SOME NON-CENTRAL DISTRIBUTION PROBLEMS IN MULTIVARIATE ANALYSIS

By A. G. Constantine

C.S.I.R.O., Adelaide[1]

1. Introduction. Many problems in multivariate analysis involve the roots of certain determinantal equations. For instance, in multivariate analysis of variance (or more generally, in testing the general linear hypothesis) one is interested in the roots r_i of the equation $\det(A - r(A + B)) = 0$, where A and B are the matrices of "between classes" and "within classes" sums of squares and products, respectively (see Fisher [5], Hsu [9] and Roy [17]). Similarly, the canonical correlations between two sets of variates are given by the roots of the equation $\det(S_{12}S_{22}^{-1}S_{21} - r^2S_{11}) = 0$, where S_{11} is the matrix of sums of squares and products of the first set of variates, etc. (see Hotelling [7]).

The distributions of the roots in the null case, i.e., when the corresponding population parameters are zero, were found by Fisher, Hsu and Roy. In this paper the distributions in the non-null cases will be derived. In particular, the density functions for the following cases are obtained:

(a) the non-central Wishart distribution (Section 6),

(b) The roots of the equation $\det((A - r(A + B)) = 0$, where the matrix A has the non-central Wishart distribution, and B has the central Wishart distribution (Section 7), and

(c) the canonical correlations (Section 8). In addition, the moments of the "generalised variance" and of the likelihood ratio criterion for testing the general linear hypothesis are given.

It seems difficult to give explicit expressions for the corresponding cumulative distribution functions. However, a particular case of the probability integral of the Wishart distribution is easily evaluated. If the matrix S has the Wishart distribution, and if Ω is a given positive definite symmetric matrix, then a series expansion for the probability $\Pr\{S < \Omega\}$ will be given in Section 9, where $S < \Omega$ means $\Omega - S$ is positive definite.

A. T. James, [12], [13], has derived the distribution of the characteristic roots of the covariance matrix and the non-central Wishart distribution. He expressed the density functions of these distributions as series of "*zonal polynomials*". The zonal polynomials of a symmetric matrix S are certain homogeneous symmetric polynomials in the characteristic roots of S and may be taken as a basis for such polynomials. The distributions derived in this paper will also be expressed as series of zonal polynomials.

Many of the distributions arising in univariate normal sampling theory, such

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as non-central χ^2, non-central F, etc., can be expressed in terms of the classical hypergeometric functions ${}_pF_q(Z)$ (see Erdelyi et al., [4]). Herz [6] has defined hypergeometric functions ${}_pF_q(Z)$ of a complex symmetric matrix Z by means of a multidimensional form of the Laplace transform. He showed that the non-central Wishart distribution involves the function ${}_0F_1$, generalising the result that the non-central χ^2 distribution involves the classical Bessel function. In Section 5, hypergeometric functions of a matrix will be defined as certain series of zonal polynomials. The equivalence of the two sets of functions follows from the important integral identity

$$(1)\quad \int_{S>0} \exp(\operatorname{tr} - RS)(\det S)^{t-\frac{1}{2}(m+1)} C_\kappa(ST)\, dS = \Gamma_m(t, \kappa) C_\kappa(R^{-1}T)(\det R)^{-t},$$

where $C_\kappa(S)$ is the zonal polynomial corresponding to the partition $\kappa = (k_1, k_2, \cdots, k_m)$ of the integer k, t is a complex number satisfying $R(t) > \frac{1}{2}(m-1)$, and

$$(2)\quad \Gamma_m(t, \kappa) = \pi^{\frac{1}{4}m(m-1)} \prod_{i=1}^{m} \Gamma(t + k_i - \tfrac{1}{2}(i-1)).$$

The integration is over the space of positive definite symmetric $m \times m$ matrices. The identity (1) will be used repeatedly in the present work. It will be proved in Section 3.

2. Zonal polynomials. A detailed discussion of zonal polynomials may be found in James [12], [14]. For convenience, their definition and principal properties will be restated here.

Let S be a positive definite, symmetric $m \times m$ matrix, and $\varphi(S)$ a polynomial in the elements of S. Then, the transformation

$$(3)\quad \varphi(S) \to \varphi(L^{-1}SL'^{-1}), \qquad L \,\varepsilon\, GL(m),$$

defines a representation of the real linear group $GL(m)$ in the vector space of all polynomials in S. The space V_k of homogeneous polynomials of degree k is invariant under the transformations (3) and decomposes into the direct sum of irreducible subspaces $V_k = \sum_\kappa \oplus V_{k,\kappa}$ where $\kappa = (k_1, k_2, \cdots, k_m)$, $k_1 \geqq k_2 \geqq \cdots \geqq k_m \geqq 0$, runs over all partitions of k into not more than m parts. In each $V_{k,\kappa}$, the irreducible representation $\{2\kappa\} = \{2k_1, 2k_2, \cdots, 2k_m\}$ of $GL(m)$ acts, each of these representations occurring exactly once in the decomposition. Each $V_{k,\kappa}$ contains a unique one dimensional subspace invariant under the orthogonal group $O(m)$. These subspaces are generated by the *zonal polynomials*, $Z_\kappa(S)$. Being invariant under the orthogonal group, i.e.,

$$(4)\quad Z_\kappa(H'SH) = Z_\kappa(S), \qquad H \,\varepsilon\, O(m)$$

they are homogeneous symmetric polynomials in the characteristic roots of S.

James normalised $Z_\kappa(S)$ by assuming that the coefficient of s_1^k in $Z_\kappa(S)$ is unity, where s_1 denotes the sum of the roots of S. However, to simplify later formulae, a different normalisation will be adopted here, namely,

$$C_\kappa(S) = c(\kappa)\, Z_\kappa(S)/1.3 \cdots (2k-1), \tag{5}$$

where $c(\kappa)$ is the degree of the representation $[2\kappa]$ of the symmetric group on $2k$ symbols (see Formulae (23) – (25) in [14]).

The zonal polynomials were defined above only for positive definite symmetric matrices S. However, since they are polynomials in the characteristic roots of S, their definition may be extended to arbitrary complex symmetric matrices. Furthermore, if S is a symmetric matrix, and R is a positive definite symmetric matrix, then the roots of RS are the same as those of $R^{\frac{1}{2}}SR^{\frac{1}{2}}$ where $R^{\frac{1}{2}}$ is the (unique) positive definite square root of R. Hence, one may define $C_\kappa(RS) = C_\kappa(R^{\frac{1}{2}}SR^{\frac{1}{2}})$.

The fundamental property of the zonal polynomials is given by the following integral, proved in [12]:

$$\int_{O(m)} C_\kappa(H'SHT)\, d(H) = C_\kappa(S)C_\kappa(T)/C_\kappa(I), \tag{6}$$

where I is the identity matrix, and $d(H)$ is the invariant Haar measure on the orthogonal group, normalised to make the volume of the group manifold unity.

In order to evaluate the integral (1), further information about $C_\kappa(S)$ is needed. Unfortunately, an explicit expression for $C_\kappa(S)$ is not yet known, though James has calculated and tabulated them up to order $k = 5$. However, the following lemma, due essentially to Hua [10], is sufficient for the purpose. First, order the partitions of k lexicographically, i.e., if $\kappa = (k_1, k_2, \cdots, k_m)$ and $\tau = (t_1, t_2, \cdots, t_m)$ are two partitions of k, then define $\kappa > \tau$ if $k_1 = t_1, \cdots, k_i = t_i, k_{i+1} > t_{i+1}$. Then

LEMMA 1. *Let S be positive definite, κ a partition of k, and $C_\kappa(S)$ the corresponding zonal polynomial. Then*

$$C_\kappa(S) = \sum_{\tau \leqq \kappa} d_{\kappa,\tau}\, \chi_\tau(S), \qquad d_{\kappa,\kappa} \neq 0, \tag{7}$$

where $\chi_\tau(S)$ is the character of the representation $\{\tau\}$ of $GL(m)$, and the summation is over all partitions τ with $\tau \leqq \kappa$ in the sense of the above ordering of the partitions.

PROOF. Let $A_\kappa(S)$ denote the matrix representing S in the representation $\{\kappa\}$ of $GL(m)$, so that $\chi_\kappa(S) = \operatorname{tr} A_\kappa(S)$. The elements of $A_\kappa(S)$ are homogeneous polynomials of degree k in S. Let V_κ be the vector space of polynomials generated by these elements. V_κ is clearly invariant under the transformation

$$S \to L^{-1}SL'^{-1}, \qquad L \,\varepsilon\, GL(m), \tag{8}$$

and decomposes into a direct sum of subspaces associated with the representations $\{2\tau\}$ of $GL(m)$. Only those representations for which $\tau < \kappa$ can occur in this decomposition, and the representation $\{2\kappa\}$ *must* occur. Now $\chi_\kappa(S)$ is an element of V_κ and is invariant under (8) if L is restricted to be orthogonal. Hence, in the decomposition of V_κ under (8), $\chi_\kappa(S)$ must be a sum of invariant vectors, i.e., zonal polynomials. In other words, $\chi_\kappa(S) = \sum_{\tau \leqq \kappa} c_{\kappa,\tau} C_\tau(S)$, and $c_{\kappa,\kappa} \neq 0$. Inverting this linear relationship gives (7). Q.E.D.

Now let $s_1, \cdots, s_m$ denote the characteristic roots of S, and $\kappa = (k_1, \cdots, k_m)$ as before. Then

$$\chi_\kappa(S) = \sum c_{n_1 \cdots n_m} s_1^{n_1} \cdots s_m^{n_m}, \tag{9}$$

where the coefficients in (9) are non-negative integers. If the monomials in this expression are ordered lexicographically, the term of highest weight occurring is $s_1^{k_1} \cdots s_m^{k_m}$ with coefficient 1 (see Weyl [19], p. 134). Hence

$$C_\kappa(S) = d_{\kappa,\kappa} s_1^{k_1} \cdots s_m^{k_m} + \text{"lower terms"}. \tag{10}$$

Now, if S is a symmetric matrix of rank $h < m$, then $C_\kappa(S) = 0$ if $k_{h+1} \neq 0$. Furthermore, it follows directly from (7) that $C_\kappa(S) \equiv 0$ for partitions κ into *more* than m non-zero parts, the same being true of the characters.

Finally, since the linear relation (7) is non-singular, and since the characters are linearly independent, the zonal polynomials are also linearly independent. Hence, they may serve as a basis for symmetric functions of S.

3. Evaluation of the integral (1). Before proving the identity (1), let us note one further consequence of Lemma 1. From the theory of characters (Littlewood [16]) it is known that

$$\chi_\kappa(S) = a_1^{k_1-k_2} a_2^{k_2-k_3} \cdots a_m^{k_m} + \text{"lower terms"},$$

where a_p is the pth elementary symmetric function of the roots of S and where by "lower terms" is meant monomials in the a_p similar to the one displayed but corresponding to partitions $\tau < \kappa$. Expanding a_p as the sum of the principal $p \times p$ minors of S gives

$$C_\kappa(S) = d_{\kappa,\kappa}(\det S_1)^{k_1-k_2} (\det S_2)^{k_2-k_3} \cdots (\det S_m)^{k_m} + \cdots$$

where $S_p = (s_{ij})$, $i, j, = 1, 2, \cdots, p$. Hence, if T is a diagonal matrix with elements $t_1, \cdots, t_m$, then

$$\begin{aligned} C_\kappa(T) &= d_{\kappa,\kappa} t_1^{k_1} \cdots t_m^{k_m} + \text{"lower terms"}, \\ C_\kappa(ST) &= d_{\kappa,\kappa} t_1^{k_1} \cdots t_m^{k_m} (\det S_1)^{k_1-k_2} \cdots (\det S_m)^{k_m} + \cdots . \end{aligned} \tag{11}$$

THEOREM 1. *Let R be a complex symmetric matrix whose real part is positive definite, and let T be an arbitrary complex symmetric matrix. Then*

$$\int_{S>0} \exp(\operatorname{tr} - RS)(\det S)^{t-\frac{1}{2}(m+1)} C_\kappa(ST)\, dS = \Gamma_m(t, \kappa)(\det R)^{-t} C_\kappa(TR^{-1}), \tag{12}$$

the integration being over the space of positive definite $m \times m$ matrices, and valid for all complex numbers t satisfying $R(t) > \frac{1}{2}(m - 1)$. The constant $\Gamma_m(t, \kappa)$

is given by

$$\Gamma_m(t, \kappa) = \pi^{\frac{1}{4}m(m-1)} \prod_{i=1}^{m} \Gamma(t + k_i - \tfrac{1}{2}(i-1)). \tag{13}$$

PROOF. (12) is first proved for the special case $R = I$, the $m \times m$ identity matrix. Put

$$f(T) = \int_{S>0} \exp(\operatorname{tr} - S)(\det S)^{t-\frac{1}{2}(m+1)} C_\kappa(ST)\, dS. \tag{14}$$

$f(T)$ is clearly a symmetric function of T (in fact, a homogeneous symmetric polynomial). Hence, making the transformation $T \to H'TH$ and integrating H over $O(m)$, using (6), gives $f(T) = [f(I)/C_\kappa(I)]\, C_\kappa(T)$. Consequently, to evaluate $\Gamma_m(t, \kappa) = f(I)/C_\kappa(I)$, it is sufficient to compare the coefficients of a suitable monomial in T on both sides of (14). Assuming that T is diagonal, and comparing coefficients of $t_1^{k_1} \cdots t_m^{k_m}$ on both sides of (14), using (11), it follows that

$$\begin{aligned}\Gamma_m(t, \kappa) &\\ &= \int_{S>0} \exp(\operatorname{tr} - S)(\det S)^{t-\frac{1}{2}(m+1)}(\det S_1)^{k_1-k_2}(\det S_2)^{k_2-k_3} \\ &\qquad \cdots (\det S_m)^{k_m}\, dS.\end{aligned} \tag{15}$$

To evaluate this last integral, put $S = R'R$, where R is an upper triangular matrix, $r_{ij} = 0$ if $i > j$, $r_{ii} > 0$. Then $\det S_k = r_{11}^2 \cdots r_{kk}^2$, and the Jacobian of the transformation is $2^m \prod_{i=1}^m r_{ii}^{m-i+1}$, (see [3]). Substituting in (15),

$$\begin{aligned}\Gamma_m(t, \kappa) &= \int \cdots \int \exp\,(\sum_{i \leq j} r_{ij}^2) \prod_{i=1}^{m} (r_{ii}^2)^{t+k_i-\frac{1}{2}(i+1)} \prod dr_{ii}^2 \prod_{i<j} dr_{ij} \\ &= \pi^{\frac{1}{4}m(m-1)} \prod_{i=1}^{m} \Gamma(t + k_i - \tfrac{1}{2}(i-1)),\end{aligned}$$

the range of integration being $0 < r_{ii} < \infty$, $-\infty < r_{ij} < \infty$. This proves (12) for $R = I$. For the general case, substitute $R^{\frac{1}{2}}SR^{\frac{1}{2}}$ for S in (14), the Jacobian being $(\det R)^{\frac{1}{2}(m+1)}$.

4. The Laplace transform. In the next section, hypergeometric functions ${}_pF_q(S)$ of a matrix will be defined by series of zonal polynomials. C. S. Herz [6] has also studied such functions, defining them by means of the Laplace transform. The identity (1) can be interpreted as a Laplace transform and will be used to show that Herz's functions are the same as those defined in Section 5.

Let $f(S)$ be a function of the positive definite symmetric $m \times m$ matrix S. *The Laplace transform of* $f(S)$ is defined to be

$$g(Z) = \int_{S>0} \exp(\operatorname{tr} - SZ) f(S)\, dS, \tag{16}$$

where $Z = X + iY$ is a complex symmetric matrix, $R(Z) = X$ and Y real, and it is assumed that the integral converges in the "half-plane" $R(Z) = X > X_0$ for some positive definite X_0 (the notation $X > X_0$ means $X - X_0$ is positive definite). If this is so, $g(Z)$ is an analytic function of Z in the half-plane, and, if $g(Z)$ satisfies the conditions

$$\int |g(X + iY)| \, dY < \infty, \tag{17}$$

and

$$\lim_{x \to \infty} \int |g(X + iY)| \, dY = 0, \tag{18}$$

then the inverse formula

$$f(S) = \frac{2^{\frac{1}{2}m(m-1)}}{(2\pi i)^{\frac{1}{2}m(m+1)}} \int_{R(Z) > X_0 > 0} \exp(\operatorname{tr} SZ) g(Z) \, dZ \tag{19}$$

holds. In (19) the integration is taken over $Z = X + iY$, with $X > X_0$ and fixed and Y ranges over all real symmetric matrices. For details, see Herz [6], and references quoted therein.

From (1), the following theorem immediately follows.

THEOREM 2. *The Laplace transform of* $(\det S)^{t-\frac{1}{2}(m+1)} C_\kappa(S)$ *is given by*

$$\int_{S>0} \exp(\operatorname{tr} - SZ)(\det S)^{t-\frac{1}{2}(m+1)} C_\kappa(S) \, dS = \Gamma_m(t, \kappa)(\det Z)^{-t} C_\kappa(Z^{-1}), \tag{20}$$

valid for $R(t) > \frac{1}{2}(m - 1)$, *and the corresponding inverse transform is given by*

$$\frac{2^{\frac{1}{2}m(m-1)}}{(2\pi i)^{\frac{1}{2}m(m+1)}} \int_{R(Z)>0} \exp(\operatorname{tr} SZ)(\det Z)^{-t} C_\kappa(Z^{-1}) \, dZ = \frac{1}{\Gamma_m(t, \kappa)} (\det S)^{t-\frac{1}{2}(m+1)} C_\kappa(S). \tag{21}$$

PROOF. To prove (20), put $T = I$, $R = Z$ in (1). To prove (21), it has to be shown that $(\det Z)^{-t} C_\kappa(Z^{-1})$ satisfies (17) and (18). To do this, note that since $R(Z) > 0$, $C_\kappa(Z^{-1})$ is bounded, so that it is sufficient to verify the results for $(\det Z)^{-t}$. This was done by Herz [6]. Q.E.D.

The Laplace transform defined above also satisfies the convolution theorem: if $g_1(Z)$, $g_2(Z)$ are the Laplace transforms of $f_1(S)$, $f_2(S)$, then $g_1(Z)g_2(Z)$ is the Laplace transform of $f(R) = \int_0^R f_1(S) f_2(R - S) \, dS$, the integration being over all S for which $0 < S < R$. Using the convolution theorem, an analogue of the beta-function integral can be derived.

THEOREM 3. *If* R *is a positive definite* $m \times m$ *matrix, then*

$$\int_0^I (\det S)^{t-\frac{1}{2}(m+1)} \det (I - S)^{u-\frac{1}{2}(m+1)} C_\kappa(RS) \, dS = \frac{\Gamma_m(t, \kappa)\Gamma_m(u)}{\Gamma_m(t + u, \kappa)} C_\kappa(R), \tag{22}$$

where $\Gamma_m(u) = \pi^{\frac{1}{4}m(m-1)} \prod_{i=1}^m \Gamma(u - \frac{1}{2}(i - 1))$.

PROOF. The left hand side of (22) is a symmetric function $F(R)$ of R, so that, as in the proof of Theorem 1.

$$F(R) = [F(I)/C_\kappa(I)]\, C_\kappa(R). \tag{23}$$

On the other hand, making the transformation $S = R^{-\frac{1}{2}}\, TR^{-\frac{1}{2}}$,

$$F(R)(\det R)^{t+u-\frac{1}{2}(m+1)} = \int_0^R (\det T)^{t-\frac{1}{2}(m+1)} \det (R - T)^{u-\frac{1}{2}(m+1)}\, C_\kappa(T)\, dT \tag{24}$$

Taking the Laplace transform of both sides of (24) using the convolution theorem to evaluate the transform of the right hand side, gives

$$[F(I)/C_\kappa(I)]\, \Gamma_m(u + t, \kappa)\, (\det Z)^{-t-u}\, C_\kappa(Z^{-1})$$
$$= \Gamma_m(t, \kappa)\, (\det Z)^{-t}\, C_\kappa(Z^{-1})\, \Gamma_m(u)\, (\det Z)^{-u}.$$

Solving this equation for $F(I)/C_\kappa(I)$ and substituting in (23) gives the result. Q.E.D.

5. Hypergeometric functions. In analogy with the classical hypergeometric functions of a single complex variable, we define

$${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; Z) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa(Z)}{k!}, \tag{25}$$

where

$$(a)_\kappa = \prod_{i=1}^{m} (a - \tfrac{1}{2}(i - 1))_{k_i}, \qquad \kappa = (k_1, \cdots, k_m) \tag{26}$$

and, as usual, $(x)_n = x(x + 1) \cdots (x + n - 1)$, $(x_0)_0 = 1$. If a is such that the gamma functions are defined, then (26) may be written as

$$(a)_\kappa = \Gamma_m(a, \kappa)/\Gamma_m(a). \tag{27}$$

In (25), Z is a complex symmetric $m \times m$ matrix, and it is assumed that $p \leqq q + 1$, otherwise the series may only converge for $Z = 0$. For $p = q + 1$, the series converge for $\|Z\| < 1$, where $\|Z\|$ denotes the maximum of the absolute values of the characteristic roots of Z. For $p \leqq q$, the series converge for all Z. The parameters a_i and b_j are arbitrary complex numbers so long as none of the b_j is an integer or half integer $\leqq \frac{1}{2}(m - 1)$ (otherwise some of the denominators in (25) will vanish). Finally, note that if one of the a_i is a negative integer say $a_1 = -n$ then for $k \geqq mn + 1$, all coefficients in (25) vanish, so that the function reduces to a *polynomial* of degree mn.

Using the results of Theorem 2, integrating the series term-by-term, it follows that

$${}_{p+1}F_q(a_1, \cdots, a_p, c; b_1, \cdots, b_q; -Z^{-1})(\det Z)^{-c}$$
$$= \frac{1}{\Gamma_m(c)} \int_{S>0} \exp(\operatorname{tr} - SZ)\, {}_pF_q(a, \cdots, a_p; b_1, \cdots, b_q; -S) \cdot (\det S)^{c-\frac{1}{2}(m+1)}\, dS \tag{28}$$

and

$$
\begin{aligned}
&{}_pF_{q+1}(a_1, \cdots, a_p; b_1, \cdots, b_q, c; -S)(\det S)^{c-\frac{1}{2}(m+1)} \\
(29) \quad &= \Gamma_m(c) \frac{2^{\frac{1}{2}m(m-1)}}{(2\pi i)^{\frac{1}{2}m(m+1)}} \int_{R(Z)=X_0} \exp(\operatorname{tr} SZ)\, {}_pF_q(a_1, \cdots a_p; b_1, \cdots, b_q; -Z^{-1}) \\
&\qquad \cdot (\det Z)^{-c}\, dZ.
\end{aligned}
$$

These are exactly the formulae used by Herz [6] to define the hypergeometric functions, so that the functions defined by (25) are identical with those defined by Herz. In what follows, we shall often make use of some of Herz's results. The reader is referred to his paper where a detailed theory, including many topics not touched upon here, may be found.

In conclusion, we note the following two particular cases of (25).

$$
\begin{aligned}
(30) \quad {}_0F_0(Z) &= \sum_{k=0}^{\infty} \sum_{\kappa} C_\kappa(Z)/k! \\
&= \exp(\operatorname{tr} Z),
\end{aligned}
$$

which is proved in [14]. Substituting for ${}_0F_0(Z)$ in (28),

$$
\begin{aligned}
{}_1F_0(a; Z) &= \frac{1}{\Gamma_m(a)} \int_{S>0} \exp(\operatorname{tr} - S) \exp(\operatorname{tr} SZ) (\det S)^{a-\frac{1}{2}(m+1)}\, dS \\
&= \det (I - Z)^{-a}
\end{aligned}
$$

so that

$$
(31) \quad \det (I - Z)^{-a} = \sum_{k=0}^{\infty} \sum_{\kappa} (a)_\kappa [C_\kappa(Z)/k!],
$$

which generalises the binomial series.

6. The non-central Wishart distribution. The non-central Wishart distribution is the distribution of the matrix $S = XX'$, when the $m \times n$ matrix X has the normal distribution

$$
(32) \quad (\det 2\pi\Sigma)^{-\frac{1}{2}n} \exp(\operatorname{tr} - \tfrac{1}{2} \Sigma^{-1} (X - M)(X - M)').
$$

Herz [6] expressed the distribution as a "Bessel function", ${}_0F_1$, by inverting the moment generating function (Laplace transform) of the distribution. James [13], [14] gave a zonal polynomial expansion by proving

$$
(33) \quad \int_{V(n,m)} \exp(\operatorname{tr} V'X)\, dV = \frac{2^m \pi^{\frac{1}{2}mn}}{\Gamma_m(\frac{1}{2}n)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(XX')}{Z_\kappa(I_n) 2^k k!},
$$

where $V(n, m)$ is the "Stiefel manifold" of orthogonal m-frames in n-space, the connection between C_κ and Z_κ being given by (5). The derivation here will follow that of Herz.

Let X have the distribution (32), and let Z be a complex symmetric $m \times m$

matrix. If $f(S)$ denotes the probability density function of $S = XX'$, then the Laplace transform of $f(S)$ is $g(Z) = \mathfrak{E}\,[\exp(\text{tr} - ZXX')]$, $\mathfrak{E}$ denoting the expectation. Multiplying (32) by $\exp(\text{tr} - ZXX')$ and integrating over X, using the well-known result

$$\int \exp(\text{tr} - RXX')\exp(\text{tr}\, SX')\, dX = \pi^{\frac{1}{2}mn}\exp\left(\text{tr}\,\frac{1}{4}R^{-1}SS'\right)(\det R)^{-\frac{1}{2}n}$$

(see, for example, [6], p. 481), gives

$$(34)\quad \begin{aligned} g(Z) = (\det 2\Sigma)^{-\frac{1}{2}n}&\exp(\text{tr} - \tfrac{1}{2}\Sigma^{-1}MM') \\ &\cdot \exp(\text{tr}\,\tfrac{1}{4}\Sigma^{-1}MM'\Sigma^{-1}(Z + \tfrac{1}{2}\Sigma^{-1})^{-1})\det(Z + \tfrac{1}{2}\Sigma^{-1})^{-\frac{1}{2}n}. \end{aligned}$$

Putting $W = Z + \frac{1}{2}\Sigma^{-1}$ in (34), and inverting gives

$$(35)\quad \begin{aligned} f(S) &= \frac{2^{\frac{1}{2}m(m-1)}}{(2\pi i)^{\frac{1}{2}m(m+1)}}\int_{R(Z)>0}\exp(\text{tr}\, SZ)g(Z)\, dZ \\ &= (\det 2\Sigma)^{-\frac{1}{2}n}\exp(\text{tr} - \Omega)\exp\left(\text{tr} - \frac{1}{2}\Sigma^{-1}S\right) \\ &\cdot\frac{2^{\frac{1}{2}m(m-1)}}{(2\pi i)^{\frac{1}{2}m(m+1)}}\int_{R(W)=W_0>0}\exp(\text{tr}\, WS)\exp\left(\text{tr}\,\frac{1}{2}\Sigma^{-1}\Omega W^{-1}\right)(\det W)^{-\frac{1}{2}n}\, dW, \end{aligned}$$

where $\Omega = \frac{1}{2}\Sigma^{-1}MM'$. Observing that $\exp(\text{tr}\,\frac{1}{2}\Sigma^{-1}\Omega W^{-1}) = {}_0F_0(\frac{1}{2}\Sigma^{-1}\Omega W^{-1})$, we have, from (29) or equivalently by expanding the exponential according to (30) and integrating term-by-term using Theorem 2, that

$$(36)\quad \begin{aligned} \frac{2^{\frac{1}{2}m(m-1)}}{(2\pi i)^{\frac{1}{2}m(m+1)}}&\int_{R(W)=W_0}\exp(\text{tr}\, WS)\exp\left(\text{tr}\,\frac{1}{2}\Sigma^{-1}\Omega W^{-1}\right)(\det W)^{-\frac{1}{2}n}\, dW \\ &= \frac{1}{\Gamma_m(\frac{1}{2}n)}\,{}_0F_1\left(\frac{1}{2}n; \frac{1}{2}\Sigma^{-1}\Omega S\right)(\det S)^{\frac{1}{2}n-(m+1)} \\ &= \frac{1}{\Gamma_m(\frac{1}{2}n)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\frac{1}{2}\Sigma^{-1}\Omega S)}{(\frac{1}{2}n)_\kappa k!}. \end{aligned}$$

Hence

THEOREM 4. *The density function of the matrix* $S = XX'$, *when* X *has the distribution* (32) *is given by*

$$(37)\quad \begin{aligned} (\Gamma_m(\tfrac{1}{2}n))^{-1}(\det 2\Sigma)^{-\frac{1}{2}n}&\exp(\text{tr} - \Omega)\exp(\text{tr} - \tfrac{1}{2}\Sigma^{-1}S) \\ &\cdot(\det S)^{\frac{1}{2}(n-m-1)}\,{}_0F_1(\tfrac{1}{2}n; \tfrac{1}{2}\Sigma^{-1}\Omega S). \end{aligned}$$

The distribution is called the non-central Wishart distribution on n degrees of freedom with matrix of non-centrality parameters $\Omega = \frac{1}{2}MM'\Sigma^{-1}$.

COROLLARY 1. *Comparing coefficients of* C_κ *in* (36) *and in* [13], *Theorem III, p.* 877, *gives*

$$(38)\quad Z_\kappa(I_n) = 2^k(\tfrac{1}{2}n)_\kappa.$$

(This result appears difficult to establish more directly.)

COROLLARY 2. *The moments of the "generalised variance"*, det S, *are given by*

(39) $$\mathfrak{E}[(\det S)^t] = [\Gamma_m(t + \tfrac{1}{2}n)/\Gamma_m(\tfrac{1}{2}n)]\,(\det 2\Sigma)^t\,{}_1F_1\,(-t; \tfrac{1}{2}n; -\Omega).$$

PROOF. Multiplying (37) by $(\det S)^t$ and integrating it follows from (28) that

$$\mathfrak{E}[(\det S)^t] = [\Gamma_m(t + \tfrac{1}{2}n)/\Gamma_m(\tfrac{1}{2}n)]\,(\det 2\Sigma)^t \exp(\mathrm{tr} - \Omega)\,{}_1F_1\,(\tfrac{1}{2}n + t; \tfrac{1}{2}n; \Omega).$$

Applying the Kummer transformation formula, [6], p. 488,

(40) $${}_1F_1\,(a; b; \Omega) = \exp(\mathrm{tr}\,\Omega)\,{}_1F_1\,(b - a; b; -\Omega),$$

to this last expression gives the result. If t is an integer, ${}_1F_1\,(-t; \tfrac{1}{2}n; -\Omega)$ is a polynomial of degree mt.

7. The non-central means case in multivariate analysis of variance. In this section we shall derive the distribution of the characteristic roots r_i of the matrix $A(A + B)^{-1}$. In multivariate analysis of variance, A is the "between classes" matrix of sums of squares and products, and B is the "within classes" matrix. The likelihood ratio criterion for testing equality of the mean vectors is then $W = \prod (1 - r_i) = \det (B(A + B)^{-1})$, see Wilks [20]. Lawley [15] and Hotelling [8] have proposed the criterion $T_0^2 = \sum r_i/(1 - r_i) = \mathrm{tr}\,AB^{-1}$, and Roy [18] has discussed tests based on the largest or smallest of the roots.

In the non-null case, B has the Wishart distribution and A has the non-central Wishart distribution with matrix of noncentrality parameters Ω. Bartlett [1] and Constantine and James [2] expressed the distribution of the r_i as a multiple power series and calculated the coefficients for the terms up to fourth order. Here, we shall give a zonal polynomial expansion for the distribution.

THEOREM 5. *If the $m \times m$ matrix A has the non-central Wishart distribution on s degrees of freedom and matrix of noncentrality parameters Ω, and B has the Wishart distribution on t degrees of freedom, the covariance matrix in each case being Σ, then the probability density function of the roots of the matrix $R = A(A + B)^{-1}$ is given by*

(41) $$\frac{\pi^{\frac{1}{2}m^2}\Gamma_m((s + t)/2)}{\Gamma_m(\frac{1}{2}s)\Gamma_m(\frac{1}{2}t)\Gamma_m(\frac{1}{2}m)} \exp(\mathrm{tr} - \Omega)\,(\prod r_i)^{\frac{1}{2}(s-m-1)} \prod (1 - r_i)^{\frac{1}{2}(t-m-1)} \prod_{i<j} (r_i - r_j) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}(s + t))_\kappa}{(\frac{1}{2}s)_\kappa} \frac{C_\kappa(\Omega)C_\kappa(R)}{C_\kappa(I)k!}.$$

PROOF. The roots are clearly invariant under the simultaneous transformations $A \rightarrow \frac{1}{2}\Sigma^{-\frac{1}{2}} A\Sigma^{-\frac{1}{2}}$, $B \rightarrow \frac{1}{2}\Sigma^{-\frac{1}{2}} B\Sigma^{-\frac{1}{2}}$, so that the joint distribution of A and B may be assumed to be

$$[\exp(\mathrm{tr} - \Omega)/\Gamma_m(\tfrac{1}{2}s)\,\Gamma_m(\tfrac{1}{2}t)]\exp(\mathrm{tr} - A - B)\,(\det A)^{\frac{1}{2}(s-m-1)}\,(\det B)^{\frac{1}{2}(t-m-1)} \cdot {}_0F_1\,(\tfrac{1}{2}s; \Omega A).$$

Transform to variables $G = A + B$, $R = G^{-\frac{1}{2}} AG^{-\frac{1}{2}}$. The Jacobian is

$(\det G)^{\frac{1}{2}(m+1)}$, and hence, the joint distribution of G and R is

$$[\exp(\mathrm{tr} - \Omega)/\Gamma_m(\tfrac{1}{2}s)\,\Gamma_m(\tfrac{1}{2}t)] \exp(\mathrm{tr} - G)\,(\det G)^{\frac{1}{2}(s+t-m-1)}\,{}_0F_1(\tfrac{1}{2}s; \Omega G^{\frac{1}{2}} R G^{\frac{1}{2}}) \cdot (\det R)^{\frac{1}{2}(s-m-1)} \det(I - R)^{\frac{1}{2}(t-m-1)},$$

the range of the variables being $G > 0$, $0 < R < I$. It would be of interest to be able to integrate out G from this expression and so obtain the distribution of the matrix R, but this appears difficult unless Ω is either a scalar matrix or of rank 1.

R, being symmetric, can be diagonalised by an orthogonal transformation, i.e., there is an orthogonal matrix H such that $H'RH$ is a diagonal matrix with the roots r_i down the diagonal. In order to specify this transformation uniquely, it is assumed that the roots are arranged in decreasing order, $r_1 > r_2 > \cdots > r_m$, and the elements in the first column of H are positive. Under this transformation, the volume element dR becomes

$$dR = \prod_{i<j} (r_i - r_j) \prod dr_i\, d(H), \tag{43}$$

where the measure $d(H)$ is that derived by the exterior product of differential forms on the orthogonal group. With this measure

$$\int_{O(m)} d(H) = \frac{2^m \pi^{\frac{1}{2}m^2}}{\Gamma_m(\frac{1}{2}m)}. \tag{44}$$

(43) and (44) are proved in James [11]. Substituting for R in (42), and integrating over H, gives the joint distribution of $r_1, \cdots, r_m$ and G,

$$\begin{aligned} &\frac{\exp(\mathrm{tr} - \Omega)}{\Gamma_m(\frac{1}{2}s)\Gamma_m(\frac{1}{2}t)} \left(\prod r_i\right)^{\frac{1}{2}(s-m-1)} \prod (1 - r_i)^{\frac{1}{2}(t-m-1)} \prod_{i<j} (r_i - r_j) \\ &\cdot \exp(\mathrm{tr} - G)\,(\det G)^{\frac{1}{2}(s+t-m-1)} \frac{1}{2^m} \int_{O(m)} {}_0F_1\left(\frac{1}{2}s; \Omega G^{\frac{1}{2}} H R H' G^{\frac{1}{2}}\right) d(H). \end{aligned} \tag{45}$$

The factor 2^{-m} multiplying the integral arises from the restriction that the elements in the first column of H are positive, so that the integral is actually only over this 2^{-m}th part of the orthogonal group. Expanding the function ${}_0F_1$ as a series of zonal polynomials and integrating over $O(m)$, using (6),

$$\frac{1}{2^m} \int_{O(m)} {}_0F_1\left(\frac{1}{2}s; \Omega G^{\frac{1}{2}} H R H' G^{\frac{1}{2}}\right) d(H) = \frac{\pi^{\frac{1}{2}m^2}}{\Gamma_m(\frac{1}{2}m)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(\Omega G) C_\kappa(R)}{(\frac{1}{2}s)_\kappa C_\kappa(I) k!}. \tag{46}$$

The symbol R has been retained in (45) and (46) since $C_\kappa(R)$ is a polynomial in $r_1, \cdots, r_m$. Finally, substituting (46) in (45) and integrating out G using (1) gives the required result. Q.E.D.

COROLLARY. *The moments of the likelihood ratio criterion,* $W = \prod (1 - r_i) = \det (I - R)$, *are given by*

$$\mathcal{E}[W^h] = \frac{\Gamma_m(h + \frac{1}{2}t)\Gamma_m(\frac{1}{2}(s + t))}{\Gamma_m(\frac{1}{2}t)\Gamma_m(h + \frac{1}{2}(s + t))}\, {}_1F_1\left(h; h + \frac{1}{2}(s + t); -\Omega\right). \tag{47}$$

PROOF. Multiply (42) by det $(I - R)^h$ and integrate over R and G, using (22) and (1).

8. The distribution of the canonical correlation coefficients. The canonical correlations were introduced by Hotelling [7] as a measure of the relation between two sets of variates. Let $x = (x_1, \cdots, x_p)$, $y = (y_1, \cdots, y_q)$, $p \leqq q$, denote the two vector variates, Σ_{11} and Σ_{22} their respective covariance matrices and Σ_{12} the $p \times q$ matrix of covariances between the components of x and the components of y. The $(p + q)$-vector variate $(x_1, \cdots, x_p, y_1, \cdots, y_q)$ then has covariance matrix

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \qquad \Sigma_{21} = \Sigma_{12}'.$$

Hotelling showed that there exist non-singular linear transformations

$$x \to N_1 x = u, \qquad y \to N_2 y = v, \tag{48}$$

such that the new variates (u, v) have covariance matrix

$$\begin{pmatrix} I & \mathrm{P} & 0 \\ \mathrm{P} & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{49}$$

where P is the $p \times p$ diagonal matrix with elements $\rho_1, \cdots, \rho_p$ arranged in decreasing order down the diagonal. The ρ_i are the *canonical correlations*. They are easily seen to satisfy the determinantal equation

$$\det(\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} - \rho^2\Sigma_{11}) = 0. \tag{50}$$

Any function of the elements of Σ which is invariant under the transformations (48) must be a function of the ρ_i so that (49) is the canonical form of Σ under such transformations.

Given a sample from the population, the maximum likelihood estimates $r_1, \cdots, r_p$ of the ρ_i satisfy the determinantal equation

$$\det(S_{12}S_{22}^{-1}S_{21} - r^2 S_{11}) = 0, \tag{51}$$

where the S_{ij} are the corresponding submatrices of the sample covariance matrix S.

We shall derive the distribution of the canonical correlation coefficients $r_1, \cdots, r_p$ assuming that the variates (x, y) are jointly normally distributed with covariance matrix Σ. Without loss of generality it may be supposed that the means are zero. The sample may then be represented by the $(p + q) \times n$ partitioned matrix $\begin{pmatrix} X \\ Y \end{pmatrix}$, where X and Y are $p \times n$ and $q \times n$ matrices, respectively.

The sample covariance matrix is then

$$\begin{pmatrix} XX' & XY' \\ YX' & YY' \end{pmatrix}$$

so that (51) may be written as

$$\det (XY' (YY')^{-1} YX' - r^2 XX') = 0. \tag{52}$$

The roots r_i^2 of (52) are clearly invariant under the transformations (48), so that Σ may be assumed to be of the form (49).

The distribution of $r_1, \cdots, r_p$ will be found first conditional upon $y_1, \cdots, y_q$ being given. The author would like to thank a referee for suggesting this method of obtaining the distribution.) If y is given, the conditional distribution of the matrix X is normal with covariance matrix $\Omega = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ and mean matrix $\Sigma_{12}\Sigma_{22}^{-1}Y = \Lambda Y$; i.e.,

$$(\det 2\pi\Omega)^{-\frac{1}{2}n} \exp (\mathrm{tr} - \tfrac{1}{2}\Omega^{-1} (X - \Lambda Y) (X - \Lambda Y)'). \tag{53}$$

Since Σ is assumed to be (49),

$$\Omega = \begin{pmatrix} 1 - \rho_1^2 & & \\ & \ddots & \\ & & 1 - \rho_p^2 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \rho_1 & & & \\ & \ddots & & 0 \\ & & \rho_p & \end{pmatrix} = (\mathrm{P} \vdots 0). \tag{54}$$

Y is of rank q, so that there is an $n \times n$ orthogonal matrix H such that $YH = (Y_1 \vdots 0)$ where Y_1 is a $q \times q$ non-singular matrix. To do this, choose any $n - q$ orthonormal vectors in the orthogonal complement of the space spanned by the rows of Y as the last $n - q$ columns of H. Then, putting $W = XH$, Equation (52) becomes

$$\det \left(XH \begin{pmatrix} Y_1' \\ 0 \end{pmatrix} (Y_1Y_1')^{-1}(Y_1 \vdots 0)H'X' - r^2 XHH'X' \right) = 0$$

i.e.

$$\det \left(W \begin{pmatrix} I_q & 0 \\ 0 & 0 \end{pmatrix} W' - r^2 WW' \right) = 0$$

or

$$\det (UU' - r^2(UU' + VV')) = 0 \tag{55}$$

where W has been partitioned in the form $W = (U \vdots V)$, U a $p \times q$ matrix and V a $p \times (n - q)$ matrix. Furthermore,

$$\begin{aligned} \mathrm{tr}\, \Omega^{-1}(X - \Lambda Y)(X - \Lambda Y)' &= \mathrm{tr}\, \Omega^{-1}(XH - \Lambda YH((XH - YH)' \\ &= \mathrm{tr}\Omega^{-1}(W - (\Lambda Y_1 \vdots 0)(W - (\Lambda Y_1 \vdots 0))' \\ &= \mathrm{tr}\, \Omega^{-1}(U - \Lambda Y_1)(U - \Lambda Y_1)' + \mathrm{tr}\, \Omega^{-1}VV'. \end{aligned} \tag{56}$$

Substituting (56) in (53), it follows that U and V are independently distributed (in the conditional distribution given Y), the distribution of U being normal with mean matrix ΛY_1, and that of V normal with mean matrix 0, the covariance matrix in both cases being Ω. Hence, the distribution of UU' is the non-central Wishart distribution on q degrees of freedom and with matrix of non-centrality parameters $\frac{1}{2}\Omega^{-1}\Lambda Y_1 Y_1' \Lambda' = \frac{1}{2}\Omega^{-1}\Lambda YY'\Lambda'$, and the distribution of VV' is the Wishart distribution on $n - q$ degrees of freedom. Then, according to Theorem 5, the conditional distribution of $r_1^2, \cdots, r_p^2, r_1^2 > \cdots > r_p^2$, given $y_1, \cdots, y_q$ is

$$(57)\quad \begin{aligned} &\frac{\pi^{\frac{1}{2}p^2}\Gamma_p(\frac{1}{2}n)}{\Gamma_p(\frac{1}{2}q)\Gamma_p(\frac{1}{2}(n-q))\Gamma_p(\frac{1}{2}p)} \exp\left(\mathrm{tr} - \frac{1}{2}\Omega^{-1}\Lambda YY'\Lambda'\right) \left(\prod r_i^2\right)^{\frac{1}{2}(q-p-1)} \\ &\prod (1 - r_i^2)^{\frac{1}{2}(n-q-p-1)} \prod_{i<j} (r_i^2 - r_j^2) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}n)_\kappa C_\kappa(\frac{1}{2}\Omega^{-1}\Lambda YY'\Lambda')}{(\frac{1}{2}q)_\kappa C_\kappa(I_p) k!} C_\kappa(R^2) \end{aligned}$$

where

$$R^2 = \begin{pmatrix} r_1^2 & & \\ & \ddots & \\ & & r_p^2 \end{pmatrix}.$$

To find the unconditional distribution of $r_1, \cdots, r_p$, multiply (57) by the marginal distribution of Y and integrate over Y. However, (57) is a function of YY', so that it is sufficient to integrate over the distribution of YY'. Now the marginal distribution of Y is normal with covariance matrix $\Sigma_{22} = I$, so that YY' has the Wishart distribution on n degrees of freedom, i.e.,

$$(58)\qquad [2^{-\frac{1}{2}nq}/\Gamma_q(\tfrac{1}{2}n)] \exp(\mathrm{tr} - \tfrac{1}{2}YY') (\det YY')^{\frac{1}{2}(n-q-1)}.$$

Multiplying (57) by (58) and integrating term-by-term leads to the integral

$$\int_{YY'>0} \exp\left(\mathrm{tr} - \frac{1}{2}(I + \Lambda'\Omega^{-1}\Lambda)YY'\right) (\det YY')^{\frac{1}{2}(n-q-1)} \cdot C_\kappa\left(\frac{1}{2}\Lambda'\Omega^{-1}\Lambda YY'\right) d(YY').$$

Substituting for Ω and Λ from (54), the integral is, by (1),

$$\Gamma_q(\tfrac{1}{2}n)\, (\tfrac{1}{2}n)_\kappa\, 2^{\frac{1}{2}nq} \prod (1 - \rho_i^2)^{\frac{1}{2}n}\, C_\kappa(\mathrm{P}^2)$$

where P^2 is the diagonal matrix with elements $\rho_1^2, \cdots, \rho_p^2$. Hence

THEOREM 6. *Suppose the variates $x_1, \cdots, x_p, y_1, \cdots, y_q$ $p \leqq q$, are normally distribution with zero means and covariance matrix Σ. If $\rho_1^2, \cdots, \rho_p^2$ are the roots of* (50), *the maximum likelihood estimates $r_1^2, \cdots, r_p^2$ from a sample of size n, $n \geqq p + q$, are given by the roots of* (51), *and have the density*

$$(59)\quad \begin{aligned} &\frac{\pi^{-p^2}\Gamma(\frac{1}{2}n)}{\Gamma_p(\frac{1}{2}q)\Gamma_p(\frac{1}{2}(n-q))\Gamma_p(\frac{1}{2}p)} \prod (1 - \rho_i^2)^{\frac{1}{2}n} \left(\prod r_i^2\right)^{\frac{1}{2}(q-p-1)} \\ &\cdot \prod (1 - r_i^2)^{\frac{1}{2}(n-q-p-1)} \prod_{i<j} (r_i^2 - r_j^2) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}n)_\kappa (\frac{1}{2}n)_\kappa}{(\frac{1}{2}q)_\kappa} \frac{C_\kappa(R^2) C_\kappa(\mathrm{P}^2)}{C_\kappa(I_p) k!}, \end{aligned}$$

where R^2 and P^2 are diagonal matrices with elements r_i^2 and ρ_j^2 respectively.

9. Incomplete gamma and beta functions. We conclude by giving the cumulative distribution functions of the Wishart distribution and the multivariate beta distribution. More precisely, we shall give zonal polynomial expansions for the probabilities $\Pr\{S < \Omega\}$ where the matrix S has either of the distributions just mentioned.

THEOREM 7.

$$\begin{aligned}\int_0^\Omega &\exp(\mathrm{tr} - \Lambda S)(\det S)^{t-\frac{1}{2}(m+1)}\, dS \\ &= \frac{\Gamma_m(t)\Gamma_m((m+1)/2)}{\Gamma_m(t+\frac{1}{2}(m+1))}(\det \Omega)^t\, {}_1F_1\left(t; t+\frac{1}{2}(m+1); -\Omega\Lambda\right),\end{aligned} \tag{60}$$

$$\begin{aligned}\int_0^\Omega (\det R)^{s-\frac{1}{2}(m+1)} \det(I-R)^{t-\frac{1}{2}(m+1)}\, dR &= \frac{\Gamma_m(s)\Gamma_m(\frac{1}{2}(m+1))}{\Gamma_m(s+\frac{1}{2}(m+1))} \\ \cdot(\det\Omega)^s\, {}_2F_1\left(s, -t+\frac{1}{2}(m+1); s+\frac{1}{2}(m+1); \Omega\right)&, \\ \text{for} \quad 0 < \Omega < I.&\end{aligned} \tag{61}$$

PROOF. (60) is proved by putting $S = \Omega^{\frac{1}{2}}T\Omega^{\frac{1}{2}}$, $dS = (\det\Omega)^{\frac{1}{2}(m+1)}\, dT$, $0 < T < I$, expanding $\exp(\mathrm{tr}\Omega^{\frac{1}{2}}\Lambda\Omega^{\frac{1}{2}}T)$ as a series of zonal polynomials and integrating term-by-term using (22). Similarly (61) is proved by putting $R = \Omega^{\frac{1}{2}}T\Omega^{\frac{1}{2}}$, expanding $\det(I - \Omega T)^{t-\frac{1}{2}(m+1)} = {}_1F_0(-t+\frac{1}{2}(m+1); \Omega T)$ and integrating.

Substituting the appropriate parameters for t and Λ in (60), gives $\Pr\{S < \Omega\}$ when S has the Wishart distribution, namely, if $t = \frac{1}{2}n$, and $\Lambda = \frac{1}{2}\Sigma^{-1}$, where n is the number of degrees of freedom and Σ is the covariance matrix, then

$$\begin{aligned}\Pr\{S < \Omega\} = [(\det \tfrac{1}{2}\Sigma^{-1}\Omega)^{\frac{1}{2}n}\, &\Gamma_m(\tfrac{1}{2}(m+1))/\Gamma_m(\tfrac{1}{2}(n+m+1))] \\ &{}_1F_1(\tfrac{1}{2}n; \tfrac{1}{2}(n+m+1); -\tfrac{1}{2}\Sigma^{-1}\Omega).\end{aligned} \tag{62}$$

In the univariate case, $m = 1$, the series in (60) and (61) can be reduced to *polynomials* (truncated exponential and binomial series) when s and t are integers or half-integers. For $m \geqq 2$, this does not seem to be so. Furthermore, for $m \geqq 2$, $\Pr(S < \Omega) \neq 1 - \Pr\{S > \Omega\}$, since the set of S where neither of the relations $S < \Omega$ nor $S > \Omega$ holds is not of measure zero. The complementary probabilities $\Pr\{S > \Omega\}$ seem difficult to evaluate.

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