# EIGENVALUES OF NON-NEGATIVE MATRICES[1]

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**1. Introduction.** Let $P = (p_{ij})$, $i, j = 0, 1, 2, \cdots$, be a matrix with non-negative entries. $P$ is said to be irreducible if for every pair $i, j$, there is a finite sequence of integers $k_1, k_2, \cdots, k_n$ such that $p_{ik_1}p_{k_1k_2}\cdots p_{k_nj} > 0$. An alternative definition is given in Gantmacher's book ([4], p. 50).

The point of view adopted here is to consider an irreducible matrix $P$ as an operator acting on column vectors having non-negative entries. A necessary and sufficient condition for there to be a solution of $Px = \lambda x$, i.e. for $\lambda$ to be an eigenvalue, is obtained. The principal tool is the theorem of Harris [5] and Veech [7] which gives a necessary and sufficient condition for the existence of a stationary measure for a transient Markov chain.

The relationship between the $R$-recurrent matrices studied by Vere-Jones [8] and Kingman [6] and recurrent matrices is investigated in the final section. In the stochastic case, this investigation is related to the eigenvalue problem described above.

**2. The eigenvalue problem.** The method to be used is to transform $P$ into a substochastic matrix so that the Harris-Veech theorem may be applied. The first step is the observation that an eigenvector can only have positive components.

Lemma 1. *If $P$ is irreducible and $sPx \leqq x$ for some $s > 0$ and nontrivial $x$, then $x_j > 0$ for all $j$.*

Proof. For any $i, j$, $sp_{ij}x_j \leqq x_i$, so by induction for any sequence $\{k_n\}$, $s^{n+1}p_{k_0k_1}p_{k_1k_2}\cdots p_{k_nk_{n+1}}x_{k_{n+1}} \leqq x_{k_0}$. Now $x_j > 0$ for some $j$ and then for any $i$ let $\{k_n\}$ be the sequence guaranteed by the definition of irreducibility. Letting $k_0 = i$, $k_{n+1} = j$ in the above inequality yields the positivity of $x_i$.

The next lemma proves the existence of all iterates of $P$ provided there is an eigenvalue. Let $p_{ij}^{(0)} = \delta_{ij}$, $p_{ij}^{(n)} = \sum_k p_{ik}p_{kj}^{(n-1)}$, $P_{ij}(s) = \sum_{n=0}^{\infty} p_{ij}^{(n)}s^n$, and $R_{ij}$ equal the radius of convergence of this power series.

Lemma 2. *If $P$ is irreducible and $sPx \leqq x$ for some $s > 0$ and nontrivial $x$, then $p_{ij}^{(n)} < \infty$ for all $i, j$, and $n$, and $R_{ij} \geqq s$.*

Proof. First $sp_{ij}x_j \leqq x_i$ and using the given inequality in an induction yields $s^np_{ij}^{(n)}x_j \leqq x_i$. This suffices for the first part since $x_j > 0$ by Lemma 1. Finally $\{p_{ij}^{(n)}\}^{1/n} \leqq s^{-1}(x_i/x_j)^{1/n}$ so that $\limsup \{p_{ij}^{(n)}\}^{1/n} \leqq s^{-1}$ or $R_{ij} \geqq s$.

In view of Lemma 2 and our interest in $P$ having eigenvalues, it will be assumed henceforth that all iterates of $P$ are finite and that $R_{ij} > 0$. Vere-Jones [8] has shown in this case that $R_{ij} = R$, independent of $i$ and $j$, and that the series $P_{ij}(R)$ converge or diverge together. In the first case $P$ is called $R$-transient

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and in the other $R$-recurrent. A substochastic matrix $P$ (for which $R \geqq 1$) is called recurrent if $P_{ij}(1)$ diverges and transient if it converges. Thus for substochastic matrices recurrence is equivalent to 1-recurrence.

It will be convenient to introduce the analogues of the taboo probabilities of Chung [1]. Define ${}_kp_{ij}^{(1)} = p_{ij}$ and for $n \geqq 1$,

$$ {}_kp_{ij}^{(n+1)} = \sum_{\alpha \neq k} p_{i\alpha}\, {}_kp_{\alpha j}^{(n)}. $$

The defining sums will converge since ${}_kp_{ij}^{(n)} \leqq p_{ij}^{(n)}$. The usual convention that ${}_kp_{ij}^{(0)} = \delta_{ij}(1 - \delta_{ik})$ will be used. The power series ${}_kP_{ij}(s) = \sum_{n=0}^{\infty} {}_kp_{ij}^{(n)} s^n$ will converge at least for $s < R$.

At this point the method of Harris and Veech could be imitated for the case at hand. However, some additional information can be obtained by first considering the system of inequalities $sPx \leqq x$. Theorem 1 relates this system to the radius of convergence $R$ and the classification of $P$ as to its $R$-recurrence or $R$-transience.

Theorem 1. *The system $sPx \leqq x$ has infinitely many (linearly independent) solutions for $0 < s < R$, no solutions for $s > R$, and for $s = R$*

(a) *infinitely many solutions if $P$ is $R$-transient, or*

(b) *a unique solution (up to positive multiples) if $P$ is $R$-recurrent and this solution satisfies $RPx = x$.*

Proof. For $s < R$ (and $s = R$ if $P$ is $R$-transient),

$$ s \sum_{n=0}^{\infty} \sum_{k} p_{ik} p_{ka}^{(n)} s^n = \sum_{n=0}^{\infty} p_{ia}^{(n+1)} s^{n+1} $$

or $s \sum_k p_{ik} P_{ka}(s) = P_{ia}(s) - \delta_{ia} \leqq P_{ia}(s)$ so that for every $a$, $x^a = \{x_k^a\}$ where $x_k^a = P_{ka}(s)$ is a solution of the system. To see that these solutions are linearly independent, suppose that $x^a = \sum \beta_j x^j$, i.e. $x^a$ is a (finite) linear combination of the other $x^j$. Then

$$ x_a^a - 1 = (sPx^a)_a = \sum \beta_j (sPx^j)_a = \sum \beta_j x_a^j = x_a^a $$

which is the desired contradiction. (This short proof of the linear independence was suggested by S. Orey.)

The absence of solutions for $s > R$ follows from Lemma 2 so only (b) when $s = R$ remains. Vere-Jones [8] has shown that ${}_aP_{ka}(R) < \infty$, ${}_aP_{aa}(R) = 1$, and for fixed $a$, $\{{}_aP_{ka}(R)\}$ is the only solution of $RPx = x$ when $P$ is $R$-recurrent. Because the interest here is in solutions of the system $RPx \leqq x$, the uniqueness question must still be examined. We apply a technique used by Kingman for a similar purpose in [6]. Let $x$ be any solution of $RPx \leqq x$. Since $x_k \geqq Rp_{ka}x_a$, suppose that $x_k \geqq x_a \sum_{n=1}^{N} {}_ap_{ka}^{(n)} R^n$. Then

$$ x_k \geqq R \sum_{\alpha} p_{k\alpha} x_\alpha \geqq R x_a \Big(p_{ka} + \sum_{\alpha \neq a} p_{k\alpha} \sum_{n=1}^{N} {}_ap_{\alpha a}^{(n)} R^n\Big) = x_a \sum_{n=1}^{N+1} {}_ap_{ka}^{(n)} R^n. $$

Therefore the inequality is valid for all $N$ and $x_k \geqq x_a\, {}_aP_{ka}(R)$. Now if $x$ is not to be a multiple of the given solution, then for some $b$, $x_b > x_a\, {}_aP_{ba}(R)$. By irreducibility there is an $n$ such that $p_{ab}^{(n)} > 0$. Finally,

$$x_a \geqq R^n \sum_k p_{ak}^{(n)} x_k > R^n \sum_k p_{ak}^{(n)} x_a\, {}_aP_{ka}(R) = x_a\, {}_aP_{aa}(R) = x_a ,$$

which is the contradiction. Thus any solution is a multiple of $\{{}_aP_{ka}(R)\}$.

The theorem of Harris and Veech is as follows: An irreducible, substochastic, transient matrix $P$ has a positive solution to the equation $\mu P = \mu$ if and only if there is an infinite set of integers $K$ such that

$$\lim_{j\to\infty, k\to\infty, k\varepsilon K} \frac{\sum_{\alpha=j}^{\infty} {}_iP_{k\alpha}(1) p_{\alpha i}}{{}_iP_{ki}(1)} = 0$$

for $i = 0, 1, 2, \cdots$. Harris and Veech may have in mind only the stochastic case but their proof goes through without change for the substochastic case. This theorem will now be applied to yield

THEOREM 2. *The system $sPx = x$ has a solution if and only if*

(i) *$s = R$ and $P$ is $R$-recurrent, or*

(ii) *when either $s < R$ or $s = R$ and $P$ is $R$-transient, there is an infinite sequence of integers $K$ such that*

$$\lim_{j\to\infty, k\to\infty, k\varepsilon K} \frac{\sum_{\alpha=j}^{\infty} p_{i\alpha}\, {}_iP_{\alpha k}(s)}{{}_iP_{ik}(s)} = 0$$

*for every $i$.*

PROOF. By Theorem 1 there can be no solution for $s > R$ and there is always a solution in case (i). Therefore the situation must be examined only for $s < R$ and $s = R$ when $P$ is $R$-transient. Let $P^T$ denote the transpose of $P$ and $y$ satisfy $sP^Ty \leqq y$. Define a matrix $Q = (q_{ij})$ with $q_{ij} = sp_{ji}y_j/y_i$. (A similar technique was used by Derman [2] and has since been used by many authors including Vere-Jones [8] and Kingman [6].) Then $Q$ is irreducible and substochastic, and since $q_{ij}^{(n)} = s^n p_{ji}^{(n)} y_j/y_i$, $Q$ is transient. The theorem of Harris and Veech states that there is a positive solution of $\mu Q = \mu$ if and only if there is an infinite set of integers $K$ such that

$$\lim_{j\to\infty, k\to\infty, k\varepsilon K} \frac{\sum_{\alpha=j}^{\infty} {}_iQ_{k\alpha}(1) q_{\alpha i}}{{}_iQ_{ki}(1)} = 0 \tag{2.1}$$

for every $i$. But the equation $\mu Q = \mu$ is equivalent to

$$s \sum_i p_{ji}(\mu_i/y_i) = \mu_j/y_j$$

so that condition (2.1) is equivalent to the existence of a solution of $sPx = x$. Finally interpreting (2.1) in terms of $P$ yields the condition of the theorem.

COROLLARY. *If, for each $i$, $p_{i\alpha} = 0$ except for a finite set of $\alpha$ values, then $sPx = x$ has a solution for $0 < s \leqq R$.*

The corollary gives a class of examples where there is an interval of eigenvalues. For examples at the opposite extreme consider $P$ with $p_{0i} = r_i$ where $r_i > 0$, $\sum r_i < \infty$, and $p_{i,i-1} = 1$, $p_{ij} = 0$ for $j \neq i - 1$ when $i > 0$. Then $sPx = x$ has a solution only for $s = R$ and then only if $P$ is $R$-recurrent.

One final point of interest is that since only symmetric assumptions have been made on the rows and columns of $P$, there is no need to prove the usual dual theorem concerning solutions of $s\mu P = \mu$. It suffices to apply the stated theorem to $P^T$.

**3. Transformation of substochastic matrices.** The purpose of this section is to relate those substochastic matrices with radius of convergence $R > 1$ to substochastic matrices with $R = 1$. The primary motivation here is to see what the relation is between $R$-recurrent and recurrent matrices. Whenever $sPx \leqq x$, define $P(s, x) = (p_{ij}(s, x))$ with $p_{ij}(s, x) = sp_{ij}x_j/x_i$. The situation in the stochastic case is described in

THEOREM 3. *Suppose $P$ is $R$-recurrent ($R$-transient) and stochastic. Then if $x$ is any solution of $RPx \leqq x$, $P(R, x)$ is 1-recurrent (1-transient) and has eigenvalue $R$. Conversely, if $Q$ is 1-recurrent (1-transient) with eigenvalue $R > 1$ and corresponding eigenfunction $y$, then $Q(R^{-1}, y)$ is $R$-recurrent ($R$-transient) and stochastic.*

The proof of each statement is a straightforward computation and will be omitted. It is interesting to note that since recurrent (substochastic) matrices are automatically stochastic the above transformations are from stochastic matrices to stochastic matrices in the recurrent part of the theorem.

The situation when $P$ is allowed to be substochastic is briefly as follows. First, if it is $R$-recurrent ($R$-transient) it may be transformed into a 1-recurrent (1-transient) matrix as above but there is no additional information about eigenvalues. If, on the other hand, $Q$ is an arbitrary 1-recurrent (1-transient) matrix and for any $s < 1$ $x$ is a solution of $sQx \leqq x$, then $Q(s, x)$ is $s^{-1}$-recurrent ($s^{-1}$-transient) and substochastic.

## REFERENCES

[1] CHUNG, K. L. (1960). *Markov Chains with Stationary Transition Probabilities*. Springer, Berlin.

[2] DERMAN, C. (1954). A solution to a set of fundamental equations in Markov chains. *Proc. Amer. Math. Soc.* **5** 332–334.

[3] FELLER, W. (1950). *An Introduction to Probability Theory and Its Applications*. Wiley, New York.

[4] GANTMACHER, F. R. (1959). *The Theory of Matrices*, **2**. Chelsea, New York.

[5] HARRIS, T. E. (1957). Transient Markov chains with stationary measures. *Proc. Amer. Math. Soc.* **8** 937–942.

[6] KINGMAN, J. F. C. (1963). The exponential decay of Markov transition probabilities. *Proc. London Math. Soc.* (3). **13** 337–358.

[7] VEECH, W. (1963). The necessity of Harris' condition for the existence of a stationary measure. *Proc. Amer. Math. Soc.* **14** 856–860.

[8] VERE-JONES, D. (1962). Geometric ergodicity in denumerable Markov chains. *Quart. J. Math. Oxford Ser.* **13** 7–28.