ON THE LINE GRAPH OF THE COMPLETE BIPARTITE GRAPH[1]

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1. Introduction and summary. In an interesting recent article [4], J. W. Moon has given a list of properties of the graph $L(B_{mn})$ (to be defined more precisely below) and investigated the question of whether these properties characterize the graph. In case $m = n$, this question had been settled by Shrikhande [5] (see also [1], [2] and [3]), who proved that the answer was yes unless $m = n = 4$, when there is exactly one exception. In case $m > n$, Moon shows that the answer is yes unless $(m, n) = (5, 4)$ or $(4, 3)$, which cases were left unsettled in [4]. The purpose of this note is to show that, in those cases as well, the answer is yes, thus completing Moon's discussion.

Now to define our problem more exactly. The line graph of the complete bipartite graph on sets with m and n vertices, denoted by $L(B_{mn})$, is the graph with mn vertices given by all ordered pairs (i, j), $1 \leqq i \leqq m$, $1 \leqq j \leqq n$. Two vertices (i, j) and (i', j') are joined by an edge if $i = i'$ or $j = j'$, but not both. The graph $L(B_{mn})$ has the following properties:

(1.1) It has mn vertices.

(1.2) Each vertex has valence $m + n - 2$.

(1.3) If two vertices are not adjacent, there are exactly two vertices adjacent to each.

(1.4) Of the $\frac{1}{2} mn(m + n - 2)$ pairs of adjacent vertices, exactly $n\binom{m}{2}$ pairs are each adjacent to exactly $m - 2$ vertices, the remaining $m\binom{n}{2}$ pairs are each adjacent to exactly $n - 2$ vertices.

We now assume $m > n$, and let G_{mn} be any graph satisfying (1.1)–(1.4). Our object is to prove that when $(m, n) = (5, 4)$ or $(4, 3)$, $G_{mn} = L(B_{mn})$. Moon has established $G_{mn} = L(B_{mn})$ in all other cases.

2. Preliminaries. Let A be the adjacency matrix of G_{mn}; i.e., number the mn vertices of G_{mn} arbitrarily, and define $A = (a_{ij}) = 1$ if i and j are adjacent, 0 otherwise. Define $B = (b_{ij}) = 1$ if i and j are adjacent, and there are $m - 2$ vertices adjacent to i and j, 0 otherwise; $C = (c_{ij}) = 1$ if i and j are adjacent, and there are $n - 2$ vertices adjacent to i and j, 0 otherwise. We then have from (1.2)–(1.4)

$$A^2 = (m + n - 2)I + (m - 2)B + (n - 2)C + 2(J - I - A),$$

where J is the square matrix of order mn every entry of which is unity. Since $A = B + C$, this may be rewritten

$$(2.1) \qquad A^2 = (m + n - 4)I + (n - 4)A + (m - n)B + 2J.$$

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Because A commutes with the left side of (2.1), A commutes with the right side of (2.1). But A also commutes with J, since, by (1.2), $AJ = JA = (m + n - 2)J$. Therefore $A = B + C$ commutes with B, so B and C commute.

Because B and C commute, they have a common set of eigenvectors, which are also eigenvectors of A. Now $u = (1, 1, \cdots, 1)$ is an eigenvector of A corresponding to the eigenvalue $m + n - 2$. Since A is an irreducible nonnegative matrix with $m + n - 2$ the dominant eigenvalue, u is the only eigenvector (apart from multiples of u) corresponding to $m + n - 2$. But this means that u is also an eigenvector of B and of C. Thus every row sum of B is the same. By (1.4), the number of 1's in B is $nm(m - 1)$, and since B is of order mn, every row sum of B is $m - 1$. Similarly, every row sum of C is $n - 1$.

It is easy to show that if, for each i, $\{j \mid c_{ij} = 1\}$ is a clique of G_{mn}, then $G_{mn} = L(B_{mn})$. This we shall show in the next two sections for the cases $(m, n) = (4, 3)$ and $(5, 4)$.

3. The case $(m, n) = (4, 3)$**.** Both in this and the next section, we shall say that (i, j) is a B-edge (C-edge) if $b_{ij}(c_{ij})$ is 1. Let 0 be a vertex of G_{43}, let $(0, 1)$, $(0, 2)$ be the C-edges adjacent to 0, $(0, 1')$, $(0, 2')$, $(0, 3')$ the B-edges adjacent to 0. If 1 and 2 are not adjacent, then by (1.4), 1 is adjacent to one vertex of $\{1', 2', 3'\}$, and 2 is adjacent to one vertex of $\{1', 2', 3'\}$. Without loss of generality, 1 is adjacent to $1'$, and 2 is adjacent to $1'$ or $2'$. We also know from (1.4) that the total number of edges joining vertices in $S = \{1, 2, 1', 2', 3'\}$ to vertices in S is 4. Thus the total number of edges joining vertices in $\{1', 2', 3'\}$ to vertices in $\{1', 2', 3'\}$ is 2.

Now the edge $(1, 1')$ must be a B-edge. For if it were a C-edge, we would have $(0, 1')$ and $(1', 1)$ successive B- and C-edges. Since $BC = CB$, we would have to have $(2, 1)$ a B-edge. But we are assuming 1 and 2 are not adjacent. Similarly, $(2, 1')$ or $(2, 2')$ is a B-edge. The vertex $3'$ is not joined to 1 or 2, so by (1.4) it must be joined to $1'$ and $2'$. Hence, $1'$ and $2'$ are not adjacent, since we have already identified four edges which have both endpoints in S. Since $(0, 1)$ and $(1, 1')$ are successive C- and B-edges, it follows from $BC = CB$, that $(3', 1')$ is a C-edge. Then $(0, 1')$ and $(1', 3')$ are successive B- and C-edges, so from $BC = CB$, there must be successive C- and B-edges starting at 0 and ending at $3'$. Hence, there must be an edge joining 1 or 2 with $3'$. This is a contradiction, however, so our original assumption that 1 and 2 are not adjacent must be false. This completes the discussion of the (4, 3) case.

4. The case $(m, n) = (5, 4)$**.** Let 0 be a vertex of G_{54}, and let $T = \{1, 2, 3\}$ be the set of vertices joined to 0 by C-edges, and $V = \{1', 2', 3', 4'\}$ be the set of vertices joined to 0 by B-edges. We wish to show that T is a clique. From (1.4), we know that the number of edges with both ends in $T \cup V$ is 9.

(a) Suppose that no vertices of T are adjacent. Then by (1.4) there are six edges joining vertices in T with vertices in V. By the reasoning in Section 3, all these edges are B-edges. If a vertex of V were an end of more than one of these six edges, then from $BC = CB$, it would follow that the vertex was adjacent to at least two vertices of V, as well as two vertices of T. This would violate (1.4). So none of the four vertices of V can be an end of more than one of these six edges, an impossibility.

(b) Suppose T contains exactly one edge. Without loss of generality, assume it is (2, 3). Then there are two edges joining 1 with points of V, and an edge each from 2 and 3 to V, and no other edges joining T and V. Hence, there are four edges joining vertices in V to vertices in V. They cannot form a quadrilateral, say $(1', 2')$, $(2', 3')$, $(3', 4')$, $(4', 1')$, for then $1'$ and $3'$, which are not adjacent, would each be adjacent to 0, $2'$, and $4'$, violating (1.3). Hence, without loss of generality, the edges in V are $(1', 2')$, $(1', 4')$, $(2', 4')$ and $(3', 4')$. By (1.4), $4'$ is not adjacent to any vertex of T; in particular $4'$ and 1 are not adjacent. But 0 and the two other ends of the edges joining 1 to V are three vertices adjacent to 1 and $4'$, a violation of (1.3).

(c) Suppose T contains two edges. Then it is easy to see that V contains five edges, so (without loss of generality), we may infer that $1'$ and $2'$ are not adjacent, but are each adjacent to $3'$ and $4'$. Since they are also each adjacent to 0, this violates (1.3).

It follows that T is a clique.

5. Conclusion. We can therefore assert, on the basis of [4], [5] and the foregoing:

THEOREM. *A graph satisfying* (1.1)–(1.4) *is* $L(B_{mn})$, *unless* $m = n = 4$, *when there is exactly one other graph satisfying* (1.1)–(1.4), *described in* [5].

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