrete distributions it may occur that there exist u_1 and u_2, $0 < u_1 < u_2 < \infty$, such that $P_i(u_1 < Y_n < u_2) = 0$ for all n (for each n this probability is either 0 for both i or positive for both i). Then if $u_1 < u < u' < u_2$, two SPRT's with the same upper stopping bound v (and same stopping rule at v) but lower stopping bounds u, u', respectively, can differ only on a set of sample sequences of P_i probability 0 for both i. We shall call two such tests *equivalent*. A similar situation may obtain at the upper stopping bound. Clearly, two equivalent tests have the same error point. Consequently, any uniqueness has to be understood up to an equivalence. For instance, the uniqueness property of $T(u, v)$ is of that nature [1] [13]. We shall deal with the uniqueness question more fully in Section 3. It will be shown that, up to an equivalence, a SPRT with given error point is in general not unique, but a test of the form $R(s, t)$ is. On the other hand, a SPRT enjoys a certain amount of uniqueness, in that its stopping bounds are unique, up to an equivalence.

The uniqueness and nonuniqueness results follow from a monotonicity property of tests $R(s, t)$ (Theorem 3) and from the geometry of the sets $A(u, v)$. In Section 4 it is shown that the extreme points of the convex set $A(u, v)$ are the four points $\alpha(u(-), v(+))$. Thus, if these four points are distinct, and α^* is an interior point of $A(u, v)$, there are many mixtures of the four tests $T(u(-), v(+))$, and thus many nonequivalent SPRT's, all having error point α^*. However, all these SPRT's have stopping bounds u, v.

2. An existence theorem. The introduction of losses for wrong decisions, costs for making observations, and an a priori distribution on the two hypotheses, has shown itself a valuable mathematical tool in proving properties of SPRT'S. One of the uses to which it can be put is to prove the existence of a SPRT with given error point. Ghosh made an application of this in [4].

If the cost per observation is constant and the same for both hypotheses, then Wald and Wolfowitz [9] and Arrow, Blackwell and Girshick [2] have shown that a test is Bayes if and only if it is equivalent to a SPRT. One of the assumptions made in [8] is that the P_i are either discrete or absolutely continuous with respect to Lebesgue measure (Assumption 3.1 in [8]). This, however, can easily be verified to be irrelevant. In other words, the characterization of Bayes tests as SPRT's remains valid for any pair of distributions.

A little less obvious is the question of compactness of the class of all measurable tests if the distributions are arbitrary. In [8] the definition of regular convergence of a sequence of decision functions is given separately for discrete and for continuous distributions. However, it is obvious how to generalize this notion in the case of a dominated family of distributions. In particular, this can be done when there are only two distributions. It follows as a special case of a theorem by LeCam [6] (Theorem 2) that the class of sequential tests is compact relative to regular convergence (this can also be proved very simply directly). Furthermore, if $T_j \rightarrow T$ in the regular sense, then we have convergence of the joint distribution (for both i) of sample size and terminal decision under T_j to that of T. As a consequence, we have $\liminf \alpha_i(T_j) \geqq \alpha_i(T)$, $\liminf \nu_i(T_j) \geqq \nu_i(T)$, where, for any test T, $\nu_i(T)$ stands for the expected sample size of T under H_i. That is, regular convergence implies weak intrinsic convergence in the sense of [4] Lemma 1 (see also [8] Theorem 3.2). Thus, the class of all sequential tests is compact in the sense of weak intrinsic convergence, and the same is true for the class of all tests with $n \geqq 1$. (It is of some interest to note that if $T_j \rightarrow T$ and T has a closed stopping rule, then $\alpha_i(T_j) \rightarrow \alpha_i(T)$.)

Ghosh [4] used the decision theoretic approach to study several properties of SPRT's. One of the results (Theorem 2.3 in [4]) is the existence of a SPRT with given error point, under certain conditions. As indicated in [4] in a remark, most of these conditions can be relaxed to obtain the same result. However, Assumption 3.1 of [8] was made throughout. This, too, turns out to be unnecessary, by the above remarks on regular and weak intrinsic convergence. Thus, the existence proof outlined by Ghosh in [4] in his remark is valid without any assumptions, provided the proper definition of regular convergence is used. For the sake of easier accessibility, the proof is given in some detail below, constituting the proof of the "if" part of Theorem 2. One of the main devices in the proof is borrowed from Kiefer and Weiss [5], Section 4 (this device was also used in [4] to obtain Theorem 2.3, but the proof of the latter is somewhat different from the proof of Theorem 2 below).

THEOREM 2. *There is a* SPRT *with* $n \geqq 0$ *and error point* α^* *if and only if* $\alpha_1^* + \alpha_2^* \leqq 1$. *There is a* SPRT *with* $n \geqq 1$ *and error point* α^* *if and only if* $\alpha^* \varepsilon D$.

PROOF. The "only if" in the first sentence is immediate, for every SPRT has $\alpha_1 + \alpha_2 \leqq 1$. In the second sentence, if a SPRT T with $n \geqq 1$ would have $\alpha(T) \not\varepsilon D$, then there is another SPRT T^* that takes exactly one observation, so that $\nu_i(T^*) \leqq \nu_i(T)$, and such that $\alpha_i(T^*) \leqq \alpha_i(T)$, with at least one of these four inequalities strict. This, however, violates the optimum property (OP) of SPRT's with $n \geqq 1$, [3] [4] (it would be sufficient to consider the OP of SPRT's with $n \geqq 1$ within its own class [13]).

To prove the "if" part in the first sentence of the theorem, consider first the case $\alpha_1^* = 0$. A SPRT with $v = \infty$ will have $\alpha_1 = 0$, and as u increases from 0 to ∞, α_2 increases from 0 to 1. Thus, $\alpha_2 = \alpha_2^*$ for some u, with possibly randomization at u. The case $\alpha_2^* = 0$ is analogous. Now suppose $\alpha_i^* > 0$. Take any (g_1, g_2), with $0 < g_i < 1$ and $g_1 + g_2 = 1$, and, for any test T, put $\nu_0(T) = \sum g_i\nu_i(T)$. The set S of all $(\alpha_1, \alpha_2, \nu_0)$ of tests T is convex. Since $\alpha_i^* > 0$, there are tests with $\alpha_i \leqq \alpha_i^*$ and $\nu_i < \infty$, for instance fixed sample size tests. Let n_0 be the infimum of $\nu_0(T)$, taken over all T for which $\alpha_i(T) \leqq \alpha_i^*$. Take a sequence $\{T_j\}$ such that $\nu_0(T_j) \to n_0$, then by weak intrinsic compactness there is a subsequence converging to some T^*, and we find

$$\alpha_i(T^*) \leqq \alpha_i^*, \qquad \nu_0(T^*) \leqq n_0.$$

But the latter inequality is obviously an equality, and the first two are also equalities, using the argument in [5], Section 4, for otherwise mixing with a test that takes no observation could produce a test having $\alpha_i = \alpha_i^*$ and having a smaller ν_0. Thus, T^* has error point α^*, and it remains to show that T^* is equivalent to a SPRT. Now since there is no point in S with $\alpha_i \leqq \alpha_i^*$, $\nu_0 < n_0$, there is in $(\alpha_1^*, \alpha_2^*, n_0)$ a supporting hyperplane of S with nonnegative direction numbers. That is, there exist a_1, a_2, a_3, all $\geqq 0$, such that $(\alpha_1^*, \alpha_2^*, n_0)$ minimizes $a_1\alpha_1 + a_2\alpha_2 + a_3\nu_0$ among all $(\alpha_1, \alpha_2, \nu_0)$ in S. From $\alpha_i^* > 0$ it follows that $a_3 > 0$. Put $W_i = a_i/g_ia_3$, $i = 1, 2$, then T^* minimizes

$$\sum g_i[W_i\alpha_i(T) + \nu_i(T)]$$

among all T. Since this expression is the risk of T when the losses for wrong decision are W_i, the cost per observation is one unit, and the a priori distribution is (g_1, g_2), it follows from [2] [9] that T^* is equivalent to a SPRT with $n \geqq 0$. The proof of the "if" part in the second sentence of the theorem is identical to the one just presented, after restricting the tests to $n \geqq 1$ and letting the tests that take exactly one observation play the same role as tests that take no observation did before.

3. Monotonicity and uniqueness. The following theorem is an extension to randomized stopping bounds of the monotonicity theorem (Theorem 2) in [13]. It is the main tool in establishing uniqueness.

THEOREM 3. *Let* $R(s, t)$ *and* $R(s', t')$, *with either* $n \geqq 0$ *or* $n \geqq 1$, *be nonequivalent, with* $0 \leqq s \leqq s' \leqq t' \leqq t \leqq \infty$ *and* $s = (u, \lambda)$, $t = (v, \mu)$. *Let* $\Delta\alpha = \alpha(s, t) - \alpha(s', t')$, *then*

$$u\Delta\alpha_1 + \Delta\alpha_2 \leqq 0 \tag{1}$$

$$\Delta\alpha_1 + (1/v)\Delta\alpha_2 \leqq 0 \tag{2}$$

and at least one of the inequalities (1), (2) *is strict.*

The steps in the proof that have to be verified, using Theorem 2 in [13], are straightforward and will be omitted. Note that (2) follows from (1) by interchanging the roles of H_1 and H_2, and vice versa. If in Theorem 3 $\Delta\alpha_1 = 0$, then from (1) and (2) and strict inequality in one of them it follows that $\Delta\alpha_2 < 0$. Similarly with subscripts 1 and 2 interchanged. Thus we have

COROLLARY 1. *If* $R(s, t)$ *and* $R(s', t')$ *are as in Theorem* 3, *then* $\Delta\alpha_1 = 0$ *implies* $\Delta\alpha_2 < 0$ *and* $\Delta\alpha_2 = 0$ *implies* $\Delta\alpha_1 < 0$.

This corollary, applied to tests with $n \geqq 1$, is a generalization to randomized stopping bounds and arbitrary distributions of a result of Weiss [10], who showed in the continuous case (and nondegenerate intervals carrying positive probability) that if the stopping bounds of a SPRT are separated in such a way that one of the error probabilities remains constant, then the other decreases strictly.

THEOREM 4. *Up to an equivalence, there is at most one test* $R(s, t)$ *with* $n \geqq 0$ *and given error point. Similarly for a test* $R(s, t)$ *with* $n \geqq 1$.

PROOF. Suppose $R(s, t)$ and $R(s', t')$ are not equivalent. If we can pass from one test to the other by changing the randomized stopping bounds in the same direction, then by the monotonicity of the α_i at least one of the α_i changes. If we pass from one to the other by changing the bounds in opposite directions, then by Corollary 1 at least one of the α_i changes. Therefore, the two tests cannot have the same error point.

We shall make use now of the result of Theorem 6 in the next section, which states that the four points $\alpha(u(-), v(+))$ are the extreme points of $A(u, v)$. This implies that if $\alpha^* \varepsilon A(u, v)$ for some u, v, then not only is there a SPRT with error point α^*, but there is even a test $R(s, t)$ with that property.

THEOREM 5. *If there is a* SPRT *with given error point* α^*, *its stopping bounds are unique up to an equivalence.*

PROOF. If α^* is in only one set $A(u, v)$, the stopping bounds u, v are unique. If $\alpha^* \varepsilon A(u, v)$ and $\varepsilon A(u', v')$, with $(u, v) \neq (u', v')$, then there are $R(s, t)$ and $R(s', t')$, both with error point α^*, where $s = (u, \lambda)$, etc. By Theorem 4, $R(s, t)$ and $R(s', t')$ are equivalent, so that the probability that Y_n is between

u and u' is 0 for each n and both i, and similarly for v and v'. Thus, the stopping bounds are determined up to an equivalence.

Considerations similar to those of the above proof lead to the following observation on the sets $A(u, v)$: if $A(u, v)$ and $A(u', v')$ are distinct, they can at most have boundary points in common. In fact, two such sets are either disjoint, or their intersection is one vertex, or one edge, of both.

4. The extreme points of $A(u, v)$. In the following theorem $A(u, v)$ is the set of error points of SPRT's with either $n \geqq 0$ or $n \geqq 1$, and stopping bounds u, v.

THEOREM 6. *$A(u, v)$ is convex and closed, and its extreme points are $\alpha(u, v)$, $\alpha(u-, v)$, $\alpha(u, v+)$, $\alpha(u-, v+)$ if $u < v$, and $\alpha(u-, u)$, $\alpha(u, u+)$, $\alpha(u-, u+)$ if $u = v$.*

REMARK. Some or all of the extreme points listed may coincide. For instance, in the case of a SPRT with $n \geqq 0$ and $u = 1 < v$, $T(u, v)$ and $T(u, v+)$ both coincide with the test that accepts H_1 without observation, so that $\alpha(u, v) = \alpha(u, v+) = (0, 1)$.

PROOF. We shall give the proof for SPRT's with $n \geqq 1$ and $0 < u < v < \infty$. If $P_i(Y_n = u) = 0$ and $P_i(Y_n = v) = 0$ for all n (this is true for $i = 1$ if and only if it is true for $i = 2$) then $A(u, v)$ is a single point and the theorem is trivially true. If exactly one of these equalities is violated (for both i), only two of the four points that are claimed to be extreme are distinct. It will follow from the remainder of the proof that $A(u, v)$ is the segment between these two points. Assume then that $P_i(Y_n = u) > 0$ for some n, and $P_i(Y_n = v) > 0$ for some n, then all four points $\alpha(u(-), v(+))$ are distinct. These points are the vertices of a quadrangle, which can easily be shown to be convex. Therefore, this quadrangle is the intersection of four half spaces. The theorem will be proved if we show that the error point of an arbitrary member of $C(u, v)$ lies in each of these half spaces. The proof being practically identical for each of the four half spaces, we shall restrict the proof to the half space determined by the vertices $\alpha(u, v)$ and $\alpha(u-, v)$, i.e. the set of error points α for which

$$m\Delta\alpha_1 + \Delta\alpha_2 \leqq 0, \tag{3}$$

where $\Delta\alpha = \alpha - \alpha'$, α' being any point on the segment between $\alpha(u, v)$ and $\alpha(u-, v)$, and $m \geqq v$ is the negative of the slope of this segment.

Let $T \varepsilon C(u, v)$, and let $T' \varepsilon C(u, v)$ stop sampling whenever $Y_n = v$, but have the same stopping rule as T at the lower bound u. Then

$$\alpha_1(u, v) \leqq \alpha_1(T') \leqq \alpha_1(u-, v), \qquad \alpha_2(u, v) \geqq \alpha_2(T') \geqq \alpha_2(u-, v). \tag{4}$$

We first show that $\alpha(T')$ is on the segment between $\alpha(u, v)$ and $\alpha(u-, v)$. The sample sequences that lead to different decisions under T' and under $T(u, v)$ are those sequences that reach u before stopping, and subsequently equal or exceed v before stopping at or below u. Let N be the random sample size when T is used. Define the following events for $n = 1, 2, \cdots : A_n = [N > n, Y_n = u]$,

$B_n = [u < Y_j < v$ for $n < j < N$, $Y_N \geqq v]$. Then

$$\alpha_1(T') - \alpha_1(u, v) = \sum P_1(A_n B_n) = \sum P_1(A_n) P_1(B_n \mid A_n) = \alpha_1(1, v/u) \sum P_1(A_n).$$

Similarly, $\alpha_2(u, v) - \alpha_2(T') = (1 - \alpha_2(1, v/u)) \sum P_2(A_n)$. Since

$$P_2(A_n)/P_1(A_n) = u$$

for all n, we get

$$\frac{\alpha_2(u,v) - \alpha_2(T')}{\alpha_1(T') - \alpha_1(u,v)} = u \, \frac{1 - \alpha_2(1, v/u)}{\alpha_1(1, v/u)} . \tag{5}$$

The right hand side of (5) is independent of T', that is, the result is valid for any test $T' \varepsilon C(u, v)$ that stops sampling whenever $Y_n = v$. Equating the left hand side of (5) to the expression obtained by replacing T' by $T(u-, v)$, and observing (4), proves the claim that $\alpha(T')$ is on the segment between $\alpha(u, v)$ and $\alpha(u-, v)$.

Putting $\alpha(T) = \alpha$, $\alpha(T') = \alpha'$, $\alpha - \alpha' = \Delta\alpha$, it is only left to be shown that $\Delta\alpha$ satisfies (3). Any sample sequence that accepts H_2 when T is used does so when T' is used. Denote by B the set of sample sequences that accept H_2 when T' is used, but accept H_1 when T is used. Every sequence in B terminates at a value $\leqq u$ when T is used, so that $P_2(B)/P_1(B) \leqq u$. Since $P_1(B) = \alpha_1' - \alpha_1$ and $P_2(B) = \alpha_2 - \alpha_2'$, we have $\Delta\alpha_1 \leqq 0$ and $u\Delta\alpha_1 + \Delta\alpha_2 \leqq 0$. Then (3) is satisfied since $m \geqq v > u$.

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