# ASYMPTOTIC EXTREMES FOR $m$-DEPENDENT RANDOM VARIABLES

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**1. Introduction.** Let $\{X_j\}$ be a stationary sequence of random variables, i.e.

$$(1) \qquad P\{X_j \leqq x_j, X_k \leqq x_k, \cdots\} = P\{X_{j+1} \leqq x_j, X_{k+1} \leqq x_k, \cdots\},$$

which are $m$-dependent, i.e. if $A_k$ are sets of integers, $E_k$ is any event in the field of events generated by the $X_j$, $j \varepsilon A_k$, and every element of $A_k$ differs by more than $m$ from any element of $A_l$, $l \neq k$, then the events $E_k$ are mutually independent (events that occur more than a "time" $m$ apart are independent). The extreme

$$(2) \qquad Y_n = \max_{1 \leqq j \leqq n} X_j$$

of the finite sequence $X_j$, $1 \leqq j \leqq n$ has a distribution function

$$(3) \qquad F_n(y) = P\{Y_n \leqq y\} = P\{X_1 \leqq y, X_2 \leqq y, \cdots, X_n \leqq y\}.$$

This paper deals with the limit behavior of $F_n(y)$ for large $n$ and the extent to which it differs from the special case of independent $X_j$ for which $F_n(y)$ is the $n$th power of some distribution function, specifically the distribution function of $X_j$. The following theorem is proved.

Theorem. *If $B_j(y)$ denotes the event*

$$(4) \qquad B_j(y) = \{X_j > y\} \bigcap_{k=1}^{m} \{X_{j+k} \leqq y\}$$

*and $y_n$ is any sequence of real numbers such that*

$$(5) \qquad nP\{X_j > y_n\} < M$$

*for all $n$, then*

$$(6) \qquad F_n(y_n) \exp [nP\{B_j(y_n)\}] \rightarrow 1$$

*for $n \rightarrow \infty$.*

This extends a theorem by G. S. Watson [4] who proved a result essentially equivalent to the above for the special case in which $y_n$ is chosen so that $nP\{X_j > y_n\}$ has a finite limit $\xi$ and the process $\{X_j\}$ has the property that

$$(7) \qquad nP\{B_j(y_n)\} \sim nP\{X_j > y_n\} \rightarrow \xi$$

for $n \rightarrow \infty$, i.e. in the limit of rare events, $P\{X_j > y_n\} \rightarrow 0$, there is probability

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zero that if one of the events $X_j > y_n$ is known to occur, a second such event will also occur at a neighboring time point. Actually Watson gives a much weaker definition of $m$-dependence (only pairwise independence of $X_j$ and $X_k$ for $|j - k| > m$) than that used here but he seems to assume the above definition in the proof.

Several more recent papers have also been written on various aspects of limit behavior of extremes for dependent random variables or processes [1]–[3].

**2. Proof.** We first write $F_n(y) = 1 - P\{X_j > y \text{ for some } j, 1 \leqq j \leqq n\}$. The event that $X_j > y$ for some $j$, $1 \leqq j \leqq n$, is equivalent to the event that either some $B_j = B_j(y)$, $1 \leqq j \leqq n$, occur (some $X_j > y$ and the $m$ succeeding $X_k$ are $\leqq y$) or some $X_j > y$ occur and each occurrence of an $X_j > y$ is followed by another event $X_k > y$ with $1 \leqq k - j \leqq m$ (no $B_j$ occurs). The latter of these two possibilities, however, can happen only if $X_j > y$ for some $j$ with $n - m + 1 \leqq j \leqq n$, the last such event with $j \leqq n$. The probability for this is certainly less than the sum of probabilities for these values of $j$, namely $mP\{X_j > y\}$.

If follows from (5) that $mP\{X_j > y_n\} < Mm/n \to 0$ for $n \to \infty$, and so we conclude that

$$F_n(y_n) = 1 - P\left\{\bigcup_1^n B_j(y_n)\right\} + O(1/n) \qquad \text{for} \quad n \to \infty \tag{8}$$

which with the principle of inclusion and exclusion gives

$$\begin{aligned} F_n(y_n) = 1 - \sum_{j=1}^n P\{B_j(y_n)\} + \sum_{j<k} P\{B_j(y_n)B_k(y_n)\} \\ - \sum_{j<k<l} P\{B_j(y_n)B_k(y_n)B_l(y_n)\} + \cdots + O(1/n). \end{aligned} \tag{9}$$

The various types of terms, singlet events, pairs, etc. will be considered separately. For the singlet terms, the stationarity implies $\sum_1^n P\{B_j\} = nP\{B_j\} = nP\{B_1\}$. The pair terms can be separated into those with (a) $k - j \leqq m$, (b) $m + 1 \leqq k - j \leqq 2m$, and (c) $2m + 1 \leqq k - j$. In case (a) $B_jB_k$ is the null event and $P\{B_jB_k\} = 0$. In case (b),

$$P\{B_jB_k\} \leqq P\{X_j > y_n, X_k > y_n\} = P^2\{X_j > y_n\} < M^2/n^2$$

since the $X_j$ and $X_k$ are independent. There are less than $nm$ such terms, however, and their total contribution to $F_n(y_n)$ is also of order $1/n$. In case (c), the events $B_j$ and $B_k$ are independent so that $P\{B_jB_k\} = P^2\{B_j\}$. The number of these terms is $\frac{1}{2}n^2[1 + O(1/n)]$, therefore $\sum_{j<k} P\{B_jB_k\} = \frac{1}{2}n^2P^2\{B_j\} + O(1/n)$.

A similar decomposition applies to all higher order sums in (9). There are approximately $n^s/s!$ terms involving the intersections of $s$ of the $B_j$. Of these all but a fraction of order $1/n$ involve only independent $B_j$. These contribute to $F_n(y_n)$ a quantity $(-1)^s(n^s/s!)P^s\{B_j\} + O(1/n)$. The terms involving dependent $B_j$ are either null if any pair of indices differ by $m$ or less or all pairs of indices differ by more than $m$ and $P\{B_jB_kB_l \cdots\} \leqq P\{X_j > y_n, X_k > y_n, \cdots\} = P^s\{X_j > y_n\} < M^s/n^s$. The number of such terms is of order $n^{s-1}$ and their total contribution is of order $1/n$.

Since this last inequality is also valid for arbitrary $j, k, l, \cdots$, the expansion (9) for $F_n(y_n)$ is dominated by the series

$$F_n(y_n) < \sum_0^n \binom{n}{s} \frac{M^s}{n^s} < \sum_0^\infty \frac{M^s}{s!}$$

uniformly in $n$. To within a remainder $o(1)$, the series for $F_n(y)$ can therefore be approximated by the contribution from at most $o(n)$ many values of $s$, including, in particular, the $O(1/n)$ terms. Thus for $n \to \infty$, $F_n(y_n) = 1 - nP\{B_j) + (n^2/2)P^2\{B_j\} + \cdots + o(1) = \exp[-nP\{B_j\}] + o(1)$.

We have not specified that $nP\{B_j(y_n)\}$ must have a limit for $n \to \infty$ but in any case $nP\{B_j(y_n)\} \leqq nP\{X_j > y_n\} < M$, and therefore $F_n(y_n) \exp[nP\{B_j(y_n)] \to 1$ for $n \to \infty$, which proves the theorem.

**3. Examples.** For many processes that arise in practical applications the Condition (7) assumed by Watson is true. As an example of when it is not true, suppose that $Z_j$ is a sequence of independent identically distributed random variables with a distribution function $F_z(z)$, and the $X$-process is $X_j = \max(Z_j, Z_{j-1})$.

The distribution for $Y_n$ is very simple since $Y_n = \max_{1 \leqq j \leqq n} X_j = \max_{0 \leqq j \leqq n} Z_j$, and so for large $n$

$$F_n(y_n) = [F_z(y_n)]^{n+1} \sim \exp\{-n[1 - F_z(y_n)]\}. \tag{10}$$

The process $X_j$ is 1-dependent and

$$P\{X_1 > y, X_2 \leqq y\} = P\{\max(Z_1, Z_0) > y, \max(Z_2, Z_1) \leqq y\}$$
$$= P\{Z_0 > y, Z_2 \leqq y, Z_1 \leqq y\} = F_z^2(y)[1 - F_z(y)].$$

Therefore

$$nP\{B_1\} = nP\{X_1 > y_n, X_2 \leqq y_n\} \sim n[1 - F_z(y_n)]$$

and the theorem (6) is consistent with (10). However

$$P\{X_1 > y_n\} = P\{\max(Z_1, Z_0) > y_n\} = 1 - F_z^2(y_n)$$

and

$$nP\{X_1 > y_n\} \sim 2n[1 - F_z(y_n)] \nsim nP\{B_1\}.$$

The $m$-dependent processes often arise as functions of independent random variables $X_j = G(Z_j, Z_{j-1}, \cdots, Z_{j-m})$ and one can easily construct any number of processes for which (7) is false, particularly if the function $G$ is symmetric to interchange of its arguments.

## REFERENCES

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