ON THE MULTIVARIATE ANALYSIS OF WEAKLY STATIONARY STOCHASTIC PROCESSES

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0. Summary. The existence of the class of orthogonal projections which map an arbitrary q-variate weakly stationary stochastic process again into a q-variate process contained in the span of $p(\leq q)$ of its component processes is established. Mimicking the definitions of the partial and multiple correlation coefficients (e.g., Anderson, 1958), these projections are used to define partial and multiple coefficients of coherence, thus providing the foundation for the multivariate covariance and correlation analyses for weakly stationary processes employed in special cases by Tick (1963) and Jenkins (1963). Some of the properties of the partial and multiple correlation coefficients are established for the corresponding coefficients of coherence. In particular, formulas are established for generating these parameters iteratively. When used for the sample coefficients of coherence, these formulas provide useful methods of defining and constructing estimates of the multiple and partial coefficients of coherence from the usual estimates of the ordinary coefficient of coherence. Results due to Goodman (1963) concerning the distributions of these estimators when the process is Gaussian are indicated.

1. Introduction. In statistical applications it is often important to be able to deal with such questions concerning the distribution of random variables $X_1, \cdots, X_q$ as the following: How strong is the linear relationship between X_i and X_j? How strong is the residual linear relationship between X_i and X_j after the effects of the linear regression of these random variables on certain of the remaining variables has been removed? What proportion of the variance of X_i is attributable to the linear regression of X_i on one or more of the other random variables? What is the value of the residual variance of X_i after its regression on the remaining random variables has been removed? It is well known that these questions can be given formal meaning in terms of the variance-covariance structure of the distribution of $X_1, \cdots, X_q$. To each question there corresponds a parameter, depending only on the variance-covariance matrix, which provides the appropriate measure of the desired quantity. In the same order as the questions, these are the ordinary correlation coefficient, the partial correlation coefficient, the multiple correlation coefficient and the residual or conditional variance.

Several important structural relationships exist between these parameters. Of particular importance are inductive formulae which permit one to construct the

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last three, which are otherwise of a rather complicated form, from the ordinary correlation coefficients which are very simple functions of the variances and covariances.

In scientific applications of time series analysis, where $X_1(t), \cdots, X_q(t)$ are interpreted as non-deterministic functions of the time variable t, questions of a similar nature are asked. The variance of a random variable is replaced by the seemingly different concept of power, which is, intuitively and imprecisely,

$$\lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} X_i^2(t)\, dt$$

for the ith time series. Moreover, the magnitude of the linear relationship between time series is generally thought of as the degree to which one series can be transformed into the other by a linear filter.

A satisfactory mathematical model for the idealization of non-deterministic time series within which these concepts can be made precise is available in the structure of weakly stationary stochastic processes. The variance-covariance matrix of the random variables $X_1, \cdots, X_q$ has a natural analog in the spectral density matrix of the q-variate process $X_1(t), \cdots, X_q(t)$. This matrix-valued function measures the variation of the power (now also the variances of the $X_i(t)$'s) in the component time series and the linear interrelationships between the time series as a function of frequency. The coefficient of coherence, a version of which was originally defined by Wiener, is the natural analog to the ordinary correlation coefficient and, in fact, is virtually the same function of the spectral density matrix as the correlation coefficient is of the variance-covariance matrix. In (Koopmans, 1964) the coefficient of coherence was shown to enjoy all of the important properties of the correlation coefficient.

The purpose of this paper is to establish, within the same framework, the existence of the appropriate frequency dependent analogs of the partial and multiple correlation coefficients and the residual variance. The important structural relationships that hold in the case of the correlation coefficients, including the inductive formulae for constructing the partial and multiple correlation coefficients, are established for the partial and multiple coefficients of coherence. A more formal outline of the necessary tools and of the results is given in the next section.

2. Preliminaries. Let $\mathbf{X} = \{X(t) : -\infty < t < \infty\}$ be a q-variate, continuous-in-the-mean, weakly stationary stochastic process. That is, $X(t)$ is a column vector $(X_1(t), \cdots, X_q(t))'$ of complex-valued random variables over a probability space $(\Omega, \mathcal{A}, P)$ such that

(i) $EX_j(t) = 0, j = 1, 2, \cdots, q, -\infty < t < \infty$ and

(ii) $EX_j(t + \tau)\overline{X_k(t)}$ is finite in absolute value for $-\infty < t, \tau < \infty$ and depends, functionally, on τ alone for $j, k = 1, 2, \cdots, q$.

The components of $\mathbf{X}$ are, thus, elements of the Hilbert Space $\mathcal{L}_2(P)$ of complex-valued random variables, X, for which $EX = 0$ and inner product is given by $\langle X, Y\rangle = EX\bar{Y}$.

We shall assume that the spectral distribution of **X** is absolutely continuous with respect to a given Lebesgue-Stieltjes measure, μ, and shall denote the spectral density matrix of **X** with respect to μ by $\Sigma(\lambda)$. Thus, $\Sigma(\lambda) = [\sigma_{jk}(\lambda)]$ is a $q \times q$ matrix of complex-valued, measurable functions which is non-negative definite and hermitian almost everywhere with respect to μ, (abbreviated a.e. (μ)).

Let $\mathbf{G}(\Sigma)$ denote the class of all $q \times q$ matrices, $\Lambda(\lambda)$, of complex-valued, measurable functions for which

$$\|\Lambda\|^2 = \operatorname{tr} \int \Lambda\Sigma\Lambda^* \, d\mu < \infty,$$

where * denotes conjugate transpose and tr denotes trace. The range of integration is always taken to be $(-\infty, \infty)$. Elements Λ and Π of $\mathbf{G}(\Sigma)$ are identified if $\|\Lambda - \Pi\| = 0$. Then, $\mathbf{G}(\Sigma)$ is a Hilbert Space over the complex field relative to the inner product

$$\langle \Lambda, \Pi \rangle = \operatorname{tr} \int \Lambda\Sigma\Pi^* \, d\mu.$$

Similarly, the class $\mathbf{H}(\Sigma)$ of row vectors $x(\lambda) = (x_1(\lambda), \cdots, x_q(\lambda))$ of measurable, complex-valued functions for which

$$\|x\|^2 = \int x\Sigma x^* \, d\mu < \infty,$$

(again identified by norm equivalence) is a Hilbert Space with inner product $\langle x, y \rangle = \int x\Sigma y^* \, d\mu$.

If h and g are arbitrary measurable functions of λ, we write $h = g$ a.e.(Σ) if $\operatorname{tr} \int_A \Sigma \, d\mu = 0$, where $A = [h \neq g]$.

We denote by $\mathfrak{M}$ the linear manifold in $\mathfrak{L}_2(P)$ spanned by the components of **X** (i.e. $\mathfrak{M}$ consists of all finite linear combinations of the form

$$\sum_{i=1}^{m} \sum_{j=1}^{q} a_{ij} X_j(t_i),$$

where the a_{ij} are complex constants and m is an arbitrary integer) and by $\bar{\mathfrak{M}}$ its $\mathfrak{L}_2(P)$ closure which will be termed the subspace spanned by **X**. Similarly, $\mathfrak{M}_{i_1,\cdots,i_p}$ and $\bar{\mathfrak{M}}_{i_1,\cdots,i_p}$ will denote the manifold and subspace spanned by $\mathbf{X}_{i_1,\cdots,i_p} = \{(X_{i_1}(t), \cdots, X_{i_p}(t))' : -\infty < t < \infty\}$.

There exists an inner product preserving linear isomorphism, $\leftrightarrow$, between the subspace $\bar{\mathfrak{M}}$ and $\mathbf{H}(\Sigma)$ defined by

$$X \leftrightarrow x \quad \text{if} \quad X = \int x(\lambda) \, dZ(\lambda),$$

where $Z(\lambda) = (Z_1(\lambda), \cdots, Z_q(\lambda))'$ is the spectral process determined by **X**.

Let **U** denote the group of unitary transformations, U_t, on $\bar{\mathfrak{M}}$ onto $\bar{\mathfrak{M}}$ determined by the equations

$$U_t X_j(u) = X_j(t + u), \qquad j = 1, 2, \cdots, q, \qquad -\infty < t, u < \infty.$$

By an immediate extension of Theorem A of (Koopmans, 1964) a linear isomorphism (also denoted by $\leftrightarrow$) can be established between $\mathbf{G}(\Sigma)$ and the class, $\mathbf{T}(X)$, of all linear transformations which transform $\mathbf{X}$ again into q-variate, continuous-in-the-mean, weakly stationary processes relative to the group, $\mathbf{U}$. That is, if $T \varepsilon \mathbf{T}(X)$, then the elements, $TX_j(t)$, of the transformed process $T\mathbf{X}$ are in $\mathfrak{M}$ and satisfy the equations

$$U_t TX_j(u) = TX_j(u + t), \quad j = 1, 2, \cdots, q, \quad -\infty < t, u < \infty.$$

This implies that $R\mathbf{X}$ and $T\mathbf{X}$ are stationarily correlated with each other for all R, $T \varepsilon \mathbf{T}(X)$ and, in particular, $\mathbf{X}$ is stationarily correlated with every $T\mathbf{X}$.

The isomorphism between $\mathbf{G}(\Sigma)$ and $\mathbf{T}(X)$ can be described as follows. If $T \leftrightarrow \Lambda$, then for every $X \varepsilon \mathfrak{M}$ for which $X \leftrightarrow x \varepsilon \mathbf{H}(\Sigma)$ and

$$\int x\Lambda\Sigma\Lambda^* x^* \, d\mu < \infty$$

we have

$$TX = \int x\Lambda \, dZ.$$

I.e. $\mathfrak{D}(T)$, the domain of T, is isomorphic to $\mathbf{H}(\Lambda\Sigma\Lambda^*)$ and T acts as an integrated matrix operator on this domain. We will treat T as though it were a matrix operator when applied to the elements of $\mathbf{X}$ and define the Gramian of q-vectors of elements of $\mathfrak{L}_2(P)$, $X = (X_1, \cdots, X_q)'$, $Y = (Y_1, \cdots, Y_q)'$, as the $q \times q$ matrix $\langle\langle X, Y\rangle\rangle$ whose i, jth element is $\langle X_i, Y_j\rangle$. The parts of Theorem A (Koopmans, 1964) that will be required in this paper are stated in the appropriate form as follows.

THEOREM A 1. *If $T \varepsilon \mathbf{T}(X)$, $\Lambda \varepsilon \mathbf{G}(\Sigma)$ and $T \leftrightarrow \Lambda$, then the q-variate, weakly stationary process $T\mathbf{X}$ has the spectral representation*

$$TX(t) = \int e^{it\lambda}\Lambda(\lambda) \, dZ(\lambda), \quad -\infty < t < \infty.$$

Moreover, if $x \leftrightarrow X \varepsilon \mathfrak{D}(T)$,

$$TX = \int x(\lambda)\Lambda(\lambda) \, dZ(\lambda).$$

The spectral distribution of $T\mathbf{X}$ is absolutely continuous with respect to μ and has spectral density $\Lambda\Sigma\Lambda^*$.

2. *Let $T \leftrightarrow \Lambda$ and $R \leftrightarrow \Pi$. Then if $x \leftrightarrow X \varepsilon \mathfrak{D}(T)$, $y \leftrightarrow Y \varepsilon \mathfrak{D}(R)$,*

$$\langle TX, RY\rangle = \int x\Lambda\Sigma\Pi^* y^* \, d\mu.$$

In particular, the cross covariances of $T\mathbf{X}$ and $R\mathbf{X}$ have the spectral representation,

$$\langle\langle TX(t), RX(s)\rangle\rangle = \int e^{i(t-s)\lambda}\Lambda(\lambda)\Sigma(\lambda)\Pi^*(\lambda) \, d\mu(\lambda), \quad -\infty < t, s < \infty.$$

3. *Further, if the matrix product* $\Lambda\Pi \,\varepsilon\, \mathbf{G}(\Sigma)$ *and* $\mathfrak{D}(T) = \mathfrak{D}(R) = \bar{\mathfrak{M}}$, *then the composite operator* RT *is in* $\mathbf{T}(X)$ *with* $\mathfrak{D}(RT) = \bar{\mathfrak{M}}$ *and* $RT \leftrightarrow \Lambda\Pi$.

A corollary of this theorem, which will be needed in the sequel, characterizes the class, $\mathbf{T}^\pi(X)$, of orthogonal projections in $\mathbf{T}(X)$. The proof given in (Koopmans, 1964) for the bivariate processes can be easily extended to the q-variate case.

COROLLARY A. *Let* $\mathbf{G}^\pi(\Sigma)$ *be the collection of* $\Lambda \,\varepsilon\, \mathbf{G}(\Sigma)$ *for which*

(i) $\Lambda^2 = \Lambda$, *and*

(ii) $\Lambda\Sigma = \Sigma\Lambda^*$ a.e.(Σ) *and such that*

(iii) $\int x\Lambda\Sigma\Lambda^* x^* \, d\mu \leqq \int x\Sigma x^* \, d\mu$ *for all* $x \,\varepsilon\, \mathbf{H}(\Sigma)$. *Then,* $T \,\varepsilon\, \mathbf{T}^\pi(X)$ *if and only if* $T \leftrightarrow \Lambda$ *for some* $\Lambda \,\varepsilon\, \mathbf{G}^\pi(\Sigma)$. *Moreover,* $\Lambda \,\varepsilon\, \mathbf{G}^\pi(\Sigma)$ *and* $\Lambda \leftrightarrow T$ *implies* $I_q - \Lambda \,\varepsilon\, \mathbf{G}^\pi(\Sigma)$, *where* I_q *is the* $q \times q$ *identity matrix, and* $I_q - \Lambda \leftrightarrow I - T$. *The spectral density of* $(I - T)\mathbf{X}$ *is*

$$\Sigma - \Lambda\Sigma\Lambda^* \text{ a.e.}(\mu).$$

(That $\Lambda \,\varepsilon\, \mathbf{G}(\Sigma)$ is an easy consequence of Condition iii so that this need not be specified in the statement of the Corollary.)

To motivate the definitions of the partial and multiple coefficients of coherence it is useful to recall in a somewhat different form than is usually given, the definitions of the partial and multiple correlation coefficients for random variables $X_1, X_2, \cdots, X_q$ in $\mathfrak{L}_2(P)$.

Let $\mathfrak{M}_{i_1,\cdots,i_p}$ be the collection of all complex linear combinations of the random variables $X_{i_1}, \cdots, X_{i_p}$. Then $\hat{\mathfrak{M}}_{i_1,\cdots,i_p}$ is a linear subspace of $\hat{\mathfrak{M}} = \hat{\mathfrak{M}}_{1,\cdots,q}$ and there exists a unique orthogonal projection $\hat{\Pi}_{i_1,\cdots,i_p}$ on $\hat{\mathfrak{M}}$ with range equal to $\hat{\mathfrak{M}}_{i_1,\cdots,i_p}$.

DEFINITION 1. *The partial correlation coefficient* $\hat{\rho}_{ij\cdot i_1,\cdots,i_p}$, *of* X_i *and* X_j *given* $X_{i_1}, \cdots, X_{i_p}$ *is then the ordinary correlation coefficient of* $(I - \hat{\Pi}_{i_1,\cdots,i_p}) X_i$ *and* $(I - \hat{\Pi}_{i_1,\cdots,i_p})X_j$, *where* I *is the identity operator on* $\hat{\mathfrak{M}}$.

DEFINITION 2. *The multiple correlation coefficient,* $\hat{R}_{i\cdot i_1,\cdots,i_p}$, *of* X_i *given* $X_{i_1}, \cdots, X_{i_p}$ *is the ordinary correlation coefficient of* X_i *and* $\hat{\Pi}_{i_1,\cdots,i_p}X_i$.

The properties of and interrelationships between these correlation coefficients are determined by the projections $\hat{\Pi}_{i_1,\cdots,i_p}$ and the properties of the ordinary correlation coefficients. Due to the close analogy between the ordinary correlation coefficient and the coefficient of coherence for pairs of stationarily correlated, weakly stationary processes (Koopmans, 1964) it is reasonable to expect that a similar theory can be derived for multivariate time series provided the orthogonal projection, $\Pi_{i_1,\cdots,i_p}$, on $\bar{\mathfrak{M}}$ with range $\bar{\mathfrak{M}}_{i_1,\cdots,i_p}$, maps $\mathbf{X}$ into a q-variate weakly stationary process which is stationarily correlated with $\mathbf{X}$; i.e. provided $\Pi_{i_1,\cdots,i_p} \,\varepsilon\, \mathbf{T}^\pi(X)$. Section 3 is devoted to establishing this result. The partial and multiple coefficients of coherence are introduced in Section 4 and several of the properties with correlation coefficient analogs are derived. The close resemblance between the coherence expressions and those for the correlation coefficients are demonstrated when $\det \Sigma(\lambda) > 0$.

3. Proof that $\Pi_{i_1,\cdots,i_p} \,\varepsilon\, \mathbf{T}^\pi(X)$. It suffices to establish the result for projections of the form $\Pi_r = \Pi_{r,r+1,\cdots,q}$ since the general projection can be reduced to this form by relabeling the component processes of $\mathbf{X}$. We will show that $\Pi_r \,\varepsilon\, \mathbf{T}^\pi(X)$ by establishing that there always exist a $G \,\varepsilon\, \mathbf{G}^\pi(\Sigma)$ such that the range, $\mathcal{R}(T)$, of $T \leftrightarrow G$ is $\mathfrak{M}_{r,r+1,\cdots,q}$. The uniqueness of orthogonal projection will then imply $T = \Pi_r$.

A $q \times q$ matrix, G, will be said to be (n_1, n_2) partitioned if n_1 and n_2 are positive integers such that $n_1 + n_2 = q$ and

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

where G_{ij} is a matrix of dimension $n_i \times n_j$, $1 \leqq i, j \leqq 2$. Similarly, a q row (column) vector x is said to be (n_1, n_2) partitioned if $x = (x_1, x_2)((x_1, x_2)')$ and x_i is a n_i row (column) vector. Unless otherwise specified, throughout the remainder of this section the matrix valued function Σ and the elements of $\mathbf{G}(\Sigma)$ and $\mathbf{H}(\Sigma)$ will be $(r - 1, q - r + 1)$ partitioned.

Lemma 1.

$$G_r(\lambda) = \begin{bmatrix} 0 & A(\lambda) \\ 0 & I \end{bmatrix} \varepsilon\, \mathbf{G}^\pi(\Sigma)$$

if and only if $A(\lambda)$ *is a measurable solution of the matrix equation*

$$A(\lambda)\Sigma_{22}(\lambda) = \Sigma_{12}(\lambda) \text{ a.e.}(\mu), \tag{1}$$

where $\Sigma(\lambda) = [\Sigma_{ij}(\lambda)]i, j = 1, 2$; 0 *denotes a matrix of zeros*; *and* I *is the identity matrix.*

Proof. We suppress the argument λ in all proofs except where needed for clarity. If $G_r \,\varepsilon\, \mathbf{G}^\pi(\Sigma)$, then A is measurable and, by virtue of Condition (ii) of Corollary A, $G_r\Sigma = \Sigma G_r^*$ a.e.(Σ) But $\mathrm{tr}\Sigma = 0$ implies $\Sigma = 0$. Thus, $[G_r\Sigma \neq \Sigma G_r^*] \subseteq [\mathrm{tr}\Sigma \neq 0]$ which implies $G_r\Sigma = \Sigma G_r^*$ a.e.(μ). A simple multiplication establishes that,

$$G_r\Sigma = \begin{bmatrix} A\Sigma_{21} & A\Sigma_{22} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \quad \text{and} \quad \Sigma G_r^* = \begin{bmatrix} \Sigma_{12}A^* & \Sigma_{12} \\ \Sigma_{22}A^* & \Sigma_{22} \end{bmatrix}. \tag{2}$$

Thus, equating the components of these matrices, Equation (1) is seen to be satisfied.

Now, let $G_r = \begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix}$ where A is a measurable solution of Equation 1. It is easily seen that $G_r^2 = G_r$ a.e.(μ) which implies Condition (i) of Corollary A.

Since Σ is hermitian, $\Sigma_{21} = \Sigma_{12}^*$ and $\Sigma_{22}^* = \Sigma_{22}$. Thus, $A\Sigma_{22} = \Sigma_{12}$ implies $\Sigma_{22}A^* = \Sigma_{21}$. Also, $\Sigma_{12}A^* = A\Sigma_{22}A^* = (A\Sigma_{22}A^*)^* = A\Sigma_{21}$. Thus, by Equation 2, it follows that $G_r\Sigma = \Sigma G_r^*$ a.e.(μ) and Condition (ii) is established.

To establish Condition (iii), let $H = \begin{bmatrix} I & -A \\ 0 & 0 \end{bmatrix}$. Since Σ is non-negative definite hermitian a.e.(μ), $xH\Sigma H^*x^* \geqq 0$ a.e.(μ) for all $x \,\varepsilon\, \mathbf{H}(\Sigma)$. Then, applying

Equation (2),

$$H\Sigma H^* = \begin{bmatrix} \Sigma_{11} - A\Sigma_{22}A^* & 0 \\ 0 & 0 \end{bmatrix} \text{ a.e.}(\mu).$$

Thus, $xH\Sigma H^*x^* = x_1(\Sigma_{11} - A\Sigma_{22}A^*)x_1^* \geqq 0$ a.e.(μ), where $x = (x_1, x_2)$. Now,

$$G_r\Sigma G_r^* = \begin{bmatrix} A\Sigma_{22}A^* & A\Sigma_{22} \\ \Sigma_{22}A^* & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} A\Sigma_{22}A^* & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \text{ a.e.}(\mu)$$

by virtue of Equation (2). Thus,

$$x\Sigma x^* - xG_r\Sigma G_r^*x^* = x_1(\Sigma_{11} - A\Sigma_{22}A^*)x_1^* \geqq 0 \text{ a.e.}(\mu).$$

It follows that for all $x \varepsilon \mathbf{H}(\Sigma)$,

$$\int xG_r\Sigma G_r^*x^*\, d\mu \leqq \int x\Sigma x^*\, d\mu,$$

and Condition (iii) of Corollary A is satisfied.

LEMMA 2. *The matrix equation*

$$A(\lambda)\Sigma_{22}(\lambda) = \Sigma_{12}(\lambda)$$

has a solution at each λ for which $\Sigma(\lambda)$ is non-negative definite, hermitian. The class of solutions of this equation possesses a measurable member.

PROOF. Fix λ at a value for which $\Sigma = \Sigma(\lambda)$ is non-negative definite, hermitian. The eigenvalues of Σ are then non-negative real numbers which we will label to satisfy the inequalities $\xi_1 \geqq \xi_2 \geqq \cdots \geqq \xi_q \geqq 0$. The principal axis theorem for hermitian forms (Gantmacher, 1959, p. 337) yields the representation

$$\Sigma = U \operatorname{diag}\{\xi_1, \xi_2, \cdots, \xi_q\} U^*$$

where $U^*U = I$ and $\operatorname{diag}\{\xi_1, \xi_2, \cdots, \xi_q\}$ is the square matrix with main diagonal elements $\xi_1, \xi_2, \cdots, \xi_q$ and zeros elsewhere.

Let s be the integer for which $\xi_s > 0$ and $\xi_{s+1} = 0$, and define the $q \times s$ matrix, D, by

$$D = \begin{bmatrix} Q \\ 0 \end{bmatrix},$$

where $Q = \operatorname{diag}\{\xi_1^{\frac{1}{2}}, \xi_2^{\frac{1}{2}}, \cdots, \xi_s^{\frac{1}{2}}\}$ and 0 is the $(q - s) \times s$ matrix of zeros.

Then, $DD^* = \operatorname{diag}\{\xi_1, \xi_2, \cdots, \xi_q\}$ and $D^*D = \operatorname{diag}\{\xi_1, \xi_2, \cdots, \xi_s\}$ which is a non-singular $s \times s$ matrix. The principal axis representation can now be written in the form

$$\Sigma = \begin{bmatrix} \Sigma_1 \\ \Sigma_2 \end{bmatrix} [\Sigma_1^*, \Sigma_2^*] = \begin{bmatrix} \Sigma_1\Sigma_1^* & \Sigma_1\Sigma_2^* \\ \Sigma_2\Sigma_1^* & \Sigma_2\Sigma_2^* \end{bmatrix},$$

where Σ_1 is the $r - 1 \times s$ matrix and Σ_2 the $q - r + 1 \times s$ matrix defined by

$$\begin{bmatrix} \Sigma_1 \\ \Sigma_2 \end{bmatrix} = UD.$$

It follows that $\Sigma_{ij} = \Sigma_i \Sigma_j^*$, $i, j = 1, 2$ in the $(r - 1, q - r + 1)$ partition of Σ.

Now, the equation $A\Sigma_{22} = \Sigma_{12}$ has a solution if and only if the rows of Σ_{12} are contained in the linear span of the rows of Σ_{22}. A necessary and sufficient condition for this to be the case is that for every non zero $q - r + 1$ column vector, b, if $\Sigma_{22}b = 0$ then $\Sigma_{12}b = 0$. But $\Sigma_{22}b = 0$ implies $b^*\Sigma_2\Sigma_2^*b = 0$ and, thus, $\Sigma_2^*b = 0$. Then $\Sigma_{12}b = \Sigma_1\Sigma_2^*b = 0$.

The matrix A can be determined explicitly by an application of the principal axis theorem to the matrix Σ_{22}. Let

$$\Sigma_{22} = U_2 \operatorname{diag}\{\xi_1^{(2)}, \xi_2^{(2)}, \cdots, \xi_{q-r+1}^{(2)}\} U_2^*$$

where $\xi_1^{(2)} \geqq \cdots \geqq \xi_{s_1}^{(2)} > 0$, $\xi_{s_1+1}^{(2)} = \cdots = \xi_{q-r+1}^{(2)} = 0$ are the eigenvalues of Σ_{22}, and let the $q - r + 1 \times s_1$ matrix D_2 be defined as was D above. Then, if $\Sigma_{(2)} = U_2D_2$, it can be shown (see e.g. Marsaglia, 1963) that

$$\begin{aligned} A &= \Sigma_{12}\Sigma_{(2)}(\Sigma_{(2)}^*\Sigma_{(2)})^{-2}\Sigma_{(2)}^*, && \text{for} \quad s_1 > 0 \\ &= 0 && \text{for} \quad s_1 = 0. \end{aligned} \tag{3}$$

It is seen from the definition of $\Sigma_{(2)}$ that when $s_1 = q - r + 1$ (i.e. $\det\Sigma_{22} > 0$) $A = \Sigma_{12}\Sigma_{22}^{-1}$ as it should be.

This explicit expression for A is easily seen to make A a measurable function of the elements of Σ_{12}, the eigenvalues of Σ_{22} and the elements of the unitary matrix U_2. Since it it well known that the solution of an algebraic equation is a continuous function of the coefficients of the polynomial, the eigenvalues of Σ_{22} are measurable functions of the elements of Σ_{22}. Thus, the measurability of A, and consequently of G_r, depends on the measurability of the elements of U_2. In order not to further defer the main result, the existence of a measurable version of U_2 will be established in the Appendix.

THEOREM 1. *Let* $T_r \leftrightarrow G_r \varepsilon \mathbf{G}^\pi(\Sigma)$ *where* G_r *is as defined in Lemma* 1. *Then* $\mathfrak{R}(T_r) = \mathfrak{M}_{r,r+1,\cdots,q}$. *Thus* $\Pi_r = T_r \varepsilon \mathbf{T}^\pi(X)$.

PROOF. In $(r - 1, q - r + 1)$ partitioned form, the spectral representation for $T_r\mathbf{X}$ may be written

$$T_rX_1(t) = \int e^{it\lambda}A(\lambda)\, dZ_2(\lambda), \tag{4}$$

$$T_rX_2(t) = \int e^{it\lambda}\, dZ_2(\lambda) = X_2(t), \qquad -\infty < t < \infty. \tag{5}$$

The corresponding partitions of the subspaces will be written $\mathfrak{M}_1 = \mathfrak{M}_{1,\cdots,r-1}$, $\mathfrak{M}_2 = \mathfrak{M}_{r,\cdots,q}$, etc. In the proof of Lemma 1 it was shown that for every r vector $x \varepsilon \mathbf{H}(\Sigma_{11})$,

$$\int xA\Sigma_{22}A^*x^* \, d\mu < \infty.$$

This implies $xA \varepsilon \mathbf{H}(\Sigma_{22})$. By Theorem A and Equation (4) it follows that $T_r\mathfrak{M}_1 \subseteq \bar{\mathfrak{M}}_2$. By Equation (5), it is seen that $T_r\mathfrak{M}_2 \subseteq \bar{\mathfrak{M}}_2$. Thus, since T_r is continuous, $T_r\bar{\mathfrak{M}} = T_r(\overline{\mathfrak{M}_1 + \mathfrak{M}_2}) \subseteq \bar{\mathfrak{M}}_2$ or $\mathfrak{R}(T_r) \subseteq \bar{\mathfrak{M}}_2$. But Equation (5) implies that the generators of $\bar{\mathfrak{M}}_2$ are in $\mathfrak{R}(T_r)$. Since $\mathfrak{R}(T_r)$ is a closed linear subspace, $\bar{\mathfrak{M}}_2 \subseteq \mathfrak{R}(T_r)$ and the theorem is proved..

4. Applications of Theorem 1—The multivariate coefficients of coherence. Let $\Pi_{i_1,\cdots,i_p}$ be the projection in $\mathbf{T}^\pi(X)$ with range $\bar{\mathfrak{M}}_{i_1,\cdots,i_p}$ as defined above and let I denote the identity element in $\mathbf{T}^\pi(X)$. Then $(I - \Pi_{i_1,\cdots,i_p}) \varepsilon \mathbf{T}^\pi(X)$ by Corollary A. Denote the spectral density matrix of the q-variate process $(I - \Pi_{i_1,\cdots,i_p})\mathbf{X}$ by

$$\Sigma^{\perp}_{i_1,\cdots,i_p}(\lambda) = [\sigma^{\perp}_{ij\cdot i_1,\cdots,i_p}(\lambda)].$$

Then, by analogy with the partial correlation coefficient (Anderson, 1958, p. 29), we will define the *partial coefficient of coherence*, $\rho_{ij\cdot i_1,\cdots,i_p}(\lambda)$, of $\mathbf{X}_i$ and $\mathbf{X}_j$ to be the ordinary coefficient of coherence (Koopmans, 1964) of the univariate processes $(I - \Pi_{i_1,\cdots,i_p})\mathbf{X}_i$ and $(I - \Pi_{i_1,\cdots,i_p})\mathbf{X}_j$:

$$(6) \qquad \begin{aligned} \rho_{ij\cdot i_1,\cdots,i_p}(\lambda) &= \frac{|\sigma^{\perp}_{ij\cdot i_1,\cdots,i_p}(\lambda)|}{[\sigma^{\perp}_{ii\cdot i_1,\cdots,i_p}(\lambda)\sigma^{\perp}_{jj\cdot i_1,\cdots,i_p}(\lambda)]^{\frac{1}{2}}} \quad \text{when the denominator is positive,} \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

$1 \leqq i, j, i_1, \cdots, i_p \leqq q$.

Similarly, the *multiple coefficient of coherence*, $R_{i\cdot i_1,\cdots,i_p}(\lambda)$, will be defined as the ordinary coefficient of coherence of the univariate processes $\mathbf{X}_i$ and $\Pi_{i_1,\cdots,i_p}\mathbf{X}_i$. Let

$$\Sigma_{i_1,\cdots,i_p}(\lambda) = [\sigma_{ij\cdot i_1,\cdots,i_p}(\lambda)]$$

denote the spectral density matrix of $\Pi_{i_1,\cdots,i_p}\mathbf{X}$, and let

$$M_{i_1,\cdots,i_p}(\lambda) = [\mu_{ij\cdot i_1,\cdots,i_p}(\lambda)]$$

be the $q \times q$ matrix valued integrand in the spectral representation of $\langle\langle X(t), \Pi_{i_1,\cdots,i_p}X(s)\rangle\rangle$ given in Theorem A. Then $\mu_{ii\cdot i_1,\cdots,i_p}(\lambda)$ is the cross spectral density of $\mathbf{X}_i$ and $\Pi_{i_1,\cdots,i_p}\mathbf{X}_i$. The multiple coefficient of coherence can now be defined formally by the following expression:

$$(7) \qquad \begin{aligned} R_{i\cdot i_1,\cdots,i_p}(\lambda) &= \frac{|\mu_{ii\cdot i_1,\cdots,i_p}(\lambda)|}{[\sigma_{ii}(\lambda)\sigma_{ii\cdot i_1,\cdots,i_p}(\lambda)]^{\frac{1}{2}}} \quad \text{when the denominator is positive,} \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

$1 \leqq i, i_1, \cdots, i_p \leqq q$.

The remainder of this section will be devoted to establishing other expressions for the partial and multivariate coefficients of coherence and to the derivations of some of the interrelationships between these quantities which have analogs in multivariate analysis. To do this it will again be convenient to relabel the components of the process $\mathbf{X}$ so that $1 \leqq i, j \leqq r - 1$ and $\{i_1, \cdots, i_p\} = \{r, r + 1, \cdots, q\}$.

Whenever the denominator of the coefficients of coherence 6 and 7 are zero the numerators are also zero. We will hereafter eliminate the necessity of specifying separately the value of such indeterminate ratios by adopting the convention $0/0 = 0$.

In the notation of Section 3, let $\Pi_r \leftrightarrow G_r(\lambda)$ and let $G_r(\lambda)$ and $\Sigma(\lambda)$ be $(r - 1, q - r + 1)$ partitioned. $A(\lambda)$ is as defined in Lemma 1.

THEOREM 2. *Let $a_{(i)}(\lambda)$ denote the* ith *row of $A(\lambda)$. Then for $i, j = 1, 2, \cdots, r - 1$,*

$$\sigma^{\perp}_{ij\cdot r,\cdots,q}(\lambda) = \sigma_{ij}(\lambda) - a_{(i)}(\lambda)\Sigma_{22}(\lambda)a^*_{(j)}(\lambda) \text{ a.e.}(\mu).$$

PROOF. This is an immediate consequence of Corollary A. Note that on the set $[\det \Sigma_{22}(\lambda) > 0]$, $a_{(i)}\Sigma_{22}a^*_{(j)} = \sigma_{(i)}\Sigma_{22}^{-1}\sigma^*_{(j)}$ where $\sigma_{(i)}$ is the ith row of Σ_{12}. The definition of the partial coefficient of coherence given by Equation (6) then takes on the adaptation of the familiar form from multivariate analysis to complex random variables (Anderson, 1958, p. 29).

THEOREM 3. $R^2_{i\cdot r,\cdots,q}(\lambda) = \dfrac{a_{(i)}(\lambda)\Sigma_{22}(\lambda)a^*_{(i)}(\lambda)}{\sigma_{ii}(\lambda)}$ a.e.(μ).

PROOF. From the properties of the elements of $\mathbf{G}^\pi(\Sigma)$ and from Theorem A,

$$M_{r,\cdots,q} = G_r\Sigma = G_r\Sigma G_r^* = \Sigma_{r,\cdots,q} \text{ a.e.}(\mu).$$

Thus, $\mu_{ii\cdot r,\cdots,q} = \sigma_{ii\cdot r,\cdots,q} = [A\Sigma_{22}A^*]_{ii} = a_{(i)}\Sigma_{22}a^*_{(i)}$ by a simple computation. The result now follows from Equation (7).

On the set $[\det\Sigma_{22}(\lambda) > 0]$ we get the more familiar result

$$R^2_{i\cdot r,\cdots,q}(\lambda) = \frac{\sigma_{(i)}(\lambda)\Sigma_{22}^{-1}(\lambda)\sigma^*_{(i)}(\lambda)}{\sigma_{ii}(\lambda)},$$

(see Anderson, 1958, p. 32).

COROLLARY 1. $\sigma^{\perp}_{ii\cdot r,\cdots,q}(\lambda) = (1 - R^2_{i\cdot r,\cdots,q}(\lambda))\sigma_{ii}(\lambda)$ a.e.(μ).

PROOF. Immediate from Theorems 2 and 3. This is Equation (23) of (Anderson, 1958, p. 32). In the present context, $1 - R^2_{i\cdot r,\cdots,q}(\lambda)$ can be interpreted as the proportion of the spectral density of $\mathbf{X}_i$ at frequency λ which remains after removing the influence of the linear regression of $\mathbf{X}_i$ on $\mathbf{X}_r, \mathbf{X}_{r+1}, \cdots, \mathbf{X}_q$.

We now derive two relationships which make it possible to construct the multivariate coefficients of coherence inductively from the ordinary coefficients of coherence for pairs of the component processes.

THEOREM 4.

$$\gamma_{ij\cdot r,\cdots,q}(\lambda) = \frac{\gamma_{ij\cdot r+1,\cdots,q}(\lambda) - \gamma_{ir\cdot r+1,\cdots,q}(\lambda)\bar{\gamma}_{jr\cdot r+1,\cdots,q}(\lambda)}{[(1 - \rho^2_{ir\cdot r+1,\cdots,q}(\lambda))(1 - \rho^2_{jr\cdot r+1,\cdots,q}(\lambda))]^{\frac{1}{2}}} \tag{8}$$

where $\rho_{ij\cdot r,\cdots,q}(\lambda) = |\gamma_{ij\cdot r,\cdots,q}(\lambda)|$ a.e.(μ) *for* $1 \leqq i, j < r \leqq q$, *and*

$$\gamma_{ij\cdot q+1}(\lambda) = \sigma_{ij}(\lambda)/[\sigma_{ii}(\lambda)\sigma_{jj}(\lambda)]^{\frac{1}{2}}, \qquad 1 \leqq i, j \leqq q.$$

Proof. Let the projection Π_r be defined as above and let $G_r \varepsilon \mathbf{G}^{\pi}(\Sigma)$ be written in an $(r - 1, q - r + 1)$ partition as

$$G_r = \begin{bmatrix} 0 & A_r \\ 0 & I \end{bmatrix}.$$

Since $\Pi_r\Pi_{r+1} = \Pi_{r+1}$ it follows that $(I - \Pi_r) = (I - \Pi_r)(I - \Pi_{r+1})$ and, by Theorem A,

$$\Sigma_r^{\perp} = (I - G_r)\Sigma_{r+1}^{\perp}(I - G_r)^*, \qquad 1 \leqq r < q \tag{9}$$

where, as defined earlier,

$$\Sigma_r^{\perp} = (I - G_r)\Sigma(I - G_r)^*, \qquad 1 \leqq r \leqq q. \tag{10}$$

(All equations hold a.e.(μ).) From the form of G_{r+1}, it follows from Equation (10) that the $(r, q - r)$ partition of $\Sigma_{r+1}^{\perp}$ is

$$\Sigma_{r+1}^{\perp} = \begin{bmatrix} \Sigma_{11} - A_r\Sigma_{22}A_r^* & 0 \\ 0 & 0 \end{bmatrix}, \tag{11}$$

where the partition of Σ is also $(r, q - 1)$.

If the $(r - 1, q - r + 1)$ partition of $\Sigma_{r+1}^{\perp}$ is written $\Sigma_{r+1}^{\perp} = [\Sigma_{ij\cdot r+1}^{\perp}]$, $i, j = 1, 2$, then it is easy to show that

$$\Sigma_{11\cdot r+1}^{\perp} = [\sigma_{ij\cdot r+1,\cdots,q}], \qquad 1 \leqq i, j \leqq r - 1$$

$$\Sigma_{12\cdot r+1}^{\perp} = \begin{bmatrix} \sigma_{1r\cdot r+1,\cdots,q}^{\perp} & 0 & \cdots & 0 \\ \sigma_{2r\cdot r+1,\cdots,q}^{\perp} & 0 & \cdots & 0 \\ \vdots & \vdots & & \\ \sigma_{r-1,r\cdot r+1,\cdots,q}^{\perp} & 0 & \cdots & 0 \end{bmatrix}$$

and

$$\Sigma_{22\cdot r+1}^{\perp} = \begin{bmatrix} \sigma_{rr\cdot r+1,\cdots,q}^{\perp} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

An application of Equation (9) yields the $(r - 1, q - r + 1)$ partition

$$\Sigma_r^{\perp} = \begin{bmatrix} \Sigma_{11\cdot r+1}^{\perp} - A_r\Sigma_{22\cdot r+1}^{\perp}A_r^* & \Sigma_{12\cdot r+1}^{\perp} - A_r\Sigma_{22\cdot r+1}^{\perp} \\ \Sigma_{21\cdot r+1}^{\perp} - \Sigma_{22\cdot r+1}^{\perp}A_r^* & 0 \end{bmatrix}. \tag{12}$$

But, by Equation (11) with $r + 1$ replaced by r, the $(r - 1, q - r + 1)$ partition of $\Sigma_r^\perp$ is

$$\Sigma_r^\perp = \begin{bmatrix} \Sigma_{11\cdot r}^\perp & 0 \\ 0 & 0 \end{bmatrix},$$

where $\Sigma_{11\cdot r}^\perp = [\sigma_{ij\cdot r,\cdots,q}^\perp]$, $1 \leqq i, j \leqq r - 1$. Equating this expression to Equation (12) yields

$$\Sigma_{11\cdot r}^\perp = \Sigma_{11\cdot r+1}^\perp - A_r \Sigma_{22\cdot r+1}^\perp A_r^*$$

and

$$\Sigma_{12\cdot r+1}^\perp = A_r \Sigma_{22\cdot r+1}^\perp .$$

Equivalently,

(13) $$\sigma_{ij\cdot r,\cdots,q}^\perp = \sigma_{ij\cdot r+1,\cdots,q}^\perp - a_{ir}\sigma_{rr\cdot r+1,\cdots,q}^\perp \bar{a}_{jr}$$

and

(14) $$\sigma_{ir\cdot r+1,\cdots,q}^\perp = a_{ir}\sigma_{rr\cdot r+1,\cdots,q}^\perp \quad \text{for } 1 \leqq i, j \leqq r - 1,$$

by the above expressions for

$\Sigma_{ij\cdot r+1}^\perp$, $1 \leqq i, j \leqq 2$. Let $\gamma_{ij\cdot r,\cdots,q} = \sigma_{ij\cdot r,\cdots,q}^\perp / (\sigma_{ii\cdot r,\cdots,q}^\perp \sigma_{jj\cdot r,\cdots,q}^\perp)^{\frac{1}{2}}$,

(with the $0/0 = 0$ convention understood). Then Equation (14) may be rewritten

$$a_{ir}(\sigma_{rr\cdot r+1,\cdots,q}^\perp)^{\frac{1}{2}} = \gamma_{ir\cdot r+1,\cdots,q}(\sigma_{ii\cdot r+1,\cdots,q}^\perp)^{\frac{1}{2}}.$$

This equation and Equation (13) are now easily seen to imply Equation (8).

This derivation is similar to the computation leading to Equation (34) of (Anderson, 1958) which is the real variables equivalent of Equation (8). It is to be noted that only the minor problem of appropriately defining the indeterminate ratios result from the possible singularity of Σ.

The following lemma is Theorem 1 of (Koopmans, 1964).

LEMMA 4. *Let* $\mathbf{Y}$ *be a bivariate process* $(q = 2)$ *with components* $\mathbf{Y}_j$, $j = 1, 2$ *and spectral density matrix* $[\tau_{ij}(\lambda)]$, *and let* $\tau^{(j)}(\lambda) j = 1, 2$ *be the spectral densities of the univariate processes* $\mathbf{Y}^{(j)}$ *defined by*

$$\mathbf{Y}^{(1)} = P_2\mathbf{Y}_1, \ \mathbf{Y}^{(2)} = (I - P_2)\mathbf{Y}_1,$$

where P_2 *is the projection in* $\mathbf{T}^\pi(Y)$ *with range equal to the closed linear manifold spanned by* $\mathbf{Y}_2$. *Then*

$$\tau^{(1)}(\lambda) = \rho^2(\lambda)\tau_{11}(\lambda)$$

(15) *and*

$$\tau^{(2)}(\lambda) = (1 - \rho^2(\lambda))\tau_{11}(\lambda) \text{ a.e.}(\mu),$$

where $\rho(\lambda)$ *is the (ordinary) coefficient of coherence for* $\mathbf{Y}$.

THEOREM 5. $1 - R^2_{r\cdot r+1,\cdots,q}(\lambda) = (1 - \rho^2_{rr+1\cdot r+2,\cdots,q}(\lambda))(1 - R^2_{r\cdot r+2,\cdots,q}(\lambda))$ a.e.(μ) *for* $1 \leqq r \leqq q - 2$, *where*

$$R_{q-2\cdot q}(\lambda) = \rho_{q-2,q}(\lambda) \text{ a.e.}(\mu).$$

PROOF. It will be convenient, in this proof, to denote the closed linear manifolds generated by a bivariate process $\mathbf{Y}$ and its univariate components $\mathbf{Y}_j$, $j = 1, 2$ by $\mathfrak{M}\{\mathbf{Y}_1, \mathbf{Y}_2\}$ and $\mathfrak{M}\{\mathbf{Y}_j\}$, $j = 1, 2$ respectively.

If Lemma 4 is applied to $\mathbf{Y}_1 = \mathbf{X}_r$, $\mathbf{Y}_2 = \Pi_{r+1}\mathbf{X}_r$, then $P_2 = \Pi_{r+1}$ restricted to $\mathfrak{M}\{\mathbf{X}_r, \Pi_{r+1}\mathbf{X}_r\}$ and

$$\sigma[(I - \Pi_{r+1})\mathbf{X}_r](\lambda) = (1 - R^2_{r\cdot r+1,\cdots,q}(\lambda))\sigma_{rr}(\lambda) \text{ a.e.}(\mu), \tag{16}$$

by Equation (15) and the definition of the multiple coefficient of coherence. Here, $\sigma[\mathbf{U}](\lambda)$ denotes the spectral density of the univariate process U.

Similarly, a second application of Lemma 4 with $\mathbf{Y}_1 = \mathbf{X}_r$, $\mathbf{Y}_2 = (I - \Pi_{r+2})\mathbf{X}_r$ yields,

$$\sigma[(I - \Pi_{r+2})\mathbf{X}_r](\lambda) = (1 - R^2_{r\cdot r+2,\cdots,q}(\lambda))\sigma_{rr}(\lambda) \text{ a.e.}(\mu). \tag{17}$$

Let $\Delta_r = \Pi_{r+1} - \Pi_{r+2}$. Then $\Delta_r \, \varepsilon \, \mathbf{T}^{\pi}(X)$ and the relation

$$(I - \Pi_{r+1}) = (I - \Delta_{r+1})(I - \Pi_{r+2}) \tag{18}$$

is valid. Moreover, if $\Delta^{(r)}_{r+1}$ denotes the restriction of Δ_{r+1} to $\mathfrak{M}\{(I - \Pi_{r+2})\mathbf{X}_r, (I - \Pi_{r+2})\mathbf{X}_{r+1}\}$, it can easily be shown that the range of $\Delta^{(r)}_{r+1}$ is $\mathfrak{M}\{(I - \Pi_{r+2})\mathbf{X}_{r+1}\}$. Thus if Lemma 4 is applied a third time with

$$\mathbf{Y}_1 = (I - \Pi_{r+2})\mathbf{X}_r,\ \mathbf{Y}_2 = (I - \Pi_{r+2})\mathbf{X}_{r+1},$$

it follows that $P_2 = \Delta^{(r)}_{r+1}$ and, by Equation (15) and the definition of the partial coefficient of coherence,

$$\begin{aligned} \sigma[(I - \Delta_{r+1})(I - \Pi_{r+2})\mathbf{X}_r](\lambda) \\ = (1 - \rho^2_{rr+1\cdot r+2,\cdots,q}(\lambda))\sigma[(I - \Pi_{r+2})\mathbf{X}_r](\lambda) \text{ a.e.}(\mu). \end{aligned} \tag{19}$$

The theorem now follows by combining Equations (16), (17), and (19) using Equation (18) and by noting that the factor $\sigma_{rr}(\lambda)$ can be cancelled even when $\sigma_{rr}(\lambda) = 0$, since both sides of the resulting equation are then unity.

Theorem 6 in conjunction with Theorem 5 provides a means of generating the multiple coefficients of coherence. The analogous result for multivariate correlation coefficients is given in a special case on page 43 of (Anderson, 1958).

As a concomitant of the above correlation analysis, a spectral covariance analysis or cross-spectrum analysis for weakly stationary processes is immediately available. Thus, for example, the residual cross-spectrum for the processes $\mathbf{X}_i$, $\mathbf{X}_j$, after the effects of regression on the component processes $\mathbf{X}_{i_1}, \cdots, \mathbf{X}_{i_p}$ have been removed, is the i, jth element of the matrix function $\Sigma^{\perp}_{i_1,\cdots,i_p}(\lambda)$ defined above. In particular, the residual spectrum of $\mathbf{X}_i$, after the effects of regression on $\mathbf{X}_r, \cdots, \mathbf{X}_q$ have been removed, is given by Corollary 1 in terms of the multiple coefficient of coherence and the spectrum of $\mathbf{X}_i$. This type of analysis was employed by Tick (1963) for processes with discrete spectral distributions having a finite number of discontinuities (μ = point measures on a finite set of points) and by Jenkins (1963) for a class of processes with absolutely continuous, non-degenerate spectra (μ = Lebesgue measure and det $\Sigma(\lambda) > 0$ a.e.) to analyze the response characteristics of certain complex linear systems.

When μ is Lebesgue measure, det $\Sigma(\lambda) > 0$ a.e. and the distribution of $\mathbf{X}$ is Gaussian, approximate distributions of the natural estimators for $\rho^2_{ij\cdot i_1,\cdots,i_p}(\lambda)$ and $R^2_{i\cdot i_1,\cdots,i_p}(\lambda)$ are provided in the comprehensive paper of Goodman (1963). He shows that as in the case of the real multivariate normal distribution, the distribution of the estimator of the partial coefficient of coherence is identical to the distribution of the estimator of the ordinary coefficient of coherence with merely a reduction in the degrees of freedom. Thus, the tables of the distribution of this estimator, (Amos and Koopmans, 1963), can be used for the partial coefficient of coherence, as well. This makes it possible to obtain explicit solutions to inference problems in which the partial coefficient of coherence is the parameter of interest.

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5. Appendix—Continuation of Lemma 2. We now show that if $\Sigma(\lambda)$ is an a.e. (μ) non-negative definite, $p \times p$ hermitian matrix of measurable functions with (measurable) eigenvalues $\xi_1(\lambda) \geqq \xi_2(\lambda) \geqq \cdots \geqq \xi_p(\lambda)$, then there exists a matrix $U(\lambda)$ with measurable components such that $U(\lambda)^*U(\lambda) = I$ and

$$\Sigma(\lambda) = U(\lambda) \operatorname{diag} \{\xi_1(\lambda), \xi_2(\lambda), \cdots, \xi_p(\lambda)\} U^*(\lambda) \text{ a.e.}(\mu).$$

It is easily seen that if $U(\lambda) = [\varphi_1(\lambda), \varphi_2(\lambda), \cdots, \varphi_p(\lambda)]$ where, for each λ such that $\Sigma(\lambda)$ is non-negative definite, hermitian,

(i) $\{\varphi_1(\lambda), \cdots, \varphi_p(\lambda)\}$ is a complete orthonormal set of p dimensional column vectors in the inner product

$$\langle\varphi_i(\lambda), \varphi_j(\lambda)\rangle = \varphi_i^*(\lambda)\varphi_j(\lambda),$$

(ii) $\varphi_i(\lambda)$ is an eigenvector for $\Sigma(\lambda)$ corresponding to the eigenvalue $\xi_i(\lambda)$, $i = 1, 2, \cdots, p$, and

(iii) $\varphi_i(\lambda)$ has measurable components, $i = 1, 2, \cdots, p$, then $U(\lambda)$ will satisfy these conditions.

Without loss of generality we may assume that $\Sigma(\lambda)$ is non-negative definite, hermitian for all λ.

Define the sets S_n, $1 \leqq n \leqq 2^{p-1}$, by

$$S_n = [\xi_1(\lambda)\Delta_1\xi_2(\lambda)\Delta_2 \cdots \Delta_{p-1}\xi_p(\lambda)]$$

where Δ_j takes on the "values" $>$ or $=$, $\delta_j = 1$ or 0 according as Δ_j is $>$ or $=$, and $\delta_1\delta_2 \cdots \delta_{p-1}$ is the binary expansion of $n - 1$. Then S_n is measurable, and the eigenvalues have fixed multiplicities on S_n for each n.

Fix n and let $x_1(\lambda), x_2(\lambda), \cdots, x_s(\lambda)$ denote the distinct eigenvalues of $\Sigma(\lambda)$ on S_n with multiplicities

$$r_1, r_2, \cdots, r_s, \sum_{i=1}^{s} r_i = p.$$

The minimal polynomial of $\Sigma(\lambda)$ on S_n is then

$$\Psi(x, \lambda) = \prod_{i=1}^{s} (x - x_i(\lambda)).$$

Following Gantmacher (1959, **1** p. 84), define

$$\psi(x, y, \lambda) = \frac{\Psi(x, \lambda) - \Psi(y, \lambda)}{x - y} \tag{20}$$

and form the matrix polynomial

$$C(x, \lambda) = \psi(xI, \Sigma(\lambda), \lambda), \lambda \in S_n,$$

where I is the $q \times q$ identity matrix and x is a real scalar. The quantity $x - y$ is a factor of $\Psi(x, \lambda) - \Psi(y, \lambda)$, thus $\psi(x, y, \lambda)$ is everywhere continuous in x and y and is measurable in λ. It follows that $C(x_i(\lambda), \lambda)$ is a $q \times q$ matrix with measurable entries for $i = 1, 2, \cdots, s$.

Since $\Psi(x_i(\lambda), \lambda) = 0$ and $\Sigma(\lambda)$ satisfies its minimal equation, the matrix polynomials $\Psi(x_i(\lambda)I, \lambda)$ and $\Psi(\Sigma(\lambda), \lambda)$ are both zero for all $\lambda \in S_n$. Thus, by Equation (20), for each $\lambda \in S_n$,

$$(\Sigma(\lambda) - x_i(\lambda)I)C(x_i(\lambda), \lambda) = 0, \qquad i = 1, 2, \cdots, s.$$

Now, if it can be shown that the rank of $C(x_i(\lambda), \lambda)$ is r_i, it will follow that there are r_i linearly independent eigenvectors of $\Sigma(\lambda)$ corresponding to $x_i(\lambda)$ and each is some linear combination of the columns of $C(x_i(\lambda), \lambda)$. We now show this to be the case. (The proof of this result was provided by D. W. Sasser.)

SUBLEMMA. *Rank* $C(x_i(\lambda), \lambda) = r_i, i = 1, 2, \cdots, s$.

PROOF. The argument λ remains fixed throughout the proof and will be deleted. Since Σ can be put into canonical form without changing the dimension of the eigenspaces it suffices to consider the case where

$$\Sigma = \operatorname{diag}\{x_1, \cdots, x_1, x_2, \cdots, x_2, \cdots, x_s, \cdots, x_s\}.$$

In this matrix and the following diagonal matrices, the length of the kth block of symbols between the braces is r_k, $k = 1, 2, \cdots, s$. Then, for $x \neq x_i$, $i = 1, 2, \cdots, s$, $\Sigma - xI$ is non-singular and

$$\begin{aligned} C(x) = \Psi(x)(\Sigma - xI)^{-1} &= \prod_{i=1}^{s} (x - x_i) \operatorname{diag}\{(x_1 - x)^{-1}, \cdots, (x_1 - x)^{-1}, \\ &\quad (x_2 - x)^{-1}, \cdots, (x_2 - x)^{-1}, \cdots, (x_s - x)^{-1}, \cdots, (x_s - x)^{-1}\} \\ &= (-1)^s \operatorname{diag}\{\prod_{i \neq 1} (x_i - x), \cdots, \prod_{i \neq 1} (x_i - x), \prod_{i \neq 2} (x_i - x), \cdots, \\ &\quad \prod_{i \neq 2} (x_i - x), \cdots, \prod_{i \neq s} (x_i - x), \cdots, \prod_{i \neq s} (x_i - x)\}. \end{aligned}$$

Thus,

$$C(x_k) = \lim_{x \to x_k} C(x) = (-1)^s \operatorname{diag} \{0, \cdots, 0, a_k, \cdots, a_k, 0, \cdots, 0\}$$

where $a_k = \prod_{i \neq k} (x_i - x_k)$. This matrix clearly has rank r_k, $k = 1, 2, \cdots, s$.

To complete the proof of Lemma 2 it is only necessary to measurably orthonormalize the columns of $C(x_i(\lambda), \lambda)$ for each i. Since eigenvectors corresponding to different eigenvalues are orthogonal, the union of the s sets of resulting eigenvectors will be orthonormal. Also, this set will contain exactly p elements since the ranks of the eigenspaces are constant for all $\lambda \varepsilon S_n$ and are equal to the multiplicities of the eigenvalues. That this orthonormalization can be carried out in a measurable way by a simple extension of the Gram-Schmidt process is easily shown and the details will be omitted.

Now, let $\{\varphi_1^{(n)}(\lambda), \cdots, \varphi_p^{(n)}(\lambda)\}$ be the orthonormal set of eigenvectors of $\Sigma(\lambda)$ for $\lambda \varepsilon S_n$ and let $\chi_n(\lambda)$ be the set characteristic function of S_n. Then, if

$$\varphi_i(\lambda) = \sum_{n=1}^{2p-1} \chi_n(\lambda)\varphi_i^{(n)}(\lambda), \qquad i = 1, 2, \cdots, p,$$

the $p \times p$ unitary matrix

$$U(\lambda) = [\varphi_1(\lambda), \cdots, \varphi_p(\lambda)]$$

is a measurable solution to our problem.

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