LIMIT DISTRIBUTIONS OF A BRANCHING STOCHASTIC PROCESS[1]

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1. Brief summary. A population of particles is considered whose size $X_N(t)$ changes according to a branching stochastic process. The purpose of this paper is to find an approximate distribution of $X_N(t)$ when t is fixed (not necessarily large) but the initial size of the population, N, is large. If N is allowed to tend to infinity, and if the parameters of the process are made to change in a way analogous to the Poisson approximation of a binomial distribution, then it is shown that a limiting distribution of the *process* $X_N(t) - N$ exists as $N \to \infty$, and this limiting distribution is the distribution of a continuous process with independent increments. The relation between the parameters of the infinitely divisible distribution of the limiting process and the sequence of branching processes is exhibited.

2. Introduction. A continuous time branching stochastic process (to be defined below) is considered whose size (i.e., the number of particles) at time t will be denoted by $X_N(t)$. We assume that $P[X_N(0) = N] = 1$, i.e., at time $t = 0$ there is a non-random number $N > 0$ of particles in the population. We assume that all particles act independently of each other. At any time a particle might "split" into k *new* particles, where $k = 0, 1, 2, \cdots$, i.e., it dies (or vanishes) as it produces k new particles. These new particles are assumed to be stochastically independent of the parent particle and all other particles that exist at one time or another in the population.

Let $\varphi_N(t)$ be a positive continuous function of t which might or might not depend on N. It is assumed that the conditional probability that any particular particle splits into k particles during the time interval $[t, t + h]$, given that it is in existence at time t and came into existence at time $t' \leqq t$ $(t' \geqq 0)$ is equal to

$$\lambda_k \varphi_N(t)h + o(h), \tag{1}$$

where λ_k, $\varphi_N(t)$ and $o(h)$ do not depend on t', where $\lambda_k \geqq 0$ is constant and not zero for all k, and where $o(h)$ is assumed to be uniform with respect to t. The sequence $\{\lambda_k\}$ is assumed to satisfy

$$\sum_{k=0}^{\infty} \lambda_k < \infty. \tag{2}$$

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We denote

$$(3) \qquad \lambda = \sum_{k=0}^{\infty} \lambda_k ,$$

and we assume that the conditional probability that a particular particle does not "split" during $[t, t + h]$ given that it exists at time t and came into existence at time $t' \leqq t$ $(t' \geqq 0)$, is equal to

$$(4) \qquad 1 - \lambda\varphi_N(t)h + o(h),$$

where again the limit indicated by $o(h)$ is assumed to be uniform in t and does not depend on t'. (This assumption cannot be derived from (1) and therefore must be included.)

We assume each λ_k and λ to be functions of N which satisfy

$$(5) \qquad \lim_{N\to\infty} N\lambda_k = \nu_k \geqq 0, \quad \text{and}$$

$$(6) \qquad 0 < \lim_{N\to\infty} N\lambda = \nu = \sum_{k=0}^{\infty} \nu_k < \infty.$$

The final assumption is that

$$(7) \qquad \lim_{N\to\infty} \varphi_N(t) = \varphi(t) > 0$$

uniformly over every bounded interval.

The above assumptions will be shown to yield the limiting distribution of the *stochastic process* $X_N(t) - N$ (as $N \to \infty$) which is shown to be that of a stochastic process $X(t)$ with independent increments whose characteristic function at each t is given by

$$(8) \qquad \psi(u) = \exp \Phi(t) \sum_{k=0}^{\infty} (e^{iu(k-1)} - 1)\nu_k ,$$

where

$$(9) \qquad \Phi(t) = \int_0^t \varphi(\tau)\, d\tau.$$

It should be first pointed out that the branching process here differs from the usual branching process in that we do not consider spontaneously generated particles. The exact distribution of $X_N(t)$ is extremely difficult or impossible to obtain in any reasonable form. The most general result that we know about the exact distribution of a branching process was obtained by D. G. Kendall [5]. He obtained the probabilities $\{P[X_1(t) = n], n = 0, 1, \cdots\}$ in the case where $\lambda_k = 0$ for $k \not\in \{0, 2\}$ and where $\lambda_0\varphi_1(t)$ and $\lambda_2\varphi_1(t)$ were arbitrary positive continuous functions of t, $\mu(t)$ and $\nu(t)$ respectively. In his paper, Kendall used the method of replacing the differential-difference equations for the distribution of the population size by a partial differential equation for its generating function. This method breaks down in the branching process considered in this paper.

The limit theorems usually considered for branching processes are those in which the limiting distribution or some limiting probability or a limit in quadratic mean of $X_N(t)$ is obtained for N fixed but as $t \to \infty$, where $X_N(t)$ is perhaps suitably normed. (See [1]–[5] and [7]–[11].) Here t is fixed, N is allowed to tend to infinity, and the limiting distribution obtained is *not only the distribution of a random variable but the distribution of a process.*

3. Some lemmas. In this section we prove five lemmas, Lemmas 1–5, which are needed to prove both Theorem 1 and Theorem 2. Lemma 2 might be of a small amount of independent interest and is used only to prove Lemma 3, which is used to prove Theorem 1. Lemma 4 is a combination of particular cases of a Helly theorem and the Lebesgue dominated convergence theorem; it is used in the proofs of Lemma 5 and Theorem 2. Lemma 5 is used in the proof of Theorem 1.

The following lemma is used a number of times in this paper without being mentioned explicitly. In particular, it is used in the proof of Lemma 3.

LEMMA 1. *The conditional probability that a particle splits into k particles during $[t, t + h]$, given that it exists at t and at $t' < t$, is $\lambda_k \varphi_N(t)h + o(h)$, where $o(h)$ does not depend on t'. The condtional probability that it does not split during $[t, t + h]$, given that it exists at t and at $t' < t$, is $1 - \lambda \varphi_N(t)h + o(h)$, where $o(h)$ does not depend on t'.*

PROOF. We prove only the first assertion. For one particular particle, let S_k denote the event that it splits into k particles during $[t, t + h]$, let $A(\tau)$ denote the event that it is in existence at time τ, let T denote the time it came into existence (at time 0 or as the result of a particle splitting), and let $F_T(t'') = P([T \leqq t''] \mid A(t)A(t'))$. Then, by (1) in Section 2, we have

$$\begin{aligned} P(S_k \mid A(t)A(t')) &= \int_0^{t'} P(S_k \mid A(t)A(t'), T = t'')\, dF_T(t'') \\ &= \int_0^{t'} P(S_k \mid A(t), T = t'')\, dF_T(t'') \\ &= \int_0^{t'} (\lambda_k \varphi_N(t)h + o(h))\, dF_T(t'') = \lambda_k \varphi_N(t)h + o(h), \end{aligned}$$

which completes the proof.

LEMMA 2. *Let f be a real-valued function over $[0, 1]$ which is Riemann integrable. Let $0 = x_{n,0} < x_{n,1} < \cdots < x_{n,n} = 1$ be such that $\max_{1 \leqq k \leqq n}\{x_{n,k} - x_{n,k-1}\} \to 0$ as $n \to \infty$. If $\xi_{n,k} \varepsilon [x_{n,k-1}, x_{n,k}]$, then*

$$L_n = \prod_{k=1}^{n} (1 + f(\xi_{n,k})(x_{n,k} - x_{n,k-1})) \to \exp \int_0^1 f(x)\, dx$$

as $n \to \infty$.

PROOF. Riemann integrability of f implies that f is bounded. Hence for all sufficiently large values of n and for $k = 1, 2, \cdots, n$, we have $|f(\xi_{n,k})\cdot$

$(x_{n,k} - x_{n,k-1})| < 1$. Then we may write

$$\log L_n = \sum_{k=1}^{n} f(\xi_{n,k})(x_{n,k} - x_{n,k-1}) + R_n$$

where

$$|R_n| \leqq 2\sum_{k=1}^{n} f^2(\xi_{n,k})(x_{n,k} - x_{n,k-1})^2.$$

Thus $R_n \to 0$ as $n \to \infty$ and

$$\log L_n \to \int_0^1 f(x)\,dx \quad \text{as} \quad n \to \infty,$$

which was to be proved.

Let $Y_N(t)$ denote the number of particles which split during $[0, t]$.

LEMMA 3. *For every* $t \geqq 0$, $T > 0$,

$$P([Y_N(t + T) - Y_N(t) = 0] \mid [X(t) = n]) = \exp - n\lambda \int_t^{t+T} \varphi_N(\tau)\,d\tau.$$

(In case $t = 0$, then $n = N$.)

PROOF. Consider one particular particle among the n in existence at time t. If $t \leqq \tau < \tau + h$, then the conditional probability that it does not split during $[\tau, \tau + h]$, given that it still exists at time τ, is given by $1 - \lambda\varphi_N(\tau)h + o(h)$. Hence the conditional probability that it does not split during any of the intervals

$$\{[t + jT/m, t + (j + 1)T/m], j = 0, 1, \cdots, m - 1\},$$

given that it is in existence at time t, is

$$\prod_{j=0}^{m-1}(1 - \lambda\varphi_N(t + jT/m)(T/m) + o(T/m)),$$

which, by Lemma 2, converges (as $m \to \infty$) to $\exp - \lambda\int_t^{t+T}\varphi_N(\tau)\,d\tau$. The conclusion of the lemma follows because of the independence of the n particles.

LEMMA 4. *Let* $\mu, \mu_1, \mu_2, \cdots$ *be a sequence of probability measures over n-dimensional Euclidean space* $E^{(n)}$, *let* $f, f_1, f_2, \cdots$ *be a sequence of measurable functions over* $E^{(n)}$, *and assume that*

(i) $|\mu_n - \mu|(B) \to 0$ *as* $n \to \infty$ *for every bounded measurable subset* $B \subset E^{(n)}$,

(ii) $f, f_1, f_2, \cdots$ *are uniformly bounded a.e. with respect to* $\mu, \mu_1, \mu_2, \cdots$, *and*

(iii) $f_n \to f$ *as* $n \to \infty$ *uniformly over every bounded Borel set except over a subset of* μ- *and* μ_m-*measure zero*, $m = 1, 2, \cdots$. *Then* $\int f_n\,d\mu_n \to \int f\,d\mu$ *as* $n \to \infty$.

PROOF. We may assume without loss of generality that $|f_m(x)| \leqq 1$ for all $x \in E^{(n)}$ and all m. Let $0 < \epsilon < 1$. Let S be a sphere such that $\mu(S) > 1 - \epsilon/4$ and $\mu_n(S) > 1 - \epsilon/4$ for all sufficiently large values of n. Further, for all large n we have $|f_n(x) - f(x)| < \epsilon/4$ for all $x \in S$ except for a set of μ- and μ_m-measure zero, $m = 1, 2, \cdots$, and $|\mu_n - \mu|(S) < \epsilon/4$. Hence for all large n (and letting S^c denote the complement of S in $E^{(n)}$),

$$\left|\int f_n\, d\mu_n - \int f\, d\mu\right| \leqq \left|\int_S f_n\, d\mu_n - \int_S f\, d\mu_n\right| + \left|\int_S f\, d\mu_n - \int_S f\, d\mu\right| + \left|\int_{S^c} f_n\, d\mu_n\right| + \left|\int_{S^c} f\, d\mu\right| \leqq \int_S |f_n - f|\, d\mu_n + \int_S |f|\, d|\mu_n - \mu| + \mu_n(S^c) + \mu(S^c) < \epsilon,$$

which proves the assertion.

In all of the applications that will be made of Lemma 4 in this paper, μ_n and μ will be carried by the set of lattice points $\mathfrak{L}$ in $E^{(n)}$. For each such lattice point x it will turn out that $\mu_n(\{x\}) \to \mu(\{x\})$ as $n \to \infty, f, f_1, f_2, \cdots$ will be uniformly bounded, and $f_n(x) \to f(x)$ as $n \to \infty$. If one writes $f(x) = f_m(x) = 0$ for $x \not\in \mathfrak{L}$, then (i), (ii) and (iii) of Lemma 4 will be satisfied.

LEMMA 5. *Let $Q(h)$ denote the conditional probability that none of the particles produced during $[t, t + h]$ also split during $[t, t + h]$, given that one particle in existence at time t does split during $[t, t + h]$. Then $Q(h) \to 1$ as $h \to 0$ $(h > 0)$ uniformly in t over every bounded interval (for fixed N).*

PROOF. We first note that the conditional probability that a particle existing at time t splits into k particles during $[t, t + h]$, given that it does split during $[t, t + h]$, is

$$q_k(h) = [\lambda_k \varphi_N(t)h + o(h)]/[\lambda\varphi_N(t)h + o(h)] = \frac{\lambda_k}{\lambda} + o(1),$$

and $\sum_{k=0}^{\infty} q_k(h) = 1$ for all $h > 0$. Let V denote the time of splitting of the particle in existence at time t, and let $F_V(v \mid h)$ be the conditional distribution function of V, given the event B_k (which is defined below). Let A denote the event that none of the particles produced by a split during $[t, t + h]$ also split during $[t, t + h]$, and let B_k denote the event that the particular particle in existence at time t does split into k new particles during $[t, t + h]$. Then for fixed $h_0 > 0$ and $0 < h \leqq h_0$ we have

$$\begin{aligned} Q(h) &= \sum_{k=0}^{\infty} P(A \mid B_k)P(B_k) \\ &= \sum_{k=0}^{\infty} \int_t^{t+h} P(A \mid B_k, V = v)\, dF_V(v \mid h) q_k(h) \\ &= \sum_{k=0}^{\infty} \left\{ \int_t^{t+h_0} I_{[t,t+h]}(1 - \lambda_N \varphi_N(v)(t + h - v) + (t + h - v)o(1))^k\, dF_V(v \mid h) \right\} q_k(h). \end{aligned} \tag{10}$$

Now $q_k(h) \to \lambda_k/\lambda$ as $h \to 0$ which is a bona fide probability distribution. Further, each integral in (10) is absolutely bounded by 1 for every $h > 0$ and converges to 1 as $h \to 0$. Hence Lemma 4 applies, yielding the conclusion of the lemma.

Let X_n denote the change in the size of the population that results from the nth splitting that takes place.

REMARK. *The sequence* $\{X_1, X_2, \cdots\}$ *are independent, identically distributed random variables with common distribution* $P[X_n = k - 1] = \lambda_k/\lambda$, $k = 0, 1, 2, \cdots$.

PROOF. Let T_{n-1} denote the time of the $(n-1)$th splitting, and let $F_{T_{n-1}}(\tau)$ denote the conditional distribution of T_{n-1}, given the event $\bigcap_{j=1}^{n-1}[X_j = k_j]$. Thus $F_{T_{n-1}}(\tau)$ is a bona fide probability distribution. Now

$$P([X_n = k_n] \mid [X_1 = k_1] \cdots [X_{n-1} = k_{n-1}])$$
$$= \int_0^\infty P([X_n = k_n] \mid [X_1 = k_1] \cdots [X_{n-1} = k_{n-1}], T_{n-1} = \tau)\, dF_{T_{n-1}}(\tau).$$

By means of Lemmas 3 and 5 we find that the integrand above equals

$$\frac{\int_\tau^\infty \left\{\exp - m_0\lambda \int_\tau^t \varphi_N(\theta)\, d\theta\right\} m_0\lambda_{k_n+1}\varphi_N(t)\, dt}{\sum_{j=0}^\infty \int_\tau^\infty \left\{\exp - m_0\lambda \int_\tau^t \varphi_N(\theta)\, d\theta\right\} m_0\lambda_j\varphi_N(t)\, dt},$$

where $m_0 = N + k_1 + \cdots + k_{n-1}$. This ratio equals λ_{k_n+1}/λ, and this observation concludes the proof.

It should be mentioned in passing that $Y_N(t), X_1, X_2, \cdots$ are indeed random variables, i.e., $[Y_N(t) \leqq x]$, $[X_1 \leqq x]$, $[X_2 \leqq x]$, $\cdots$ are all events which are formed by countable set-theoretic operations on events whose probabilities were given in Section 2. Indeed, let k_i be an integer, $k_i \geqq -1$, $1 \leqq i \leqq m$, and let $t > 0$. Let us denote $A_{0,n}$ as the event that no particles split during $[0, (j_1 - 1)/2^n t]$, $A_{i,n}(j_i)$ as the event that no particles split during $[j_i t/2^n, (j_{i+1} - 1)t/2^n]$, $1 \leqq i < m$, $A_{m,n}(j_m)$ as the event that no particles split during $[j_m t/2^m, t]$, and $B_{i,n}(k_i)$ as the event that just one particle splits during $[(j_i - 1)t/2^n, j_i t/2^n]$ and splits into k_i particles. Then

$$\bigcap_{j=1}^m [X_j = k_j] = \bigcap_{n=1}^\infty \bigcup_{j_1=1}^\infty \bigcup_{j_2=j_1+1}^\infty \cdots \bigcup_{j_m=j_{m-1}+1}^\infty A_{0,n} \bigcap_{i=1}^{m-1} A_{i,n}(j_i) \bigcap_{r=1}^m B_{r,n}(k_r).$$

Also,

$$[Y_N(t) = m] = \bigcap_{n=1}^\infty \bigcup_{k_1=-1}^\infty \cdots \bigcup_{k_m=-1}^\infty \bigcup_{j_1=1}^{s(1)} \bigcup_{j_2=j_1+1}^{s(2)} \cdots \bigcup_{j_m=j_{m-1}+1}^{s(m)} A_{0,n} \bigcap_{i=1}^m (A_{i,n}(j_i)B_{i,n}(k_i)),$$

where $s(i) = 2^n - (m + 1) + \mathrm{i}$, $1 \leqq i \leqq m$. The above two equations verify that $Y_N(t), X_1, X_2, \cdots$ are random variables.

4. The limit distribution of the process $X_N(t)$. Let $\{U_n(t)\}$ be a sequence of stochastic processes, where $t \varepsilon T$, T being some interval of real numbers. We say that this sequence converges *in distribution* to the stochastic process $U(t)$ if for every positive integer m, every m-tuple $\{t_1, \cdots, t_m\} \subset T$, and every m-tuple $\{u_1, \cdots, u_m\} \varepsilon (-\infty, \infty)$ which is a continuity point of the joint distribution function of $U(t_1), \cdots, U(t_m)$,

$$\lim_{n\to\infty} P\left(\bigcap_{i=1}^{m} [U_n(t_i) \leqq u_i]\right) = P\left(\bigcap_{i=1}^{m} [U(t_i) \leqq u_i]\right).$$

The purpose of this section is to show that the sequence of stochastic processes $\{X_N(t) - N, N = 1, 2, \cdots\}$ converges *in distribution* as $N \to \infty$ to a stochastic process with independent increments.

THEOREM 1. *For every non-negative integer k and every integer $n_i \geqq -1, 1 \leqq i \leqq k$,*

$$\lim_{N\to\infty} P[Y_N(t) = k] = e^{-\nu\Phi(t)}(\nu\Phi(t))^k/k! \tag{11}$$

and

$$\lim_{N\to\infty} P\left(\bigcap_{i=1}^{k} [X_i = n_i] \,|\, [Y_N(t) = k]\right) = \prod_{i=1}^{k} (\nu_{n_i+1}/\nu). \tag{12}$$

PROOF. We first note that if $a_{ij}(N) \geqq 0$, $b_{ij} \geqq 0$ for $i, j, N = 1, 2, \cdots$, if $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} b_{ij} = 1$, if $a(N) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}(N) \leqq 1$ for every N, and if $a_{ij}(N) \to b_{ij}$ as $N \to \infty$ for every i, j, then $a(N) \to 1$ as $N \to \infty$ and $\sum_{i=1}^{\infty} a_{ij}(N) \to \sum_{i=1}^{\infty} b_{ij}$ as $N \to \infty$ for every j.

Let us denote

$$K = \prod_{i=1}^{k} \left\{\left(N + \sum_{j=1}^{i-1} n_j\right) \lambda_{n_i+1}\varphi_N(t_i)\right\},$$

$$L = \prod_{i=1}^{k-1} \left\{\exp - \left(N + \sum_{j=1}^{i} n_j\right) \lambda \int_{t_i}^{t_{i+1}} \varphi_N(\theta)\, d\theta\right\},$$

$$R = \exp - N\lambda \int_0^{t_1} \varphi_N(\theta)\, d\theta, \quad \text{and}$$

$$W = \exp - \left(N + \sum_{i=1}^{k} n_i\right) \lambda \int_{t_k}^{t} \varphi_N(\theta)\, d\theta.$$

Using Lemma 3 and Lemma 5 one is able to establish that

$$P\left([Y_N(t) = k] \bigcap_{i=1}^{k} [X_i = n_i]\right) = \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{k-1}}^t RKLW\, dt_k.$$

Using the Lebesgue dominated convergence theorem we obtain

$$\lim_{N\to\infty} P\left([Y_N(t) = k] \bigcap_{i=1}^{k} [X_i = n_i]\right)$$

$$= \exp\left[-\nu \int_0^t \varphi(\theta)\, d\theta\right] \left(\prod_{i=1}^{k} \nu_{n_i+1}\right) \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{k-1}}^t \left(\prod_{i=1}^{k} \varphi(t_i)\right) dt_k$$

$$= e^{-\nu\Phi(t)} \left(\prod_{i=1}^{k} \nu_{n_i+1}\right) \Phi^k(t)/k!.$$

This limit and the first sentence of this proof imply (11) and (12).

LEMMA 6. *Let* $\{(Y_{n,1}, \cdots, Y_{n,k})\}$ *be a sequence of k-dimensional random variables which converges in distribution to* $(Y_1, \cdots, Y_k)$. *If* $S_{n,j} = Y_{n,1} + \cdots + Y_{n,j}$, $1 \leqq j \leqq k$, $n = 1, 2, \cdots$, *and if* $S_j = Y_1 + \cdots + Y_j$, *then the joint distribution of* $(S_{n,1}, \cdots, S_{n,k})$ *converges to the joint distribution of* $(S_1, \cdots, S_k)$ *as* $n \to \infty$.

PROOF. By hypothesis, for every k-tuple of reals $(u_1, u_2, \cdots, u_k)$ we have

$$E\left(\exp i \sum_{j=1}^{k} u_j Y_{n,j}\right) \to E\left(\exp i \sum_{j=1}^{k} u_j Y_j\right) \quad \text{as} \quad n \to \infty.$$

Define $u_j = t_j + t_{j+1} + \cdots + t_k$, $1 \leqq j \leqq k$. Then

$$E\left(\exp i \sum_{j=1}^{k} t_j S_{n,j}\right) \to E\left(\exp i \sum_{j=1}^{k} t_j S_j\right)$$

as $n \to \infty$, for every k-tuple $(t_1, \cdots, t_k)$ of reals, which concludes the proof.

We are now able to establish the limit distribution of $X_N(t) - N$.

THEOREM 2. *Under the hypotheses of Section* 2, *the stochastic processes* $\{X_N(t) - N\}$ *converge in distribution to a stochastic process with independent increments whose distribution is determined by* (8) *and* (9).

PROOF. Let $f_N(u)$ denote the characteristic function of $X_N(t) - N$. We shall first show that, for fixed t and u, $\lim_{N\to\infty} f_N(u) = \psi(u)$, where $\psi(u)$ is defined in (8). We first observe that

$$\sum_{k=0}^{\infty}\left\{\sum{}' E\left(\exp[iu(X_N(t) - N)] \,|\, [Y_N(t) = k] \bigcap_{i=1}^{k} [X_i = n_i]\right) \cdot P\left(\bigcap_{i=1}^{k} [X_i = n_i] \,|\, [Y_N(t) = k]\right)\right\} P[Y_N(t) = k]$$

$$= \sum_{k=0}^{\infty}\left\{\sum{}' \exp\left[iu \sum_{j=1}^{k} n_j\right] P\left(\bigcap_{i=1}^{k} [X_i = n_i] \,|\, [Y_N(t) = k]\right)\right\} P[Y_n(t) = k],$$

where the sum $\sum'$ is taken over integers $n_i \geqq -1$, $1 \leqq i \leqq k$. By Theorem 1 and the Helly-Bray theorem,

$$\lim_{N\to\infty} \sum{}' \exp\left[iu \sum_{j=1}^{k} n_j\right] P\left(\bigcap_{i=1}^{k} [X_i = n_i] \,|\, [Y_N(t) = k]\right) = f^k(u),$$

where $f(u) = \sum_{m=-1}^{\infty} e^{ium} \nu_{m+1}/\nu$. Then by Lemma 4 and Theorem 1 we obtain

$$\lim_{N\to\infty} f_N(u) = \sum_{k=0}^{\infty} f^k(u) e^{-\nu\Phi(t)} (\nu\Phi(t))^k/k! = \psi(u).$$

We now prove that the process $X_N(t) - N$ converges *in distribution* to a process with independent increments. Let $0 = t_0 < t_1 < t_2 < \cdots < t_m < \cdots$ be any increasing sequence of non-negative numbers, and let $U_i = X_N(t_i) - X_N(t_{i-1})$. The joint characteristic function of $U_1, \cdots, U_m$ is given by

$$f_{U_1,\cdots,U_m}(u_1, \cdots, u_m) = \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_m=-\infty}^{\infty} \exp\left[i \sum_{j=1}^{m} u_j k_j\right] P\left(\bigcap_{s=1}^{m} [U_s = k_s]\right).$$

We shall now prove that

$$(13) \qquad \lim_{N\to\infty} f_{U_1,\cdots,U_m}(u_1, \cdots, u_m) = \prod_{i=1}^{m} f_i(u_i),$$

where $f_i(u) = \exp\{\Phi(t_i) - \Phi(t_{i-1})\} \sum_{k=0}^{\infty} (e^{i(k-1)u} - 1)\nu_k$.

We may write

$$(14) \; f_{U_1,\cdots,U_m}(u_1, \cdots, u_m) = \sum_{k_1,\cdots,k_m} \exp\left[i \sum_{j=1}^{m} u_j k_j \right] \prod_{i=1}^{m} P\left([U_i = k_i] \mid \bigcap_{j=1}^{i-1} [U_j = k_j] \right).$$

Let $X(t)$ denote a process with independent increments whose distribution is given by (8) and (9), and let $p_j(k) = P[X(t_j) - X(t_{j-1}) = k]$. Since $\lim_{N\to\infty} (N + \sum_{s=1}^{j} k_s)\lambda_n = \nu_n$ and $\lim_{N\to\infty} (N + \sum_{s=1}^{j} k_s)\lambda = \nu$ for $0 \leqq j \leqq m - 1$, we obtain by what was proved just above (13)

$$\lim_{N\to\infty} P\left([U_i = k_i] \mid \bigcap_{j=1}^{i-1} [U_j = k_j] \right) = p_i(k_i) \quad \text{for} \quad 1 \leqq i \leqq m.$$

Applying the Helly-Bray theorem to (14) we obtain (13). By Lemma 6 we may conclude that the joint distribution of $\{X_N(t_1), \cdots, X_N(t_m)\}$ converges to the joint distribution of $\{X(t_1), \cdots, X(t_m)\}$.

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