ESTIMATION OF ONE OR TWO PARAMETERS OF THE EXPONENTIAL DISTRIBUTION ON THE BASIS OF SUITABLY CHOSEN ORDER STATISTICS

By Gunnar Kulldorff

University of Lund

1. Introduction. Assume that n independent observations are made on a random variable with the exponential distribution function

$$F(x) = 1 - e^{-(x-\alpha)/\sigma} \text{ for } x \geqq \alpha, \tag{1.1}$$

where $\sigma > 0$. We are interested in the estimation of σ and/or α on the basis of k suitably chosen order statistics $x(n_1), x(n_2), \cdots, x(n_k)$, where $n_1, n_2, \cdots, n_k$ are the ranks satisfying the inequalities $1 \leqq n_1 < n_2 < \cdots < n_k \leqq n$.

The asymptotic situation $(n \to \infty)$ has been considered by Ogawa (1960) and Kulldorff (1962), while the small sample situation has been dealt with by Harter (1961) and Siddiqui (1963) for $k = 1$, and by Ukita (1955), Harter (1961), Sarhan, Greenberg and Ogawa (1963) and Siddiqui (1963) for $k = 2$. This paper deals with the small sample situation for any value of k. We shall derive the best linear unbiased estimates (BLUE's) of

(i) σ when α is known (Section 2),
(ii) α when σ is known (Section 3),
(iii) σ and α when both parameters are unknown (Section 4).

We shall also present some theoretical and numerical results concerning the optimum choice of the ranks.

In deriving the various BLUE's, we shall make use of the following results due to Gumbel (1937) and Sarhan (1954):

$$E[x(n_i)] = \alpha + \sigma \sum_{j=1}^{n_i} (n - j + 1)^{-1}, \tag{1.2}$$

$$V[x(n_i)] = \operatorname{Cov}[x(n_i), x(n_{i'})] = \sigma^2 \sum_{j=1}^{n_i} (n - j + 1)^{-2} \qquad (i < i'). \tag{1.3}$$

It will be convenient to employ the notation

$$\delta_{ri} = \sum_{j=n_{i-1}}^{n_i - 1} (n - j)^{-r} \quad (r = 1, 2; i = 1, 2, \cdots, k), \tag{1.4}$$

where $n_0 = 0$. Whenever the quantities δ_{10}/δ_{20} and $\delta_{1,k+1}/\delta_{2,k+1}$ appear in the sequel, it will be understood that they are equal to zero.

We shall also use the following well-known theorem, which is usually associated with the names of Gauss and Markoff.

Theorem 1. *Let X be a $k \times 1$ vector of observations, let θ be a $s \times 1$ vector of*

Received March 14, 1963.

unknown parameters ($s \leqq k$), *let* A *be a known* $k \times s$ *matrix of rank* s, *and let* W *be a known positive definite* $k \times k$ *matrix. Assume that* $E(X) = A\theta$ *and* $V(X) = \sigma^2 W$, *where* σ^2 *is a known or unknown real number. Then* $T = (A'W^{-1}A)^{-1} A'W^{-1}X$ *is the best linear unbiased estimate of* θ, *and* $V(T) = (A'W^{-1}A)^{-1}\sigma^2$. (The word "best" refers to the property that all the diagonal elements of $V(T)$ are the smallest possible.)

2. Estimation of σ when α is known.

THEOREM 2. *If* α *is known and the ranks* $n_1, n_2, \cdots, n_k$ *are fixed, then the BLUE of* σ *is*

$$\hat{\sigma} = b_0\alpha + \sum_{i=1}^{k} b_i x(n_i), \tag{2.1}$$

where

$$b_i = (\delta_{1i}/\delta_{2i} - \delta_{1,i+1}/\delta_{2,i+1})/K \quad \text{for} \quad i = 0, 1, \cdots, k \tag{2.2}$$

and

$$K = \sum_{i=1}^{k} \delta_{1i}^2/\delta_{2i}. \tag{2.3}$$

The variance of $\hat{\sigma}$ *is* $V(\hat{\sigma}) = \sigma^2/K$.

PROOF. We apply Theorem 1 for $X' = \{x(n_1) - \alpha, \cdots, x(n_k) - \alpha\}$ and $\theta = \sigma$. Because of relations (1.2) to (1.4) we have $A' = \{a_1, \cdots, a_k\}$ where $a_i = \sum_{h=1}^{i} \delta_{1h}$ ($i = 1, 2, \cdots, k$), and $W = \{w_{ij}\}$ where $w_{ij} = w_{ji} = \sum_{h=1}^{i} \delta_{2h}$ if $i \leqq j$ ($i, j = 1, 2, \cdots, k$). The elements w'_{ij} of W^{-1} are $w'_{ii} = \delta_{2i}^{-1} + \delta_{2,i+1}^{-1}$ for $i = 1, 2, \cdots, k-1$, $w'_{kk} = \delta_{2k}^{-1}$, $w'_{i-1,i} = w'_{i,i-1} = -\delta_{2i}^{-1}$ for $i = 2, 3, \cdots, k$, and $w'_{ij} = 0$ if $|i - j| \geqq 2$. Straightforward matrix multiplication then leads to the results in Theorem 2.

The following special cases of Theorem 2 have previously been considered:

(i) $k = n$.

(ii) $n_i - n_{i-1} = 1$ for $i = 2, 3, \cdots, k$ (Sarhan, 1955).

(iii) $n_i = i$ for $i = 1, 2, \cdots, m$ and $n_i = n - k + i$ for $i = m+1, m+2, \cdots, k$, where $1 \leqq m \leqq k - 2$ (Sarhan and Greenberg, 1962, Section 11B.5).

(iv) $k = 1$ (Harter, 1961).

(v) $k = 2$ (Harter, 1961; Sarhan, Greenberg and Ogawa, 1963).

Since $(1/n)\sum_{i=1}^{n} x(i) - \alpha$ is the BLUE of σ based on all the n observations and has the variance σ^2/n, the relative efficiency of $\hat{\sigma}$ is $\text{RE}(\hat{\sigma}) = K/n$.

The problem of finding those values of $n_1, n_2, \cdots, n_k$ which make $V(\hat{\sigma})$ as small as possible and hence make K as large as possible, has been solved numerically by Sarhan, Greenberg and Ogawa (1963) for $k = 2$ and $n = 2(1)20$, and by Harter (1961) for $k = 1, 2$ and $n = k(1)100$. They tabulated the values found together with the corresponding values of $b_1, b_2, V(\hat{\sigma})/\sigma^2$ and $\text{RE}(\hat{\sigma})$.

In order to facilitate the numerical problem of maximizing K for $k \geqq 2$, we shall now present a theorem which reduces the complex k-variate problem to a univariate problem repeated k times.

THEOREM 3.

(i) *Let* $K_0(n) = 0$ *for all* n. *For* $i = 1, 2, \cdots, k$ *let* $N_i(n)$ *be the integer value of* n_1 *which maximizes* $\delta_{11}^2/\delta_{21} + K_{i-1}(n - n_1)$ *for a given value of* n, *and let* $K_i(n)$ *be this maximum. The maximum value of* K *in* (2.3) *is then* $K_k(n)$ *and is obtained for* $n_1 = N_k(n)$ *and* $n_i = n_{i-1} + N_{k-i+1}(n - n_{i-1})$ $(i = 2, 3, \cdots, k)$.

(ii) *Let furthermore* $D_i(n)$ *be the value of* δ_{11}/δ_{21} *for* $n_1 = N_i(n)$ $(i = 1, 2, \cdots, k)$. *The coefficients* b_i *in* (2.2) *corresponding to the optimum ranks* n_i $(i = 1, 2, \cdots, k)$ *are then* $b_i = [D_{k-i+1}(n - n_{i-1}) - D_{k-i}(n - n_i)]/K_k(n)$ $(i = 0, 1, \cdots, k)$, *where* $n_0 = n_{-1} = 0$ *and* $D_0(n) = D_{k+1}(n) = 0$ *for all* n.

PROOF.

(i) The first part of the theorem is trivial for $k = 1$. Assume that it holds for $k = k_0$. In maximizing K with $k = k_0 + 1$, we first keep n_1 fixed and, noting that $\delta_{11}^2/\delta_{21}$ depends only on n_1 and n, maximize the function $\sum_{i=2}^{k_0+1} \delta_{1i}^2/\delta_{2i}$ with respect to $n_2, n_3, \cdots, n_{k_0+1}$. It is readily seen that this function has the same character as K for $k = k_0$, although the arguments are $n_i - n_1$ $(i = 2, 3, \cdots, k_0+1)$ and $n - n_1$ instead of n_i $(i = 1, 2, \cdots, k_0)$ and n. The maximum value is therefore $K_{k_0}(n - n_1)$, and is obtained for $n_i = n_{i-1} + N_{k_0-i+2}(n - n_{i-1})$ $(i = 2, 3, \cdots, k_0 + 1)$. It remains to maximize $\delta_{11}^2/\delta_{21} + K_{k_0}(n - n_1)$ with respect to n_1, which, according to the definition of $N_i(n)$ and $K_i(n)$, is accomplished for $n_1 = N_{k_0+1}(n)$ with the maximum value $K_{k_0+1}(n)$. This completes the verification that the first part of the theorem holds also for $k = k_0 + 1$, and hence for all values of k.

(ii) Substitution of $n_{i-1} + N_{k-i+1}(n - n_{i-1})$ for n_i in (1.4) and adjustment of the summation index makes

$$\delta_{ri} = \sum_{j=0}^{N_{k-i+1}(n-n_{i-1})-1} (n - n_{i-1} - j)^{-r} \qquad (i = 1, 2, \cdots, k).$$

Hence $\delta_{1i}/\delta_{2i} = D_{k-i+1}(n - n_{i-1})$ for $i = 1, 2, \cdots, k$ according to the definition of $D_i(n)$. For $i = 0$ and $i = k + 1$, the relation $\delta_{1i}/\delta_{2i} = D_{k-i+1}(n - n_{i-1})$ is trivial because both sides vanish. This completes the verification of the second part of Theorem 3.

The optimum ranks n_i $(i = 1, 2, \cdots, k)$ and the corresponding values of b_i $(i = 0, 1, \cdots, k)$, $V(\hat{\sigma})/\sigma^2 = 1/K$ and $\mathrm{RE}(\hat{\sigma}) = K/n$ have been computed for $k = 1(1)5$ and $n = k(1)100$. The results for $k = 1$ and 2 confirm Harter's (1961) results, the discrepancy being greater than one unit in the last decimal place only once, for $k = 2$ and $n = 57$, where $V(\hat{\sigma})/\sigma^2 = 0.0211897$. The results for $k = 3, 4, 5$ are given in Table 1.

3. Estimation of α when σ is known.

THEOREM 4. *If* σ *is known and the ranks* $n_1, n_2, \cdots, n_k$ *are fixed, then the BLUE of* α *is* $\hat{\alpha} = x(n_1) - \sigma\delta_{11}$ *with* $V(\hat{\alpha}) = \sigma^2\delta_{21}$.

PROOF. The proof of this theorem is a straightforward application of Theorem 1, setting $X' = \{x_1, \cdots, x_k\}$ where $x_i = x(n_i) - \sigma \sum_{h=1}^{i} \delta_{1h}$ $(i = 1, 2, \cdots, k)$ and $\theta = \alpha$. We then have $A' = \{1, 1, \cdots, 1\}$, while W and W^{-1} are the matrices described in the proof of Theorem 2.

Since δ_{21} increases with n_1, $V(\hat{\alpha})$ attains its minimum for $n_1 = 1$. Then $\hat{\alpha} = x(1) - \sigma/n$ with $V(\hat{\alpha}) = \sigma^2/n^2$. This estimate is also the BLUE of α based on all the n observations. Hence $\text{RE}(\hat{\alpha}) = 1$.

4. Estimation of σ and/or α when both parameters are unknown. In this section we shall assume that $k \geqq 2$.

THEOREM 5. *If σ and α are both unknown and the ranks $n_1, n_2, \cdots, n_k$ are fixed, then the BLUE's of σ and α are*

$$\hat{\sigma} = \sum_{i=1}^{k} b_i x(n_i) \tag{4.1}$$

and

$$\hat{\alpha} = x(n_1) - \hat{\sigma}\delta_{11}, \tag{4.2}$$

where

$$\begin{aligned} b_i &= -\delta_{12}/L\delta_{22} && \textit{for} \quad i = 1 \\ &= (\delta_{1i}/\delta_{2i} - \delta_{1,i+1}/\delta_{2,i+1})/L && \textit{for} \quad i = 2, 3, \cdots, k \end{aligned} \tag{4.3}$$

and

$$L = \sum_{i=2}^{k} \delta_{1i}^2/\delta_{2i}. \tag{4.4}$$

The variances and covariance of $\hat{\sigma}$ and $\hat{\alpha}$ are

$$V(\hat{\sigma}) = \sigma^2/L, \tag{4.5}$$

$$V(\hat{\alpha}) = \sigma^2(\delta_{21} + \delta_{11}^2/L), \tag{4.6}$$

$$\text{Cov}(\hat{\sigma}, \hat{\alpha}) = -\sigma^2\delta_{11}/L. \tag{4.7}$$

PROOF. Set $X' = \{x(n_1), \cdots, x(n_k)\}$ and $\theta' = \{\sigma, \alpha\}$ in Theorem 1. We then have $A = \{a_{ij}\}$ where $a_{i1} = \sum_{h=1}^{i}\delta_{1h}$ and $a_{i2} = 1$ $(i = 1, 2, \cdots, k)$, while W and W^{-1} are the matrices described in the proof of Theorem 2. The results in Theorem 5 are then obtained by straightforward matrix multiplication.

The BLUE of a linear function of σ and α is the same linear function of $\hat{\sigma}$ and $\hat{\alpha}$. For estimation of $\mu = \sigma + \alpha$, which is the mean of the exponential distribution, we have the following theorem, which is an immediate consequence of Theorem 5.

THEOREM 6. *If σ and α are both unknown and the ranks $n_1, n_2, \cdots, n_k$ are fixed, then the BLUE of $\mu = \sigma + \alpha$ is $\hat{\mu} = x(n_1) + \hat{\sigma}(1 - \delta_{11})$, where $\hat{\sigma}$ is given by* (4.1). *The variance of $\hat{\mu}$ is $V(\hat{\mu}) = \sigma^2[\delta_{21} + (1 - \delta_{11})^2/L]$.*

The following special cases of Theorems 5 and 6 have previously been considered:

(i) $k = n$ (Sarhan, 1954).

(ii) $n_i - n_{i-1} = 1$ for $i = 2, 3, \cdots, k$ (Sarhan, 1955).

(iii) $n_i = i$ for $i = 1, 2, \cdots, m$ and $n_i = n - k + i$ for $i = m + 1, m + 2, \cdots, k$, where $1 \leqq m \leqq k - 2$ (Sarhan and Greenberg, 1962, Section 11B.5).

(iv) $k = 2$ (Ukita, 1955; Harter, 1961; Sarhan, Greenberg and Ogawa, 1963).

If we compare the estimates $\hat{\sigma}$, $\hat{\alpha}$ and $\hat{\mu}$ with the BLUE's of σ, α and μ based on all the n observations, we know (Sarhan, 1954) that the latter estimates have variances $\sigma^2/(n - 1)$, $\sigma^2/n(n - 1)$ and σ^2/n, respectively. The relative efficiencies of $\hat{\sigma}$, $\hat{\alpha}$ and $\hat{\mu}$ are therefore $\mathrm{RE}(\hat{\sigma}) = L/(n - 1)$,

$$\text{(4.8)} \qquad \mathrm{RE}(\hat{\alpha}) = 1/n(n - 1)(\delta_{21} + \delta_{11}^2/L)$$

and

$$\text{(4.9)} \qquad \mathrm{RE}(\hat{\mu}) = 1/n[\delta_{21} + (1 - \delta_{11})^2/L].$$

The two problems of finding those values of $n_1, n_2, \cdots, n_k$ which make $V(\hat{\sigma})$ and $V(\hat{\alpha})$, respectively, as small as possible, fortunately turn out to have the same solution. This solution is very closely related to that of the problem of maximizing K in (2.3), which was considered in Section 2. The form of the relationship is described in the following theorem.

THEOREM 7. *Let $n_1', n_2', \cdots, n_{k-1}'$ be the optimum ranks when selecting $k - 1$ order statistics from a sample of size $n - 1$ for the estimation of σ when α is known, let $b_0', b_1', \cdots, b_{k-1}'$ be the coefficients of the corresponding BLUE $b_0'\alpha + \sum_{i=1}^{k-1} b_i'x(n_i')$, and let σ^2/K' be the variance of this BLUE. Then the variances $V(\hat{\sigma})$ and $V(\hat{\alpha})$ as given by (4.5) and (4.6) both attain their minima for $n_1 = 1$ and $n_i = n_{i-1}' + 1$ $(i = 2, 3, \cdots, k)$. When these values of $n_1, n_2, \cdots, n_k$ are substituted in (4.3) to (4.6) we have $b_i = b_{i-1}'$ $(i = 1, 2, \cdots, k)$, $L = K'$, $V(\hat{\sigma}) = \sigma^2/K'$ and $V(\hat{\alpha}) = \sigma^2(1 + 1/K')/n^2$.*

PROOF. We shall first keep $n_2, n_3, \cdots, n_k$ fixed in $V(\hat{\sigma})$ and $V(\hat{\alpha})$, and show that both variances increase with n_1 and attain their minima for $n_1 = 1$. (For $k = 2$, this fact has been shown by Ukita (1955). Harter (1961, p. 1083) has also claimed that, for $k = 2$ and a fixed value of n_2, the variance of $\hat{\mu}$ is smallest when $n_1 = 1$. This claim is not correct, however, as is easily seen from counter-examples. When $n = 4$ and $n_2 = 3$, for example, the variance of $\hat{\mu}$ is $0.355\ \sigma^2$ for $n_1 = 1$ but only $0.347222\ \sigma^2$ for $n_1 = 2$.) Since the quantities δ_{11} and δ_{21} appearing in (4.6) obviously increase with n_1, our task is to prove that the function L in (4.4) decreases with n_1. Only the first term in L, $\delta_{12}^2/\delta_{22}$, depends on n_1. L decreases with n_1 since, for $n_1 < n_2 - 1$, the difference

$$\frac{[\delta_{12} - (n - n_1)^{-1}]^2}{\delta_{22} - (n - n_1)^{-2}} - \frac{\delta_{12}^2}{\delta_{22}} = \left[1 + \frac{\delta_{12}^2}{\delta_{22}} - 2\delta_{12}(n - n_1)\right] \bigg/ [\delta_{22}(n - n_1)^2 - 1]$$

$$\leqq [1 - \delta_{12}(n - n_1)]/[\delta_{22}(n - n_1)^2 - 1] < 0,$$

where the inequalities follow from the fact that $\delta_{22}(n - n_1) \geqq \delta_{12} \geqq (n_2 - n_1)/(n - n_1)$. For $n_1 = 1$, we have $V(\hat{\alpha}) = \sigma^2(1 + 1/L)/n^2$. Let us now turn to the second step in minimizing $V(\hat{\sigma})$ and $V(\hat{\alpha})$, i.e., let us maximize L with respect to $n_2, n_3, \cdots, n_k$, while keeping $n_1 = 1$. If we substitute $n - 1$ for n, $k - 1$ for k and $n_{i+1} - 1$ for n_i $(i = 1, 2, \cdots, k - 1)$ in (2.3) we obtain

$$K = \sum_{i=1}^{k-1} \delta_{1i}'^2/\delta_{2i}' ,$$

where

$$\delta_{ri}' = \sum_{j=n_i-1}^{n_{i+1}-2} (n - 1 - j)^{-r} = \delta_{r,i+1} \quad (r = 1, 2; i = 1, 2, \cdots, k - 1).$$

Hence, $K = L$ (for $k = 2$, this fact was noted by Harter (1961)). Our problem of maximizing L with respect to $n_2, n_3, \cdots, n_k$ while $n_1 = 1$ is therefore equivalent to that of maximizing K (for $k - 1$ order statistics from a sample of size $n - 1$) with respect to $n_1, n_2, \cdots, n_{k-1}$. It follows that the optimum values of n_i $(i = 2, 3, \cdots, k)$ are $n_i = n_{i-1}' + 1$, and that the maximum value of L is K'. The fact that $\delta_{ri}' = \delta_{r,i+1}$ $(r = 1, 2; i = 1, 2, \cdots, k - 1)$ for the optimum ranks then leads to the relation $b_i = b_{i-1}'$ $(i = 1, 2, \cdots, k)$. This completes the proof of Theorem 7.

Because of Theorem 7, the numerical results presented in Harter's (1961) Table 1 and in our Table 1 are most useful also in the present context, and there is no need for any further tabulations. For $k = 2(1)6$ and $n = k(1)101$, those tables provide us with the optimum ranks, the coefficients b_i $(i = 1, 2, \cdots, k)$ of the corresponding BLUE of σ, and the values of $V(\hat{\sigma})/\sigma^2 = 1/L$ and $\text{RE}(\hat{\sigma})$. The BLUE of α is then easily computed from the relation $\hat{\alpha} = x(1) - \hat{\sigma}/n$, while

$$(4.10) \quad V(\hat{\alpha}) = \sigma^2(1 + 1/L)/n^2 \quad \text{and} \quad \text{RE}(\hat{\alpha}) = n/(n - 1)(1 + 1/L).$$

Although the ranks which are optimum for the estimation of σ and α are usually not optimum for the estimation of μ, the BLUE of μ corresponding to those ranks is $\hat{\mu} = x(1) + \hat{\sigma}(1 - 1/n)$ with

$$(4.11) \quad V(\hat{\mu}) = \sigma^2[1 + (n - 1)^2/L]/n^2 \quad \text{and} \quad \text{RE}(\hat{\mu}) = n/[1 + (n - 1)^2/L].$$

Example. Assume that $k = 4$ and $n = 20$. Using Table 1 for $k = 3$ and $n = 19$, we find that the optimum ranks for estimation of σ and α are 1, 11, 17 and 20, and that the BLUE of σ is

$$\hat{\sigma} = -0.757587\, x(1) + 0.441013\, x(11) + 0.240060\, x(17) + 0.076514\, x(20)$$

with $V(\hat{\sigma}) = 0.056806\, \sigma^2$ and $\text{RE}(\hat{\sigma}) = 0.9265$. The BLUE's of α and μ are then $\hat{\alpha} = x(1) - \hat{\sigma}/20$ and $\hat{\mu} = x(1) + 0.95\, \hat{\sigma}$ with the variances

$$V(\hat{\alpha}) = \frac{1 + 0.056806}{20^2}\sigma^2 = 0.002642\, \sigma^2 \quad \text{and}$$

$$V(\hat{\mu}) = \frac{1 + 19^2 \cdot 0.056806}{20^2}\sigma^2 = 0.053767\, \sigma^2$$

and the relative efficiencies $\text{RE}(\hat{\alpha}) = 0.9960$ and $\text{RE}(\hat{\mu}) = 0.9299$.

TABLE 1

The BLUE $\hat{\sigma} = b_0\alpha + \sum_{i=1}^{k} b_i x(n_i)$ *based on k order statistics with optimum ranks from a sample of size n and on a known value of* α. *When* α *is unknown the table applies to the BLUE* $\hat{\sigma} = b_0 x(1) + \sum_{i=1}^{k} b_i x(n_i + 1)$ *based on* $k + 1$ *order statistics with optimum ranks from a sample of size* $n + 1$

$k = 3$

n	n_1	n_2	n_3	b_0	b_1	b_2	b_3	$V(\hat{\sigma})/\sigma^2$	RE($\hat{\sigma}$)
3	1	2	3	−1.000000	.333333	.333333	.333333	.333333	1.0000
4	2	3	4	−.848485	.343434	.252525	.252525	.252525	.9900
5	2	4	5	−.896208	.425124	.266948	.204136	.204136	.9797
6	3	5	6	−.810242	.415228	.223841	.171173	.171173	.9737
7	4	6	7	−.746576	.404767	.193692	.148117	.148117	.9645
8	4	7	8	−.799109	.464763	.203608	.130738	.130738	.9561
9	4	7	9	−.834861	.407551	.287479	.139830	.116525	.9535
10	5	8	10	−.787920	.402347	.259400	.126173	.105144	.9511
11	6	9	11	−.748760	.396766	.236810	.115184	.095987	.9471
12	6	10	12	−.782972	.435832	.241293	.105847	.088206	.9448
13	7	11	13	−.750043	.428615	.223421	.098007	.081673	.9418
14	8	12	14	−.721399	.421724	.208301	.091375	.076146	.9381
15	8	13	15	−.753338	.455743	.212047	.085548	.071290	.9352
16	9	14	16	−.727956	.448134	.199384	.080439	.067032	.9324
17	9	14	17	−.756473	.419841	.251378	.085254	.063295	.9294
18	9	15	18	−.780330	.446573	.253090	.080667	.059889	.9276
19	10	16	19	−.757587	.441013	.240060	.076514	.056806	.9265
20	11	17	20	−.736945	.435670	.228459	.072817	.054061	.9249
21	12	18	21	−.718115	.430546	.218065	.069504	.051601	.9228
22	12	19	22	−.740758	.454300	.220021	.066438	.049325	.9215
23	13	20	23	−.723366	.448874	.210829	.063662	.047264	.9199
24	14	21	24	−.707310	.443695	.202475	.061140	.045392	.9179
25	14	22	25	−.728691	.465654	.204237	.058799	.043654	.9163
26	15	23	26	−.713634	.460283	.196717	.056634	.042047	.9147
27	15	23	27	−.733225	.440645	.233232	.059348	.040555	.9133
28	15	24	28	−.750777	.459222	.234261	.057294	.039151	.9122
29	16	25	29	−.736543	.454791	.226385	.055368	.037835	.9114
30	17	26	30	−.723196	.450522	.219090	.053584	.036616	.9104
31	18	27	31	−.710650	.446409	.212314	.051927	.035483	.9091
32	18	28	32	−.727372	.463591	.213429	.050353	.034408	.9082
33	19	29	33	−.715447	.459365	.207199	.048883	.033404	.9072
34	18	28	33	−.757467	.432459	.229223	.095785	.032461	.9061
35	18	29	34	−.770232	.446531	.230632	.093069	.031540	.9059
36	19	30	35	−.758206	.443433	.224271	.090502	.030670	.9057
37	20	31	36	−.746794	.440407	.218296	.088091	.029853	.9053
38	20	32	37	−.759225	.453786	.219627	.085812	.029081	.9049
39	20	32	38	−.770596	.440058	.242341	.088197	.028338	.9048
40	21	33	39	−.759755	.437437	.236314	.086004	.027634	.9047
41	22	34	40	−.749414	.434865	.230618	.083931	.026968	.9044
42	22	35	41	−.760397	.446787	.231665	.081945	.026330	.9043
43	23	36	42	−.750521	.444126	.226335	.080060	.025724	.9041
44	24	37	43	−.741068	.441522	.221276	.078270	.025149	.9037
45	24	38	44	−.751786	.452948	.222282	.076556	.024598	.9034
46	25	39	45	−.742714	.450275	.217522	.074917	.024071	.9031
47	26	40	46	−.734005	.447661	.212989	.073356	.023570	.9027
48	25	40	47	−.762781	.449745	.237839	.075198	.023087	.9024
49	26	41	48	−.754054	.447360	.233020	.073674	.022620	.9022
50	27	42	49	−.745656	.445021	.228417	.072219	.022173	.9020

TABLE 1. $k = 3$ (*Continued*)

n	n_1	n_2	n_3	b_0	b_1	b_2	b_3	$V(\hat{\sigma})/\sigma^2$	RE($\hat{\sigma}$)
51	28	43	50	−.737570	.442726	.224016	.070827	.021746	.9017
52	28	44	51	−.746981	.452701	.224799	.069481	.021332	.9015
53	29	45	52	−.739173	.450356	.220626	.068191	.020936	.9012
54	30	46	53	−.731638	.448058	.216626	.066955	.020557	.9009
55	30	47	54	−.740837	.457695	.217382	.065760	.020190	.9005
56	31	48	55	−.733540	.455356	.213575	.064608	.019836	.9002
57	32	49	56	−.726483	.453064	.209918	.063502	.019496	.8998
58	31	49	57	−.750888	.454930	.231014	.064944	.019166	.8996
59	32	50	58	−.743799	.452798	.227145	.063856	.018845	.8994
60	33	51	59	−.736934	.450705	.223421	.062809	.018536	.8991
61	32	50	59	−.759689	.435665	.234038	.089986	.018237	.8989
62	32	51	60	−.766920	.443624	.234764	.088532	.017943	.8989
63	33	52	61	−.760047	.441878	.231041	.087128	.017658	.8989
64	34	53	62	−.753379	.440156	.227449	.085774	.017384	.8988
65	34	54	63	−.760503	.447887	.228157	.084460	.017117	.8988
66	35	55	64	−.754030	.446129	.224716	.083186	.016859	.8987
67	36	56	65	−.747741	.444395	.221391	.081955	.016610	.8986
68	36	57	66	−.754755	.451915	.222079	.080761	.016368	.8985
69	36	57	67	−.761426	.444141	.235178	.082107	.016131	.8984
70	37	58	68	−.755308	.442541	.231830	.080938	.015902	.8984
71	38	59	69	−.749355	.440961	.228588	.079807	.015679	.8983
72	38	60	70	−.755883	.447996	.229185	.078702	.015462	.8982
73	39	61	71	−.750085	.446387	.226066	.077631	.015252	.8982
74	40	62	72	−.744436	.444800	.223043	.076593	.015048	.8980
75	40	63	73	−.750868	.451661	.223626	.075581	.014849	.8979
76	41	64	74	−.745356	.450049	.220712	.074595	.014655	.8978
77	42	65	75	−.739981	.448459	.217882	.073639	.014468	.8977
78	41	65	76	−.757720	.449737	.233193	.074790	.014284	.8975
79	42	66	77	−.752337	.448234	.230255	.073847	.014104	.8975
80	43	67	78	−.747082	.446750	.227400	.072932	.013929	.8974
81	43	68	79	−.753023	.453098	.227883	.072042	.013759	.8973
82	44	69	80	−.747887	.451594	.225125	.071169	.013593	.8972
83	45	70	81	−.742870	.450108	.222441	.070321	.013431	.8971
84	46	71	82	−.737965	.448640	.219829	.069495	.013273	.8969
85	46	72	83	−.743819	.454812	.220319	.068688	.013119	.8968
86	47	73	84	−.739017	.453327	.217790	.067899	.012968	.8966
87	48	74	85	−.734320	.451861	.215327	.067131	.012822	.8965
88	47	74	86	−.750289	.453055	.229118	.068116	.012678	.8964
89	47	74	86	−.755621	.446997	.223751	.084872	.012536	.8963
90	47	74	87	−.760726	.441159	.233637	.085930	.012397	.8963
91	48	75	88	−.756017	.439957	.231073	.084987	.012261	.8963
92	48	76	89	−.761008	.445392	.231554	.084062	.012127	.8963
93	49	77	90	−.756398	.444172	.229067	.083159	.011997	.8963
94	50	78	91	−.751881	.442963	.226640	.082278	.011870	.8962
95	50	79	92	−.756819	.448292	.227112	.081415	.011745	.8962
96	51	80	93	−.752392	.447067	.224755	.080570	.011623	.8962
97	52	81	94	−.748052	.445854	.222453	.079745	.011504	.8961
98	52	82	95	−.752936	.451083	.222917	.078937	.011388	.8961
99	52	82	96	−.757647	.445660	.232114	.079873	.011273	.8960
100	53	83	97	−.753389	.444513	.229799	.079077	.011161	.8960

TABLE 1—*Continued*
$k = 4$

n	n_1	n_2	n_3	n_4	b_0	b_1	b_2	b_3	b_4	$V(\hat{\sigma})/\sigma^2$	RE($\hat{\sigma}$)
4	1	2	3	4	−1.000000	.250000	.250000	.250000	.250000	.250000	1.0000
5	2	3	4	5	−.882353	.279412	.200980	.200980	.200980	.200980	.9951
6	2	4	5	6	−.910194	.344881	.228817	.168248	.168248	.168248	.9906
7	3	5	6	7	−.836889	.350211	.196989	.144845	.144845	.144845	.9863
8	3	5	7	8	−.866620	.307894	.265037	.166424	.127265	.127265	.9822
9	3	6	8	9	−.889078	.351833	.275324	.148422	.113499	.113499	.9790
10	4	7	9	10	−.839613	.355209	.248245	.133824	.102336	.102336	.9772
11	5	8	10	11	−.798043	.356288	.226388	.122041	.093326	.093326	.9741
12	5	9	11	12	−.823818	.391131	.234588	.112256	.085843	.085843	.9708
13	6	10	12	13	−.789477	.389045	.217100	.103888	.079444	.079444	.9683
14	6	10	13	14	−.813108	.360780	.263075	.115250	.074003	.074003	.9652
15	6	10	13	15	−.831480	.336250	.241755	.170530	.082946	.069121	.9645
16	6	11	14	16	−.847646	.361447	.248275	.160067	.077857	.064881	.9633
17	7	12	15	17	−.820246	.362311	.233842	.150762	.073331	.061109	.9626
18	8	13	16	18	−.795532	.362477	.221137	.142571	.069347	.057789	.9614
19	8	14	17	19	−.812289	.384580	.226642	.135271	.065796	.054830	.9599
20	8	14	18	20	−.826800	.363967	.257631	.142634	.062569	.052141	.9589
21	9	15	19	21	−.805333	.364214	.245544	.135942	.059633	.049694	.9582
22	10	16	20	22	−.785593	.364053	.234645	.129908	.056986	.047489	.9572
23	10	17	21	23	−.800001	.382472	.238598	.124372	.054558	.045465	.9563
24	11	18	22	24	−.782056	.381509	.228894	.119314	.052339	.043616	.9553
25	12	19	23	25	−.765385	.380350	.220029	.114693	.050312	.041927	.9540
26	12	20	24	26	−.779622	.397281	.223513	.110399	.048428	.040357	.9530
27	12	20	25	27	−.792588	.381517	.248684	.115707	.046680	.038900	.9521
28	13	21	26	28	−.777358	.380625	.240010	.111671	.045052	.037544	.9513
29	13	22	27	29	−.789647	.395622	.242563	.107921	.043539	.036283	.9504
30	14	23	28	30	−.775488	.394294	.234665	.104407	.042122	.035101	.9496
31	15	24	29	31	−.762144	.392880	.227321	.101140	.040803	.034003	.9487
32	14	23	29	32	−.797973	.368454	.245808	.139309	.044402	.032965	.9480
33	14	24	30	33	−.807722	.381219	.248279	.135148	.043076	.031980	.9476
34	15	25	31	34	−.794773	.380672	.241060	.131218	.041823	.031050	.9472
35	16	26	32	35	−.782485	.380011	.234292	.127534	.040649	.030179	.9467
36	16	27	33	36	−.792211	.392022	.236585	.124062	.039542	.029357	.9462
37	17	28	34	37	−.780632	.391069	.230303	.120767	.038492	.028578	.9457
38	18	29	35	38	−.769599	.390051	.224382	.117663	.037503	.027843	.9452
39	18	30	36	39	−.779224	.401432	.226505	.114722	.036565	.027147	.9445
40	18	30	37	40	−.788084	.390430	.243877	.118111	.035665	.026479	.9442
41	19	31	38	41	−.777694	.389568	.238034	.115282	.034811	.025844	.9437
42	20	32	39	42	−.767749	.388654	.232495	.112599	.034001	.025243	.9432
43	20	33	40	43	−.776505	.398980	.234268	.110032	.033225	.024667	.9428
44	21	34	41	44	−.767023	.397886	.229063	.107587	.032487	.024119	.9423
45	22	35	42	45	−.757921	.396762	.224112	.105262	.031785	.023598	.9417
46	22	36	43	46	−.766554	.406642	.225773	.103029	.031111	.023097	.9412
47	22	36	44	47	−.774736	.397194	.241260	.105817	.030465	.022618	.9407
48	23	37	45	48	−.766055	.396208	.236343	.103661	.029844	.022157	.9403
49	23	38	46	49	−.773943	.405386	.237714	.101595	.029249	.021715	.9398
50	24	39	47	50	−.765615	.404271	.233061	.099606	.028676	.021290	.9394

TABLE 1. $k = 4$ (*Continued*)

n	n_1	n_2	n_3	n_4	b_0	b_1	b_2	b_3	b_4	$V(\hat{\sigma})/\sigma^2$	RE($\hat{\sigma}$)
51	24	39	47	51	−.773223	.395675	.226894	.120094	.030559	.020882	.9390
52	24	39	48	52	−.780384	.387514	.240304	.122585	.029981	.020487	.9387
53	24	40	49	53	−.787186	.395772	.241685	.120305	.029424	.020106	.9384
54	25	41	50	54	−.779262	.395003	.237268	.118106	.028886	.019739	.9382
55	26	42	51	55	−.771598	.394207	.233027	.115995	.028370	.019386	.9379
56	26	43	52	56	−.778347	.402167	.234345	.113962	.027872	.019046	.9376
57	27	44	53	57	−.770960	.401269	.230303	.111997	.027392	.018718	.9373
58	28	45	54	58	−.763803	.400354	.226414	.110106	.026929	.018402	.9370
59	28	46	55	59	−.770482	.408047	.227671	.108281	.026483	.018097	.9366
60	28	46	56	60	−.776801	.400521	.239823	.110410	.026048	.017800	.9363
61	29	47	57	61	−.769910	.399696	.235956	.108630	.025628	.017513	.9361
62	30	48	58	62	−.763220	.398856	.232228	.106913	.025223	.017236	.9358
63	30	49	59	63	−.769465	.406052	.233336	.105247	.024830	.016967	.9355
64	28	47	58	63	−.799763	.386791	.241524	.122154	.049294	.016705	.9353
65	29	48	59	64	−.793000	.386359	.237822	.120281	.048538	.016449	.9353
66	30	49	60	65	−.786422	.385900	.234243	.118471	.047808	.016202	.9352
67	30	50	61	66	−.791677	.392382	.235477	.116718	.047100	.015962	.9351
68	31	51	62	67	−.785308	.391841	.232039	.115014	.046413	.015729	.9350
69	30	50	62	68	−.801636	.380067	.240742	.132577	.048250	.015503	.9348
70	31	51	63	69	−.795332	.379773	.237310	.130687	.047562	.015282	.9348
71	31	52	64	70	−.800062	.385796	.238518	.128853	.046895	.015068	.9348
72	32	53	65	71	−.793951	.385424	.235214	.127068	.046245	.014859	.9347
73	33	54	66	72	−.787991	.385029	.232010	.125337	.045615	.014657	.9346
74	33	55	67	73	−.792712	.390881	.233172	.123656	.045003	.014460	.9346
75	33	55	68	74	−.797245	.385191	.242111	.125538	.044405	.014268	.9345
76	34	56	69	75	−.791484	.384833	.238937	.123892	.043823	.014081	.9345
77	35	57	70	76	−.785859	.384454	.235854	.122294	.043258	.013899	.9344
78	35	58	71	77	−.790365	.390014	.236910	.120734	.042706	.013722	.9343
79	36	59	72	78	−.784893	.389575	.233932	.119216	.042169	.013549	.9342
80	37	60	73	79	−.779546	.389121	.231037	.117741	.041647	.013382	.9341
81	37	61	74	80	−.784033	.394541	.232056	.116299	.041137	.013218	.9340
82	37	61	75	81	−.788387	.389299	.240449	.117999	.040640	.013058	.9339
83	38	62	76	82	−.783196	.388880	.237574	.116588	.040154	.012902	.9338
84	38	63	77	83	−.787468	.394085	.238491	.115212	.039680	.012750	.9337
85	39	64	78	84	−.782408	.393616	.235707	.113867	.039217	.012601	.9337
86	40	65	79	85	−.777454	.393136	.232995	.112557	.038766	.012456	.9335
87	40	66	80	86	−.781707	.398220	.233884	.111278	.038325	.012314	.9334
88	39	65	80	87	−.794679	.388790	.240804	.125429	.039657	.012176	.9333
89	40	66	81	88	−.789778	.388433	.238108	.124025	.039213	.012039	.9333
90	41	67	82	89	−.784976	.388062	.235478	.122655	.038780	.011906	.9332
91	41	68	83	90	−.788887	.392853	.236360	.121317	.038357	.011776	.9331
92	42	69	84	91	−.784196	.392439	.233808	.120007	.037943	.011649	.9331
93	43	70	85	92	−.779597	.392015	.231316	.118728	.037538	.011525	.9330
94	43	71	86	93	−.783494	.396702	.232172	.117477	.037143	.011404	.9329
95	43	71	87	94	−.787258	.392118	.239470	.118915	.036754	.011284	.9328
96	44	72	88	95	−.782775	.391723	.236993	.117685	.036374	.011168	.9328
97	45	73	89	96	−.778376	.391318	.234573	.116483	.036002	.011054	.9327
98	45	74	90	97	−.782122	.395816	.235365	.115304	.035638	.010942	.9326
99	46	75	91	98	−.777817	.395375	.233010	.114150	.035281	.010832	.9325
100	47	76	92	99	−.773590	.394928	.230708	.113022	.034933	.010725	.9324

TABLE 1—*Continued*

$k = 5$

n	n_1	n_2	n_3	n_4	n_5	b_0	b_1	b_2	b_3	b_4	b_5	$V(\hat{\sigma})/\sigma^2$	RE($\hat{\sigma}$)
5	1	2	3	4	5	−1.000000	.200000	.200000	.200000	.200000	.200000	.200000	1.0000
6	2	3	4	5	6	−.904110	.235616	.167123	.167123	.167123	.167123	.167123	.9973
7	2	4	5	6	7	−.922411	.291977	.199637	.143599	.143599	.143599	.143599	.9948
8	2	4	6	7	8	−.936844	.255054	.258336	.171398	.126028	.126028	.126028	.9918
9	3	5	7	8	9	−.878947	.271932	.230003	.152600	.112206	.112206	.112206	.9902
10	3	6	8	9	10	−.896014	.311163	.244741	.137664	.101223	.101223	.101223	.9879
11	4	7	9	10	11	−.851399	.318773	.222886	.125371	.092184	.092184	.092184	.9862
12	4	7	9	11	12	−.868723	.292218	.204822	.176312	.110711	.084661	.084661	.9843
13	5	8	10	12	13	−.832506	.299403	.189402	.163038	.102376	.078288	.078288	.9826
14	5	8	11	13	14	−.848960	.278852	.225607	.176547	.095173	.072780	.072780	.9814
15	5	9	12	14	15	−.862631	.304950	.235934	.164887	.088887	.067973	.067973	.9808
16	6	10	13	15	16	−.833533	.310083	.221452	.154766	.083431	.063801	.063801	.9796
17	6	11	14	16	17	−.847544	.333359	.229559	.145863	.078632	.060130	.060130	.9783
18	7	12	15	17	18	−.822263	.336067	.217063	.137924	.074352	.056857	.056857	.9771
19	7	12	16	18	19	−.835925	.318232	.245790	.147417	.070543	.053944	.053944	.9757
20	8	13	17	19	20	−.813561	.321190	.233767	.140206	.067092	.051306	.051306	.9745
21	8	14	18	20	21	−.826249	.340240	.239499	.133649	.063954	.048906	.048906	.9737
22	9	15	19	21	22	−.806222	.341789	.228867	.127716	.061115	.046735	.046735	.9726
23	8	14	18	21	23	−.848031	.309926	.217609	.156456	.110361	.053680	.044733	.9719
24	9	15	19	22	24	−.829000	.313031	.208657	.150019	.105821	.051471	.042893	.9714
25	9	15	20	23	25	−.838748	.300532	.229502	.157643	.101635	.049435	.041196	.9710
26	9	16	21	24	26	−.847509	.315665	.234921	.151622	.097754	.047548	.039623	.9707
27	10	17	22	25	27	−.830609	.318247	.226315	.146068	.094173	.045806	.038171	.9703
28	10	18	23	26	28	−.839424	.332419	.231013	.140934	.090863	.044196	.036830	.9697
29	11	19	24	27	29	−.823884	.334138	.223148	.136136	.087770	.042691	.035576	.9693
30	12	20	25	28	30	−.809236	.335513	.215848	.131683	.084898	.041295	.034412	.9687
31	11	19	25	29	31	−.840450	.312106	.232583	.164632	.091146	.039983	.033319	.9682
32	12	20	26	30	32	−.826330	.314323	.225391	.159541	.088328	.038747	.032289	.9678
33	12	21	27	31	33	−.833876	.326334	.229537	.154748	.085674	.037583	.031319	.9676
34	13	22	28	32	34	−.820726	.327927	.222869	.150253	.083186	.036491	.030409	.9672
35	13	23	29	33	35	−.828259	.339352	.226565	.146029	.080847	.035465	.029554	.9667
36	14	24	30	34	36	−.815945	.340434	.220358	.142028	.078632	.034493	.028744	.9664
37	14	24	31	35	37	−.823165	.330893	.235350	.146819	.076531	.033572	.027976	.9661
38	15	25	32	36	38	−.811593	.332081	.229250	.143013	.074547	.032702	.027251	.9657
39	15	26	33	37	39	−.818679	.342398	.232344	.139399	.072663	.031875	.026563	.9653
40	16	27	34	38	40	−.807762	.343196	.226628	.135970	.070876	.031091	.025909	.9649
41	17	28	35	39	41	−.797306	.343839	.221214	.132722	.069183	.030348	.025290	.9644
42	17	29	36	40	42	−.804458	.353574	.224062	.129618	.067565	.029639	.024699	.9640
43	17	29	37	41	43	−.811182	.344972	.237572	.133660	.066018	.028960	.024133	.9636
44	17	29	37	42	44	−.817579	.336855	.231399	.150833	.070179	.028313	.023594	.9633
45	18	30	38	43	45	−.807875	.337678	.226332	.147530	.068642	.027693	.023077	.9629
46	18	31	39	44	46	−.814059	.346474	.228948	.144367	.067171	.027099	.022583	.9626
47	19	32	40	45	47	−.804818	.347027	.224152	.141343	.065764	.026532	.022110	.9623
48	19	32	41	46	48	−.810884	.339564	.236137	.144780	.064416	.025988	.021656	.9620
49	19	33	42	47	49	−.816651	.347820	.238375	.141870	.063121	.025465	.021221	.9617
50	20	34	43	48	50	−.807918	.348338	.233672	.139070	.061875	.024963	.020802	.9614

TABLE 1. $k = 5$ (*Continued*)

n	n_1	n_2	n_3	n_4	n_5	b_0	b_1	b_2	b_3	b_4	b_5	$V(\hat{\sigma})/\sigma^2$	RE($\hat{\sigma}$)
51	21	35	44	49	51	−.799482	.348760	.229168	.136390	.060683	.024482	.020401	.9611
52	21	36	45	50	52	−.805289	.356646	.231273	.133815	.059537	.024019	.020016	.9608
53	20	34	44	50	53	−.824121	.328004	.234152	.152498	.083010	.026458	.019643	.9605
54	20	35	45	51	54	−.828945	.335399	.236393	.149696	.081485	.025972	.019282	.9604
55	21	36	46	52	55	−.820830	.336197	.232124	.146993	.080014	.025503	.018934	.9603
56	22	37	47	53	56	−.812967	.336901	.228021	.144394	.078599	.025052	.018599	.9601
57	22	38	48	54	57	−.817855	.343978	.230136	.141888	.077235	.024617	.018276	.9599
58	23	39	49	55	58	−.810298	.344502	.226212	.139469	.075918	.024197	.017965	.9597
59	23	39	50	56	59	−.815117	.338446	.235879	.142353	.074648	.023792	.017664	.9595
60	24	40	51	57	60	−.807844	.339024	.231993	.140008	.073418	.023401	.017373	.9593
61	24	41	52	58	61	−.812528	.345663	.233883	.137735	.072226	.023021	.017091	.9592
62	25	42	53	59	62	−.805519	.346089	.230158	.135542	.071076	.022654	.016819	.9590
63	25	43	54	60	63	−.810176	.352560	.231931	.133422	.069964	.022300	.016556	.9588
64	26	44	55	61	64	−.803411	.352850	.228355	.131364	.068885	.021956	.016301	.9586
65	25	43	55	62	65	−.818988	.341184	.236712	.147859	.071609	.021623	.016054	.9583
66	26	44	56	63	66	−.812325	.341696	.233157	.145639	.070534	.021299	.015813	.9582
67	27	45	57	64	67	−.805834	.342148	.229718	.143491	.069493	.020984	.015579	.9580
68	27	46	58	65	68	−.810103	.348124	.231418	.141401	.068482	.020679	.015352	.9579
69	28	47	59	66	69	−.803823	.348456	.228106	.139377	.067501	.020383	.015133	.9577
70	28	48	60	67	70	−.808068	.354297	.229711	.137414	.066551	.020096	.014920	.9575
71	28	48	61	68	71	−.812156	.349033	.237960	.139722	.065625	.019816	.014712	.9573
72	29	49	62	69	72	−.806113	.349351	.234691	.137803	.064724	.019544	.014510	.9572
73	30	50	63	70	73	−.800214	.349625	.231520	.135941	.063849	.019280	.014314	.9570
74	30	51	64	71	74	−.804297	.355167	.232981	.134128	.062998	.019023	.014123	.9568
75	31	52	65	72	75	−.798572	.355346	.229919	.132364	.062169	.018773	.013937	.9567
76	32	53	66	73	76	−.792978	.355487	.226944	.130652	.061365	.018530	.013757	.9564
77	32	54	67	74	77	−.797066	.360868	.228345	.128981	.060580	.018293	.013581	.9563
78	32	54	68	75	78	−.801019	.355989	.236080	.131074	.059814	.018062	.013409	.9561
79	32	54	68	76	79	−.804864	.351270	.232550	.141254	.061954	.017836	.013242	.9559
80	33	55	69	77	80	−.799486	.351488	.229681	.139511	.061190	.017616	.013079	.9557
81	33	56	70	78	81	−.803244	.356563	.231026	.137810	.060444	.017402	.012919	.9556
82	34	57	71	79	82	−.798010	.356702	.228247	.136152	.059717	.017192	.012764	.9554
83	34	57	72	80	83	−.801722	.352209	.235452	.138066	.059007	.016988	.012612	.9553
84	34	58	73	81	84	−.805324	.357099	.236679	.136444	.058314	.016788	.012464	.9551
85	35	59	74	82	85	−.800253	.357237	.233928	.134858	.057636	.016593	.012319	.9550
86	35	59	74	82	86	−.803810	.352880	.230751	.132321	.070037	.017822	.012178	.9548
87	35	59	74	83	87	−.807226	.348629	.227725	.141216	.072038	.017619	.012039	.9547
88	35	59	75	84	88	−.810543	.344493	.234315	.143089	.071226	.017420	.011904	.9546
89	35	60	76	85	89	−.813768	.349057	.235559	.141494	.070432	.017226	.011771	.9545
90	36	61	77	86	90	−.808905	.349318	.232961	.139934	.069656	.017036	.011641	.9545
91	37	62	78	87	91	−.804135	.349549	.230426	.138411	.068898	.016851	.011515	.9543
92	37	63	79	88	92	−.807373	.353997	.231628	.136922	.068157	.016669	.011391	.9542
93	38	64	80	89	93	−.802717	.354166	.229163	.135465	.067431	.016492	.011270	.9541
94	38	64	81	90	94	−.805921	.350219	.235458	.137203	.066722	.016319	.011151	.9540
95	39	65	82	91	95	−.801373	.350417	.233007	.135774	.066027	.016149	.011035	.9539
96	39	66	83	92	96	−.804516	.354690	.234125	.134373	.065346	.015982	.010921	.9538
97	40	67	84	93	97	−.800073	.354832	.231738	.133003	.064680	.015819	.010810	.9537
98	40	68	85	94	98	−.803201	.359038	.232812	.131663	.064028	.015660	.010701	.9536
99	41	69	86	95	99	−.798856	.359127	.230488	.130349	.063389	.015503	.010594	.9535
100	42	70	87	96	100	−.794587	.359196	.228214	.129063	.062764	.015350	.010489	.9533

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