SEQUENTIAL TESTS FOR THE MEAN OF A NORMAL DISTRIBUTION II (LARGE t)

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1. Summary. Asymptotic expansions are derived for the behavior of the optimal sequential test of whether the unknown drift μ of a Wiener-Levy process is positive or negative for the case where the process has been observed for a long time. The test is optimal in the sense that it is the Bayes test for the problem where we have an *a priori* normal distribution of μ, the regret for coming to the wrong conclusion is proportional to $|\mu|$, and the cost of observation is constant per unit time. The Bayes procedure is then compared with the best sequential likelihood ratio test.

2. Introduction. In the Fourth Berkeley Symposium on Probability and Statistics, Chernoff [2] indicated that the problem of sequentially testing whether the mean drift of a Wiener-Levy process was positive or negative given a normal *a priori* probability distribution was relevant to the problem of deriving an asymptotically (as sampling cost approaches zero) optimal sequential test of whether the mean of a normally distributed variable is positive or negative. The former problem was shown to be equivalent to the solution of a free boundary problem involving a diffusion equation.

Incidentally a few properties of the solution were presented but nothing was done about a precise representation of the solution.

Since then the authors of the present paper separately and together have engaged in research and computation to characterize the solution more precisely and to investigate the relevance of these results to other related problems. These problems include testing problems for non-normally distributed random variables, sequential design problems and problems involving other loss functions. Because of the great variety of results and relevant applications it was decided that it would be appropriate to publish these results in relatively small pieces rather than to wait an unspecified time before all of the results could be reasonably completed and assembled into one opus. In the meantime overlapping results have also been obtained by Moriguti and Robbins [5], who have approached this problem through that of testing whether a binomial parameter exceeds or is less than $\frac{1}{2}$ and by Bather [1].

The easiest part of the problem seems to be that of deriving the asymptotic

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behavior for large t. That represents behavior of the Bayes strategy after a great deal of sampling has been carried out. An asymptotic expansion was derived in (1951) by D. G. Champernowne but was not published. Moriguti and Robbins simply indicate several terms of the expansion without elaborating. Neither do they nor Bather present any proof that the expansion is in fact a valid expansion in the sense of yielding asymptotic approximations to the desired optimum. The main point of this paper is to give such a proof.

3. Review of previous results and notation. In [2] the problem was essentially presented as follows. Let us observe a Wiener-Levy process $X^*_{t^*}$ with drift μ^*. That is

$$X^*_{t^*} = \mu^* t^* + Z^*_{t^*} \tag{3.1}$$

where $Z^*_{t^*}$ is a continuous process of independent Gaussian increments with mean 0 and variance determined by $E(Z^{*2}_{t^*}) = \sigma^{*2} t^{*2}$. The cost of an incorrect decision is given by $r^*(\mu^*) = k^*|\mu^*|$ and the cost of sampling is given by c^* per unit of time. The Bayes risk for an arbitrary procedure is given by $\mathcal{B}^* = E\{c^* T^* + \epsilon^*(\mu^*) k^* |\mu^*|\}$ where T^* is the observation time and $\epsilon^*(\mu^*)$ is the probability of error corresponding to the mean drift μ^*.

It was shown that the transformation

$$\begin{aligned} X_t &= c^{*\frac{1}{3}} k^{*-\frac{1}{3}} \sigma^{*-\frac{4}{3}} X^*_{t^*} \\ \mu &= c^{*-\frac{1}{3}} k^{*\frac{1}{3}} \sigma^{*-\frac{2}{3}} \mu^* \\ t &= c^{*\frac{2}{3}} k^{*-\frac{2}{3}} \sigma^{*-\frac{2}{3}} t^* \end{aligned} \tag{3.2}$$

yields the normalized problem where k^*, c^*, σ^* are replaced by k, c, and σ all equal to 1. That is to say we observe

$$X_t = \mu t + Z_t \tag{3.1$'$}$$

with $E(Z_t) = 0$, $E(Z_t^2) = t$, $r(\mu) = |\mu|$, a sampling cost of one per unit time, and a Bayes risk $\mathcal{B} = E\{T + \epsilon(\mu)|\mu|\} = c^{*-\frac{1}{3}} k^{*-\frac{2}{3}} \sigma^{*-\frac{2}{3}} \mathcal{B}^*$ where $T = c^{*\frac{2}{3}} k^{*-\frac{2}{3}} \sigma^{*-\frac{2}{3}} T^*$, $\epsilon(\mu) = \epsilon^*(\mu^*)$, and the parameter μ has a normal *a priori* probability distribution $\mathfrak{N}(\mu_0, \sigma_0^2)$ with

$$\begin{aligned} \mu_0 &= c^{*-\frac{1}{3}} k^{*\frac{1}{3}} \sigma^{*-\frac{2}{3}} \mu_0^* \\ \sigma_0 &= c^{*-\frac{1}{3}} k^{*\frac{1}{3}} \sigma^{*-\frac{2}{3}} \sigma_0^* \end{aligned} \tag{3.3}$$

where μ_0^* and σ_0^* are the mean and standard deviation of the normal *a priori* distribution of μ^*. It is convenient to make an additional simple transformation which is equivalent to starting the process from the point $x_0 = \mu_0/\sigma_0^2$ at time $t_0 = 1/\sigma_0^2$. With this convention we have the technical advantage that the *a posteriori* probability of μ given $X_t = x$ is $\mathfrak{N}(xt^{-1}, t^{-1})$ and the optimal solution for all initial *a priori* probability distributions can be represented by a single continuation set in (x, t) space. Then, if $X_t = x$, the optimal procedure calls for additional observation if and only if (x, t) is in the continuation region.

Given that $X_t = x$, the *a posteriori* risk due to accepting $H_1 : \mu \geqq 0$ is given by

$$(3.4)\quad D^+(x, t) = \int_{-\infty}^{0} r(\mu)(t/2\pi)^{\frac{1}{2}} \exp\left[-t(x/t - \mu)^2/2\right] d\mu = t^{-\frac{1}{2}}\psi^+(xt^{-\frac{1}{2}}).$$

Similarly the *a posteriori* risk due to accepting $H_2 : \mu < 0$ is given by

$$(3.4)'\quad D^-(x, t) = \int_{0}^{\infty} r(\mu)(t/2\pi)^{\frac{1}{2}} \exp\left[-t(x/t - \mu)^2/2\right] d\mu = t^{-\frac{1}{2}}\psi^-(xt^{-\frac{1}{2}}).$$

The *a posteriori* risk associated with taking the best decision is

$$(3.5)\quad D(x, t) = \min\left[D^+(x, t), D^-(x, t)\right] = t^{-\frac{1}{2}}\psi(xt^{-\frac{1}{2}})$$

where

$$(3.6)\quad \psi^+(\alpha) = \varphi(\alpha) - \alpha[1 - \Phi(\alpha)], \qquad \psi^-(\alpha) = \varphi(\alpha) + \alpha\Phi(\alpha),$$

$$(3.7)\quad \varphi(\alpha) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}\alpha^2}, \qquad \Phi(\alpha) = \int_{-\infty}^{\alpha} \varphi(\alpha')\, d\alpha',$$

and

$$(3.8)\quad \begin{aligned} \psi(\alpha) &= \psi^+(\alpha) && \text{for } \alpha > 0, \\ &= \psi^-(\alpha) && \text{for } \alpha \leqq 0. \end{aligned}$$

Now let us consider an arbitrary procedure represented by a continuation set in the (x, t) space with the understanding that when sampling is terminated the optimal terminal action is taken. That is, we accept $H_1 : \mu \geqq 0$ if and only if $X > 0$ at the time of termination. Let the *a posteriori* risk $B(x, t)$ represent the *expected additional* cost given $X_t = x$. This includes the risk due to error and the expected cost for additional sampling but does not include the cost of sampling up to time t. Then $B(x, t)$ satisfies the diffusion equation

$$(3.9)\quad 1 + B_t + (x/t)B_x + \tfrac{1}{2}B_{xx} = 0$$

in the continuation set. B satisfies the boundary condition

$$(3.10)\quad B = D$$

except possibly on sections of the boundary where x changes while t is constant. The optimal procedure simultaneously minimizes $B(x, t)$ for all (x, t) and is characterized by the property that $B(x, t) < D(x, t)$ inside the continuation region. This condition of optimality can be converted to the additional boundary condition

$$(3.11)\quad B_x = D_x$$

which serves to determine the *free* boundary. In other words the optimal procedure, and the corresponding risk $B(x, t)$ represent a solution of the free boundary problem of finding a continuation set and a function B which satisfies the partial differential equation (3.9) on the continuation set subject to the boundary conditions (3.10) and (3.11). At the same time the equations $B_t = D_t$ and the diffusion equation are satisfied on the boundary. This means that

$$1 + \tfrac{1}{2}B_{xx} = \tfrac{1}{2}D_{xx} \tag{3.12}$$

on the boundary.

Some additional facts of interest are the following. First if the optimal boundary is given by $\tilde{x}(t)$, $\tilde{a} = \tilde{x}t^{-\frac{1}{2}}$ is monotonic decreasing in t. Second, the optimal procedure does not truncate. That is to say that the continuation region has points with arbitrarily large t. Third, if the cost of sampling were given by some rate $c(x, t)$ when $X_t = x$, the diffusion equation would change to

$$c(x, t) + B_t + (x/t)B_x + \tfrac{1}{2}B_{xx} = 0. \tag{3.9$'$}$$

Fourth, if the regret due to taking the incorrect action were some other function of μ, the termination risk D would change accordingly but the boundary conditions would remain intact and D would still satisfy the "costless" diffusion equation

$$D_t + (x/t)D_x + \tfrac{1}{2}D_{xx} = 0 \tag{3.13}$$

except where $D^+ = D^-$. Fifth, if a termination risk $D(x, t)$ were prescribed by a different rule, the optimal procedure for this $D(x, t)$ would be represented by a solution of the free boundary problem given by equations (3.9), (3.10), and (3.11).

Finally, the converse inference in [2] that any solution of the free boundary problem must correspond to a solution of the optimization problem was never precisely proved. In fact it is not even true for general termination risk $D(x, t)$. On the other hand this inference will be used in Section 4. The appendix contains a lemma which is used to prove that the inference is correct where it is used in Section 4.

4. Asymptotic expansions for large t. In this section we shall indicate a formal derivation of an expansion for the optimal boundary $\tilde{x}(t)$ and the corresponding risk $B(x, t)$ as $t \to \infty$ and we shall prove that these expansions are asymptotic expansions in the sense that the error is small compared to the last term.

THEOREM 1. *Asymptotic expansions for $\tilde{x}(t)$ and $B(x, t)$ as $t \to \infty$ are given by*

$$\tilde{x}(t) \approx \frac{1}{4t}\left\{1 - \frac{1}{3}(4t^3)^{-1} + \frac{7}{15}(4t^3)^{-2} - \frac{23}{21}(4t^3)^{-3} + \frac{19{,}591}{5{,}670}(4t^3)^{-4} - \frac{2{,}085{,}862}{155{,}925}(4t^3)^{-5} + \cdots\right\} \tag{4.1}$$

and

$$\begin{aligned} B(x, t)/t \approx{} & -1 + V_0(\alpha) + (2\pi)^{-\frac{1}{2}}t^{-\frac{3}{2}} V_1(\alpha) - (16t^3)^{-1} V_2(\alpha) \\ & + \frac{5}{3}(16t^3)^{-2} V_4(\alpha) - \frac{437}{45}(16t^3)^{-3} V_6(\alpha) + \frac{28{,}991}{315}(16t^3)^{-4} V_8(\alpha) \\ & - \frac{16{,}476{,}572}{14{,}175}(16t^3)^{-5} V_{10}(\alpha) + \frac{47{,}556{,}611}{2{,}835}(16t^3)^{-6} V_{12}(\alpha) - \cdots \end{aligned} \tag{4.2}$$

where

(4.3) $$\alpha = xt^{-\frac{1}{2}}, \qquad V_r(\alpha) = F[1 - \tfrac{3}{2}r, \tfrac{1}{2}; -\tfrac{1}{2}\alpha^2],$$

and $F(\beta, \gamma; u) = 1 + (\beta/1\cdot\gamma)u + [\beta(\beta + 1)/1\cdot 2\cdot\gamma(\gamma + 1)]u^2 + \cdots$ is the confluent hypergeometric function.

For convenience we transform coordinates to $\rho = t^{-\frac{3}{2}}$, $\alpha = xt^{-\frac{1}{2}}$ and let $v(\alpha, \rho) = t^{-1}B(x, t)$. Then

(4.4) $$v_{\alpha\alpha} + \alpha v_\alpha + 2v + 2 = 3\rho v_\rho$$

subject to the boundary conditions

(4.5) $$\begin{aligned} v(\alpha, \rho) = \rho\psi(\alpha), \qquad v_\alpha(\alpha, \rho) = \rho\psi'(\alpha) &= -\rho[1 - \Phi(\alpha)] \quad \text{for } \alpha > 0 \\ &= \rho\Phi(\alpha) \qquad\qquad \text{for } \alpha < 0 \end{aligned}$$

Because of the symmetry we shall confine our attention to the *upper* boundary. Incidentally, the additional boundary condition $1 + \frac{1}{2}B_{xx} = \frac{1}{2}D_{xx}$ converts to

(4.6) $$v_{\alpha\alpha}(\alpha, \rho) = \rho\varphi(\alpha) - 2.$$

Note that as $t \to \infty$, $\rho \to 0$ and it is expected that $\alpha \to 0$, and it is therefore reasonable to expect a power series solution in α and ρ. In fact the partial differential equation is satisfied by any finite number of terms of the series

(4.7) $$\begin{aligned} v(\alpha, \rho) = -1 &+ c_0V_0(\alpha) + c_1\rho V_1(\alpha) \\ &+ c_2\rho^2V_2(\alpha) + \cdots + c_r\rho^rV_r(\alpha) + \cdots \end{aligned}$$

where $V_r(\alpha) = F[1 - \frac{3}{2}r, \frac{1}{2}; -\frac{1}{2}\alpha^2]$ is an even solution of the differential equation $V''(\alpha) + \alpha V'(\alpha) = (3r - 2)V(\alpha)$. Note that for positive even r, $F(1 - 3r/2, \frac{1}{2}; -\alpha^2/2)$ is a polynomial of degree $3r/2 - 1$ in α^2 with positive coefficients. For $r \geqq \frac{2}{3}$, the coefficient of α^2 is positive. Finally for $\alpha > 0$

$$\psi(\alpha) = -\alpha/2 + (2\pi)^{-\frac{1}{2}}F(-\tfrac{1}{2}, \tfrac{1}{2}; -\alpha^2/2).$$

The extra boundary condition (4.6) suggests $c_0 = 1$ and $c_1 = \varphi_0 = (2\pi)^{-\frac{1}{2}}$. Substituting these, our regular boundary conditions convert to

(4.8) $$\begin{aligned} -1 + V_0(\alpha) + \rho\alpha/2 + c_2\rho^2V_2(\alpha) + c_3\rho^3V_3(\alpha) + \cdots &= 0 \\ V_0'(\alpha) + \rho/2 + c_2\rho^2V_2'(\alpha) + c_3\rho^3V_3'(\alpha) + \cdots &= 0. \end{aligned}$$

If $\rho = b_1\alpha + b_2\alpha^2 + \cdots + b_r\alpha^r$ and $c_2, \cdots, c_r$ are such that these two equations are satisfied within terms of the order of magnitude of α^{r+1}, the top boundary Equation (4.8) can be used to obtain c_{r+1} and the bottom equation can be then used to obtain b_{r+1} so that these equations are satisfied within $O(\alpha^{r+2})$. In this way we derive the formal expansions for the upper boundary

(4.9) $$\rho \approx 4\alpha + \frac{4^2}{3}\alpha^3 - \frac{2}{15}4^3\alpha^5 + \frac{31}{105}4^4\alpha^7 - \frac{1679}{1890}4^5\alpha^9 + \frac{59{,}071}{17{,}325}4^6\alpha^{11} - \cdots$$

and for the risk we have

$$
\begin{aligned}
v(\alpha, \rho) \approx & -1 + V_0(\alpha) + (2\pi)^{-\frac{1}{2}} \rho V_1(\alpha) - \left(\frac{\rho}{4}\right)^2 V_2(\alpha) \\
& + \frac{5}{3}\left(\frac{\rho}{4}\right)^4 V_4(\alpha) - \frac{437}{45}\left(\frac{\rho}{4}\right)^6 V_6(\alpha) + \frac{28{,}991}{315}\left(\frac{\rho}{4}\right)^8 V_8(\alpha) \\
& - \frac{16{,}476{,}572}{14{,}175}\left(\frac{\rho}{4}\right)^{10} V_{10}(\alpha) + \frac{47{,}556{,}611}{2{,}835}\left(\frac{\rho}{4}\right)^{12} V_{12}(\alpha) - \cdots .
\end{aligned}
\tag{4.10}
$$

It remains to prove that these expansions do yield asymptotic approximations to the optimal boundary and associated v. Let

$$
v_r(\alpha, \rho) = -1 + c_0 V_0(\alpha) + \cdots + c_r \rho^r V_r(\alpha). \tag{4.11}
$$

We select a boundary $\rho_r(\alpha)$ so that the bottom boundary Equation (4.8) is satisfied exactly. Then

$$
\rho_r(\alpha) = b_1\alpha + b_2\alpha^2 + \cdots + b_r\alpha^r + O(\alpha^{r+1}) \tag{4.12}
$$

where the $O(\alpha^{r+1})$ term coincides for $\alpha > 0$ with a function which is analytic in some neighborhood of $\alpha = 0$, and hence $\rho_r(\alpha)$ is monotone increasing for positive α sufficiently small. The top boundary Equation (4.8) is not satisfied exactly. Let us change $D(x, t)$ to $D_r(x, t) = D(x, t) + t\theta_r(\rho)$ where $\theta_r(\rho) = O(\alpha^{r+1})$ is the discrepancy in the top boundary equation presented as a function of ρ along the boundary $\rho_r(\alpha)$. Then v_r and ρ_r represent the solution of the modified free boundary problem where the stopping risk is D_r and t is sufficiently large. In the appendix we prove that v_r and the region inside $\pm\rho_r$ correspond to the optimal procedure for the stopping risk D_r.

Now let us consider two alternative functions

$$
\begin{aligned}
v_r^+ &= v_r + K\rho^r V_r(\alpha) && \text{if } r \text{ is even} \\
&= v_r + K\rho^{r+1} V_{r+1}(\alpha) && \text{if } r \text{ is odd}
\end{aligned}
\tag{4.13}
$$

and

$$
\begin{aligned}
v_r^- &= v_r - K\rho^r V_r(\alpha) && \text{if } r \text{ is even} \\
&= v_r - K\rho^{r+1} V_{r+1}(\alpha) && \text{if } r \text{ is odd.}
\end{aligned}
\tag{4.14}
$$

As in the above discussion we find associated boundaries ρ_r^+ and ρ_r^- and discrepancies $\theta_r^+(\rho)$ and $\theta_r^-(\rho)$ where (taking K sufficiently large if r is odd) $\theta_r^+(\rho)$ coincides with a function which is analytic in a neighborhood of $\rho = 0$, $\rho^{-\frac{2}{3}}\theta_r^+(\rho) = O(\rho^{r-\frac{2}{3}})$ is positive and monotone increasing for sufficiently small positive ρ. Similarly $\rho^{-\frac{2}{3}}\theta_r^-(\rho) = O(\rho^{r-\frac{2}{3}})$ is negative and monotone decreasing for sufficiently small positive ρ. From the appendix it follows that ρ_r^+ and $v_r^{++} = v_r^+ - \theta_r^+(\rho_0)\rho_0^{-\frac{2}{3}}\rho^{\frac{2}{3}}$ represents an exact solution for the minimizing problem where D is replaced by $D_r^{++} = D + \theta_r^+(\rho)\rho^{-\frac{2}{3}} - \theta_r^+(\rho_0)\rho_0^{-\frac{2}{3}} \leqq D$ for $\rho \leqq \rho_0$. It then follows that $v_r^{++}(\alpha, \rho) \leqq v(\alpha, \rho)$ in the continuation region for this modified problem when $\rho \leqq \rho_0$ sufficiently small. Let α_0^+ be that value of α for which $\rho_r^+(\alpha_0^+) = \rho_0$. Then

$$(4.15) \quad \rho_0^{\frac{3}{2}} D(\alpha_0^+, \rho_0) = \rho_0^{\frac{3}{2}} D_r^{++}(\alpha_0^+, \rho_0) = v_r^{++}(\alpha_0^+, \rho_0) \leqq v(\alpha_0^+, \rho_0)$$

and it follows that (α_0^+, ρ_0) is not inside the continuation set for the original problem; i.e., $\tilde{\alpha}(\rho_0) \leqq \alpha_0^+$ where $\tilde{\alpha}$ corresponds to the optimal boundary for the original problem. Similarly, but somewhat more delicately, we define $v_r^{--} = v_r^- - \theta_r^-(\rho_0)\rho_0^{-\frac{3}{2}}\rho^{\frac{3}{2}}$ and $D_r^{--} = D + \theta_r^-(\rho)\rho^{-\frac{3}{2}} - \theta_r^-(\rho_0)\rho_0^{-\frac{3}{2}} \geqq D$ for $\rho \leqq \rho_0$ (with strict inequality for $\rho < \rho_0$) and $v_r^{--}(\alpha, \rho) > v(\alpha, \rho)$ in the continuation region for the new modified region when $\rho \leqq \rho_0$ sufficiently small. Letting α_0^- be that value of α for which $\rho_r^-(\alpha_0^-) = \rho_0$, we derive for $0 \leqq \alpha < \alpha_0^-$

$$(4.15)' \qquad \rho_0^{\frac{3}{2}} D(\alpha, \rho_0) = \rho_0^{\frac{3}{2}} D_r^{--}(\alpha, \rho_0) > v_r^{--}(\alpha, \rho_0) > v(\alpha, \rho_0)$$

and hence (α_0^-, ρ_0) is in or on the boundary of the continuation region for the original problem, i.e., $\tilde{\alpha}(\rho_0) \geqq \alpha_0^-$.

It follows that the optimal boundary for the original problem is between those represented by ρ_r^+ and ρ_r^- each of which differs from $b_1\alpha + b_2\alpha^2 + \cdots + b_r\alpha^r$ by $O(\alpha^r)$. Thus it is clear that v is approximated by v_r in the continuation set. Transforming and inverting the expansion for ρ we obtain the results of Theorem 1.

Certain remarks are in order. First, we have made use of the symmetry to concentrate our attention on the upper boundary. If D were not symmetric, as could be the case for a non-symmetric regret function $r(\mu)$, it would be necessary to consider also odd solutions of the differential equation. That is to say that we would also have to consider terms of the form $\rho^r \alpha F[\frac{1}{2}(3 - 3r), \frac{3}{2}; -\frac{1}{2}\alpha^2]$.

Second, the proof seems to rely on the analytic nature of the stopping risk. It seems evident that except for certain monotonicity requirements, there is no fundamental need for analyticity. There is a somewhat less involved proof using v_r and the monotonicity of $\tilde{\alpha}$ which may be more difficult to generalize.

5. Comparison of the optimal procedure with optimal continuation regions of the form $|X| \geqq a$. Suppose that μ has the normal *a priori* probability distribution $\mathfrak{N}(0, t^{-1})$. This is equivalent to assuming that $X_t = 0$. For comparison purposes let us consider the Wald type procedure which consists of continuing as long as $|X_{t'}| < a$, for $t \geqq t'$. For each a there is a corresponding Bayes risk $R(0, t)$. Let $a_0(t)$ be that value of a which minimizes the Bayes risk, yielding $R_0(0, t)$. We shall prove

THEOREM 2. *For t sufficiently large,*

$$(5.1) \qquad a_0 = \frac{1}{4t}\left\{1 - \frac{7}{3.4^2}t^{-3} + \frac{272}{15.4^4}t^{-6} - \cdots\right\}$$

and

$$(5.2) \qquad R_0(0, t) = \frac{1}{(2\pi t)^{\frac{1}{2}}} - \frac{1}{16t^2}\left\{1 - \frac{5}{3.4^2}t^{-3} + \frac{407}{(45).4^4}t^{-6} + \cdots\right\}.$$

Implicit in the results of Wald [6] and of Dvoretzky, Kiefer, and Wolfowitz [4] are the expression $\epsilon(\mu) = (1 + e^{2a\mu})^{-1}$ for the error probability and $T(\mu) = a \tanh (a\mu)/|\mu|$ for the expected observation time. Thus the Bayes risk corresponding to the $\mathfrak{N}(0, t^{-1})$ *a priori* distribution and the continuation region $|X| \leqq a$, is

$$(5.3) \qquad R(0,t) = 2 \int_0^\infty (t/2\pi)^{\frac{1}{2}} \exp(-t\mu^2/2) \{\mu(1 + e^{2a\mu})^{-1} + a\mu^{-1} \tanh a\mu\} \, d\mu.$$

Differentiating with respect to a, we have

$$(5.4) \qquad \int_0^\infty (\operatorname{sech}^2 v + v^{-1} \tanh v) \exp(-tv^2/2a_0^2) \, dv = (2a_0^3)^{-1} \int_0^\infty v^2 \operatorname{sech}^2 v \exp(-tv^2/2a_0^2) \, dv.$$

Expressing a_0 and $R_0(0, t)$ in terms of $\alpha_0 = a_0 t^{-\frac{1}{2}}$, we have $a_0 = \eta(\alpha_0)$, $t^{-\frac{1}{2}} = \alpha_0/\eta(\alpha_0)$, $R_0(0,t) = \xi(\alpha_0)$ where

$$(5.5) \qquad \eta^3(\alpha_0) = 2 \int_0^\infty v^2 \operatorname{sech}^2 v \exp(-v^2/2\alpha_0^2) \, dv \Big/ 2 \int_0^\infty [\operatorname{sech}^2 v + v^{-1} \tanh v] \exp(-v^2/2\alpha_0^2) \, dv$$

and

$$(5.6) \qquad \begin{aligned} \xi(\alpha_0) &= 2 \int_0^\infty (2\pi)^{-\frac{1}{2}} \exp(-v^2/2\alpha_0^2) \left[\frac{v(1 + e^{2v})^{-1}}{\alpha_0 \, \eta(\alpha_0)} + \frac{\eta^2(\alpha_0)}{\alpha_0} \frac{\tanh v}{v} \right] dv \\ &= (2/\pi)^{\frac{1}{2}} \int_0^\infty \exp(-v^2/2\alpha_0^2) \cdot \left\{ \frac{v}{2\alpha_0 \, \eta(\alpha_0)} [1 - \tanh v] + \frac{\eta^2(\alpha_0) \tanh v}{\alpha_0 v} \right\} dv. \end{aligned}$$

Finally, we may take v/α_0 as variable of integration and expand η^3 and ξ in ascending powers of α_0. Thus

$$(5.7) \qquad \eta^3(\alpha_0) = (\alpha_0^2/4)[1 - (7/3)\alpha_0^2 + (326/45)\alpha_0^4 + \cdots]$$

and hence $\frac{1}{4}t^{-\frac{3}{2}} = \alpha_0^3/4\eta^3(\alpha_0) = \alpha_0[1 - (7/3)\alpha_0^2 + (326/45)\alpha_0^4 - \cdots]^{-1}$ so that

$$(5.8) \qquad \alpha_0 = \tfrac{1}{4}t^{-\frac{3}{2}}\{1 - (7/3 \cdot 4^2)t^{-3} + [816/(45)4^4]t^{-6} - \cdots\}.$$

Theorem 2 follows immediately. Comparing the expansions of Theorem 2 with $B(0, t)$ and $\tilde{x}(t)$ in Theorem 1 we see that $B(0, t)$ first differs from $R_0(0, t)$ in the coefficient of t^{-8} where $-407/4^6(45)$ appears in place of $-437/4^6(45)$. On the other hand a_0 differs from $\tilde{x}(t)$ in the coefficient of t^{-4} where $-7/192$ appears in place of $-1/48 = -4/192$. Thus for large t, $\tilde{x}(t)$ is larger than a and $B(0, t)$ is smaller than $R_0(0, t)$ which are to be expected.

6. Appendix. In this appendix we establish a lemma which may be applied to show that in certain circumstances, the solution of the free boundary problem represents the minimizing procedure and the minimizing risk. Before doing so we point out that this is not true for general stopping risks. Counterexamples can easily be constructed by reducing the stopping risk on a small part of the interior of the continuation set or on a small part of the exterior of the continuation set. The minimizing procedure for the original stopping risk remains a solution of the modified free boundary problem. However in each case the minimizing procedure for the modified problem changes. In the first case the part of the interior where the stopping risk was reduced may fail to remain in the optimal continuation set for the modified problem. In the second case, the continuation set may be enlarged. In both cases the Bayes risk associated with the optimal procedure is reduced.

The key to our arguments is the approximation of the continuous time problem by the discrete time problem and the backward induction used in the latter. There is some minor convenience toward applying the backward induction in using the coordinates (y, t) where $y = xt^{-1} = \alpha t^{-\frac{1}{2}}$. This convenience derives from the fact that $Y_t = X_t/t$ is a Wiener Process in the $-t^{-1}$ scale. More precisely, the conditional distribution of Y_{t_0} given $Y_{t_1} = y$, $(t_1 \leqq t_0)$ is normal with mean y and variance

$$\gamma^2_{t_1, t_0} = t_1^{-1} - t_0^{-1}. \tag{6.1}$$

We shall often find it convenient to express γ without subscripts when they are understood without ambiguity.

Denoting the Bayes risk for a procedure by $b(y, t) = B(x, t)$ and the stopping risk by $d(y, t) = D(x, t)$, the diffusion equation transforms to

$$1 + b_t + b_{yy}/2t^2 = 0. \tag{6.2}$$

The discrete problem is one where we are permitted to stop observation only at a set of discrete time points. If at (y, t) we decide to continue observation until time $t + \delta$ when the risk is given by $b(y, t + \delta)$ our risk will be

$$\begin{aligned} h_\delta(y, t) &= \delta + E\{b(Y_{t+\delta}, t + \delta) \mid Y_t = y\} \\ &= \delta + \int_{-\infty}^{\infty} b(y + \epsilon\gamma, t + \delta)\varphi(\epsilon)\, d\epsilon, \qquad \gamma = \gamma_{t, t+\delta}\, . \end{aligned} \tag{6.3}$$

This function h_δ plays a key role in the backward induction for the discrete problem. Consider the truncated discrete problem where stopping is permitted only at times t_0, $t_0 - \delta$, $t_0 - 2\delta$, $\cdots$, $t_0 - (n - 1)\delta$, t where $t \geqq t_0 - n\delta$ and the risk at time t_0 is $b(y, t_0)$. Denote the optimal risk by $b_\delta(y, t; t_0)$.

The relation between the Wiener process and Equation (6.2) and the related heat equation require some additional terminology and notation.

Definition 1. A measurable function $d(y, t)$ is *regular* if for each $t > 0$, there is a continuous function $K(t) > 0$ such that

$$|d(y, t)| \leqq K(t) \exp [K^2(t) y^2]. \tag{6.4}$$

If d is regular and $2K\gamma < 1$,

$$\int_{-\infty}^{\infty} d(y + \epsilon\gamma, t)\varphi(\epsilon)\, d\epsilon \leqq 2^{\frac{1}{2}} K(t) \exp [2K^2(t)y^2]. \tag{6.5}$$

Let R be a subset of the half plane $t > 0$, $\mathcal{B}(R)$ the boundary of R and $\mathcal{I}(R)$ the interior of R. Let $R_{t_0} = R \cap \{(y, t): t \leqq t_0\}$. Let f be a solution of the diffusion equation

$$f_t + f_{yy}/2t^2 = 0 \tag{6.6}$$

on $\mathcal{I}(R_{t_0})$ subject to the boundary condition $f = d$ on $\mathcal{B}(R_{t_0})$ for the regular function d. With the appropriate interpretation of the boundary condition (see [3]), when R_{t_0} is a bounded set f is uniquely determined on R_{t_0} by

$$f(y, t) = E\{d(Y^*, T^*) \mid Y_t = y\} \tag{6.7}$$

where (Y^*, T^*) is the first point after time t where the Wiener process through (y, t) intersects $\mathcal{B}(R_{t_0})$.

Let $d(y, t)$ be a regular function and R a set such that $R_{t_0} - R_t$ is bounded when $0 < t < t_0$. Let $b_0(y, t)$ be the risk associated with the minimization problem for which d is the stopping risk. Let R_0 be the corresponding continuation set. Finally let $b(y, t)$ be a regular function equal to d on the complement of R and satisfying Equation (6.2) on R subject to the boundary condition $b = d$. Then we have

LEMMA 1. *If*

(i) $\sup_y |b(y, t) - b_0(y, t)| \to 0$ *as* $t \to \infty$

(ii) $b(y, t) < d(y, t)$ *on* R *and*

(iii) *for some* t_1 *and each* $t \geqq t_1$, *there is a* $\delta_0(t) > 0$ *and* $\delta_1(t) > 0$ *such that* $h_\delta(y, t') \geqq d(y, t')$ *whenever* $(y, t') \not\in R$, $\delta \leqq \delta_0(t)$ *and* $|t' - t| \leqq \delta_1(t)$. *Then* $b(y, t)$ *and* R *represent a solution of the minimization problem for* $t \geqq t_1$ *and* $b(y, t) = b_0(y, t)$ for $t \geqq t_1$.

PROOF. Condition (i) implies that $b_0(y, t) = \lim_{t_0\to\infty} \lim_{\delta\to 0} b_\delta(y, t; t_0)$. Applying the Heine Borel Theorem to $[t_1, t_0]$ we have the existence of $\delta(t_0) > 0$ such that $h_\delta(y, t) \geqq d(y, t)$ whenever $(y, t) \not\in R$, $\delta \leqq \delta(t_0)$ and $t_1 \leqq t \leqq t_0$. We take $\delta <$ min $[\delta(t_0), t_1^2/4K^2(t)]$ where K corresponds to the regular function d which incidentally dominates b. Then we may represent $h_{t_0-t}(y, t)$ for $0 < t < t_0$ by

$$h_{t_0-t}(y, t) = t_0 - t + E\{b(Y_{t_0}, t_0) \mid Y_t = y\} \tag{6.8a}$$

or

$$h_{t_0-t}(y, t) = t_0 - t + \int_{-\infty}^{\infty} b(z, t_0)\, \varphi\left(\frac{z-y}{\gamma}\right) \frac{dz}{\gamma}, \gamma^2 = t^{-1} - t_0^{-1} \tag{6.8b}$$

from which it follows that $h_{t_0-t}(y, t)$ satisfies (6.2) for $\{(y, t): t_0 - \delta \leqq t < t_0\}$ and may be regarded as the restriction of a regular function on that set. Hence $h_{t_0-t}(y, t) - b(y, t)$ satisfies (6.6) for $R \cap \{(y, t): t_0 - \delta \leqq t < t_0\}$ and, being non-negative on $\mathcal{B}(R_{t_0})$, is non-negative for $(y, t_0 - \delta) \in R$. This together with the Condition (ii) yields

$$(6.9)\quad b(y, t_0 - \delta) \leqq \min [h_\delta(y, t_0 - \delta), d(y, t_0 - \delta)] = b_\delta(y, t_0 - \delta; t_0).$$

This in turn implies that $b_\delta(y, t; t_0 - \delta) \leqq b_\delta(y, t; t_0)$ for $t \leqq t_0 - \delta$ and the argument leading to (6.9) gives

$$(6.10)\quad b(y, t_0 - 2\delta) \leqq b_\delta(y, t_0 - 2\delta; t_0 - \delta) \leqq b_\delta(y, t_0 - 2\delta; t_0).$$

Proceeding in this way through the time points t_0, $t_0 - \delta, \cdots, t_0 - (n-1)\delta, t_1$, where $t_1 \geqq t_0 - n\delta$, we conclude $b(y, t) \leqq b_\delta(y, t; t_0)$ and hence $b(y, t) \leqq b_0(y, t)$. But b is the risk associated with the procedure R and hence Lemma 1 follows.

Now let us apply Lemma 1 to the three cases mentioned in Section 4. These were the modified problems corresponding to D_r, D_r^{++} and D_r^{--}. In each case, the boundary conditions associated with solving the free boundary problem, knowledge of the second derivative of risk with respect to y, and a property of the unmodified stopping risk is used to establish Conditions (ii) and (iii) of the Lemma. We shall treat the three modified problems simultaneously, using an asterisk to denote the appropriate modified expressions such as b^*, d^*, v^*, etc.

At first we recall that for the unmodified problem the stopping risk $d(y, t) = t^{-\frac{1}{2}}\psi(yt^{\frac{1}{2}}) = \min (d^+, d^-)$ where d^+ corresponds to the risk of stopping and accepting $\mu > 0$, and satisfies

$$(6.11)\quad d^+(y, t - \delta) = \int_{-\infty}^{\infty} d^+(y + \epsilon\gamma, t)\varphi(\epsilon)\, d\epsilon, \qquad \gamma = \gamma_{t-\delta,t}.$$

This equation holds for general regret function since the right hand side may be interpreted as the risk associated with the following situation: After observing $Y_{t-\delta} = y$, we are allowed to observe Y_t free of charge but we have decided to accept $\mu > 0$ irrespective of the value of Y_t.

Second, for each of the modified problems we have

$$(6.12)\quad d^* - d = t\theta^*(\rho) = h^*(t) = O(t^{-\frac{3}{2}r+1})$$

and the arguments which yielded monotonicity and analyticity of $\rho_r(\alpha)$ also yield $dh^*/dt = O(t^{-\frac{3}{2}r})$.

Third, for t sufficiently large, it is easy to compute that

$$(6.13)\quad b^*_{yy} - d^*_{yy} = [2 + o(1)]t^2 \quad \text{uniformly for} \quad |y| \leqq \tilde{y}^*$$

where $\tilde{y}^*$ is the y value associated with the free boundary solution ρ^*, v^*, of the modified problem.

Equation (6.13) together with the two boundary condition imply that $b^* < d^*$ for $|y| < \tilde{y}^*$ which gives Condition (ii), of Lemma 1. These also imply that for arbitrary $\eta > 0$ and t sufficiently large

$$(6.14a)\quad |b^* - d^{+*}| \leqq (1 + \eta)t^2(y - \tilde{y}^*)^2 \qquad \text{for } 0 < y \leqq \tilde{y}^*$$

where $d^{+*} = d^+ + h^*(t)$. Since $d^* = d^{+*}$ for $y > 0$, (6.14b) $$b^* - d^{+*} = 0 \qquad \text{for } y \geqq \tilde{y}^*.$$

Finally comparing d^+ with d^-, we have for some constant K,

(6.14c) $$|b^* - d^{+*}| \leqq K|y| \qquad \text{for } y < 0.$$

We are now ready to establish Condition (iii). For $y > \tilde{y}^*$ and $\gamma = \gamma_{t-\delta,t}$

$$h_\delta^*(y, t-\delta) = \delta + \int_{-\infty}^{\infty} b^*(y + \epsilon\gamma, t)\varphi(\epsilon)\, d\epsilon$$

$$= \delta + \int_{-\infty}^{\infty} [b^*(y+\epsilon\gamma, t) - d^{+*}(y+\epsilon\gamma, t)]\varphi(\epsilon) + d^+(y, t-\delta) + h^*(t)$$

$$h_\delta^*(y, t-\delta) - d^*(y, t-\delta) = \delta + \int_{-\infty}^{\infty} [b^*(y + \epsilon\gamma, t) - d^{+*}(y+\epsilon\gamma, t)]\varphi(\epsilon)$$

$$+ h^*(t) - h^*(t-\delta)$$

(6.15) $$h_\delta^*(y, t-\delta) - d^*(y, t-\delta) \geqq \delta - \tfrac{1}{2}(1+\eta)\gamma^2 t^2 - Ko(\gamma^2) - \delta O(t^{-\frac{1}{2}r}).$$

Substituting $\gamma^2 = (t-\delta)^{-1} - t^{-1}$, Condition (iii) follows.

Condition (i) is trivial. Thus in each of the three modified problems, the free boundary problem solutions yield the minimizing procedure. It should be noted that that result has not been established for the original (unmodified) problem.

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