EXTREME VALUES IN UNIFORMLY MIXING STATIONARY STOCHASTIC PROCESSES

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1. Introduction and summary. Let $X_1, X_2, \cdots$ be a sequence of independent, identically distributed random variables, and write Z_n for the maximum of $X_1, X_2, \cdots X_n$. Then there are two well known theorems concerning the limiting behaviour of the distribution of Z_n. (See, for example, Gumbel [7].)

Firstly, if for some sequence of pairs of numbers a_n, b_n, the quantities $a_n^{-1}(Z_n - b_n)$ have a non-degenerate limiting distribution as $n \to \infty$, then this limit must take one of three forms. Secondly, if $c_n = c_n(\xi)$ is defined by $P[X > c_n] \leqq \xi/n \leqq P[X \geqq c_n]$, then $P[Z_n \leqq c_n]$ tends to $e^{-\xi}$ as $n \to \infty$.

Suppose now that we drop the assumption of independence of the X_j, and require instead that the sequence $\{X_n\}$ be a stationary stochastic process: then it might be expected that similar results will hold, at least if X_i and X_j are *nearly* independent when $|i - j|$ is large. In Section 2 it will be shown that, if the process $\{X_n\}$ is uniformly mixing, then the only possible non-degenerate limit laws of $a_n^{-1}(Z_n - b_n)$ are just those that occur in the case of independence, and that the only possible limit laws of $P[Z_n \leqq c_n]$ are of the form $e^{-k\xi}$, k being some positive constant not greater than one.

The uniform mixing property is rather strong at first sight. It is however clear that some restriction is necessary, at least for example to ergodic processes, and the parallel which exists to a certain extent with normed sums of random variables suggests that uniform mixing hypotheses may be appropriate. (Cf. Rosenblatt [9]).

In the independent case converse results hold, as we have already observed for the second problem, giving necessary and sufficient conditions for the existence of the limits. We give some results concerning this problem in Section 3, but they are not altogether satisfactory.

Berman ([2] and especially [3]) has investigated the same problem, under somewhat different conditions. His results for the Gaussian case in [3], however, include those which can be obtained from Lemmas 1 and 2 of the present paper, and in consequence we do not reproduce them here.

The second problem mentioned above was considered by Watson [10] for m-dependent stationary processes, and his paper did in fact suggest the present investigations. His results are contained in Section 3.

Certain results were announced without proof by Chibisov [4], but these appear not to overlap our results.

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2. Possible limit laws. We recall the definition of a uniformly mixing stationary stochastic process. The condition is that

$$|P(A \cap B) - P(A)P(B)| < g(k)$$

if $A \in \mathcal{B}(X_1, X_2, \cdots, X_m)$ and $B \in \mathcal{B}(X_{m+k+1}, X_{m+k+2}, \cdots)$ for some m, where $g(k) \to 0$ as $k \to \infty$; here $\mathcal{B}(\cdots)$ denotes the σ-field generated by the random variables indicated. We shall for convenience express the relationship between A and B by saying that they are separated by k.

For completeness we record the possible limit laws of the normed maxima $a_n^{-1}(Z_n - b_n)$ in the independent case, since it is these which arise in the present situation, although we do not in fact need the explicit forms. The limiting distribution functions are

I	$\Phi(x) = \exp[-e^{-x}]$		$-\infty < x < \infty$
II	$\Phi(x) = 0$		$x \leqq 0$
	$= \exp[-x^{-\alpha}]$	(where $\alpha > 0$)	$x > 0$
III	$\Phi(x) = \exp[-(-x)^{\alpha}]$	(where $\alpha > 0$)	$x < 0$
	$= 1$		$x \geqq 0$

THEOREM 1. *If $\{X_n : n \geqq 1\}$ is a uniformly mixing strictly stationary stochastic process, and $Z_n = \max(X_1, X_2, \cdots X_n)$, then the only possible non-degenerate limit laws of $a_n^{-1}(Z_n - b_n)$, where $a_n > 0$, are Types* I, II *and* III *above.*

The proof will be carried out by showing that asymptotically the lack of independence does not affect the problem.

We consider values of n of the form $n = rm$, where r is fixed, and group the variables X_i into r consecutive sets of m. Let Z_{mi} be the maximum of the ith group of m: i.e.

$$Z_{mi} = \max_{1 \leqq j \leqq m} X_{(i-1)m+j} \qquad 1 \leqq i \leqq r. \tag{1}$$

Then we have

$$Z_n = \max_{1 \leqq i \leqq r} Z_{mi}, \tag{2}$$

and consequently

$$P[Z_n \leqq a_n x + b_n] = P[\max_i Z_{mi} \leqq a_n x + b_n]. \tag{3}$$

We show that we can ignore any fixed finite number of the X_i in calculating the limiting distribution of the maxima. Specifically, if k is a fixed number, write

$$Z'_{mr} = \max_{k \leqq j \leqq m} X_{(r-1)m+j} \tag{4}$$

and

$$Z''_{mr} = \max_{1 \leqq j \leqq k} X_{(r-1)m+j}, \tag{5}$$

so that

$$(6) \qquad Z_{mr} = \max (Z'_{mr}, Z''_{mr}).$$

We denote by B_m the event $\{Z'_{mm} < Z''_{mm}\}$: it is easily shown by using the strong law of large numbers for strictly stationary processes that $\lim_m P(B_m) = 0$.

Then we have

$$(7) \quad P[\max_i Z_{mi} \leqq a_n x + b_n] = P[(\max_i Z_{mi} \leqq a_n x + b_n) \cap B_m] + P[(\max_i Z_{mi} \leqq a_n x + b_n) \cap B_m^c].$$

The first term clearly tends to zero as $m \to \infty$, and for the second term we have

$$(8) \quad \begin{aligned} P[(\max_i Z_{mi} &\leqq a_n x + b_n) \cap B_m^c] \\ &= P[(\max_{i \leqq r-1} Z_{mi} \leqq a_n x + b_n) \cap (Z'_{mm} \leqq a_n x + b_n) \cap B_m^c] \\ &= P[(\max_{i \leqq r-1} Z_{mi} \leqq a_n x + b_n) \cap (Z'_{mm} \leqq a_n x + b_n)] - \delta_m, \end{aligned}$$

where $0 \leqq \delta_m \leqq P(B_m)$. Furthermore,

$$(9) \quad \begin{aligned} P[(\max_{i \leqq r-1} Z_{mi} &\leqq a_n x + b_n) \cap (Z'_{mm} \leqq a_n x + b_n)] \\ &= P[\max_{i \leqq r-1} Z_{mi} \leqq a_n x + b_n] P[Z'_{mm} \leqq a_n x + b_n] + \eta_m, \end{aligned}$$

where $|\eta_m| \leqq g(k)$, and finally

$$(10) \qquad P[Z'_{mm} \leqq a_n x + b_n] = P[Z_{mm} \leqq a_n x + b_n] + \theta_m,$$

where $0 \leqq \theta_m \leqq P(B_m)$, by (6). Hence, since k is arbitrary,

$$(11) \quad \lim_m \{P[Z_n \leqq a_n x + b_n] - P[\max_{i \leqq r-1} Z_{mi} \leqq a_n x + b_n] P[Z_{mm} \leqq a_n x + b_n]\} = 0.$$

By induction it follows that

$$(12) \qquad \lim_m \{P[Z_n \leqq a_n x + b_n] - P^r[Z_{m1} \leqq a_n x + b_n]\} = 0.$$

But (12) gives rise at once to the functional equation which occurs in the independent case, and the usual proof applies from this point on (Gnedenko [6]).

Obviously, each of the three limit types can actually occur for a suitable sequence $\{X_n\}$.

THEOREM 2. *Let $\{X_n : n \geqq 1\}$ be a uniformly mixing strictly stationary stochastic process, and let $c_n(\xi)$ satisfy*

$$P[X_1 > c_n(\xi)] \leqq \xi/n \leqq P[X_1 \geqq c_n(\xi)].$$

Then if $Z_n = \max (X_1, X_2, \cdots, X_n)$, the only possible non-degenerate limit laws of $P[Z_n \leqq c_n(\xi)]$ are $e^{-k\xi}$, where k is a positive constant less than or equal to one.

We use the same approach as in the proof of Theorem 1, except that we can confine ourselves to the case $r = 2$. Then we find immediately that

$$(13) \qquad \lim_m \{P[Z_n \leqq c_n(\xi)] - P^2[Z_m \leqq c_n(\xi)]\} = 0.$$

Now from the very definition of $c_n(\xi)$, it follows that $c_n(\xi) = c_{2m}(\xi) = c_m(\xi/2)$.

Hence from (13)

$$\lim_m \{P[Z_n \leqq c_n(\xi)] - P^2[Z_m \leqq c_m(\xi/2)]^2\} = 0. \tag{14}$$

Consequently, if $P[Z_n \leqq c_n(\xi)]$ tends to a limit $\phi(\xi)$, we have

$$\phi(\xi) = \phi^2(\xi/2) \tag{15}$$

where we know that $0 \leqq \phi(\xi) \leqq 1$, and $\phi(\xi)$ decreases monotonically. Under these conditions the only solution of (15) is

$$\phi(\xi) = e^{-k\xi} \tag{16}$$

where $k > 0$. The proof that $k \leqq 1$ we defer to the next section.

The question of what values of k can actually arise suggests itself immediately; the author does not know, having been unable to find any process for which the limit exists with $k \neq 1$.

3. Sufficient conditions for the existence of limiting distributions. We deal first with the type of limit considered in Theorem 2, as the calculations are somewhat simpler.

LEMMA 1. *If there are sequences of integers* $\{p_m\}$, $\{q_m\}$ *satisfying* $mg(q_m) \to 0$, $q_m/p_m \to 0$ *and* $p_{m+1}/p_m \to 1$, *with the property that (writing* $p = p_m$ *and* $t = m(p + q)$*)*

$$\sum_{i=1}^{p-1} [(p - i)/p]\{P[X_1 > c_t, X_{i+1} > c_t]/P[X_1 > c_t]\} \to 0, \tag{17}$$

as $m \to \infty$, *then* $P[Z_n \leqq c_n] \to e^{-\xi}$ *as* $n \to \infty$, *where* $c_n = c_n(\xi)$ *is as defined in Theorem* 2.

It is easily seen that Watson's theorem follows from this; q_m may be chosen in this case to be a constant independent of m.

The similarity of the condition in the lemma to that in Lemma 1 of Berman [1] is apparent.

The proof is in some respects similar to the proof of the central limit theorem in the same circumstances (Rosenblatt [9]).

Let $W_1, V_1, W_2, \cdots, W_m, V_m$ be the maxima of successive groups of the X_i, of size p_m and q_m alternately.

Then $Z_t = \max_{1 \leqq i \leqq m} (W_i, V_i)$, and it follows that

$$\begin{aligned} P[Z_t \leqq c_t] &= P[\bigcap_i(W_i \leqq c_t)\bigcap_i(V_i \leqq c_t)] \\ &= P[\bigcap_i(W_i \leqq c_t)] - \delta_m, \end{aligned} \tag{18}$$

where

$$\begin{aligned} 0 \leqq \delta_m &\leqq P[\bigcup_i(V_i > c_t)] \\ &\leqq mP[V_1 > c_t] \\ &\leqq mqP[X_1 > c_t] \\ &\leqq mq\xi/t \end{aligned}$$

Hence $\delta_m \to 0$.

Furthermore

$$P[\bigcap_i (W_i \leqq c_t)] - \{P[W_1 \leqq c_t]\}^m = \eta_m, \tag{19}$$

where $|\eta_m| \leqq mg(q)$, and consequently $\eta_m \to 0$.

We also have, since $W_1 = \max(X_1, \cdots, X_p)$,

$$S_1 - S_2 \leqq P[W_1 > c_t] \leqq S_1 \tag{20}$$

where

$$S_1 = \sum_1^p P[X_i > c_t] \leqq p\xi/t \tag{21}$$

and

$$S_2 = \sum_{1 \leqq i-j \leqq p} P[X_i > c_t, X_j > c_t], \tag{22}$$

by the usual inequalities bounding the probability of the union of a number of events (Feller [5], Chapter 4).

At this stage it is easy to complete the proof of Theorem 2, by showing that if the limit exists, then $k \leqq 1$. From (18) and (19) in fact, the limit must also be the limit of $\{P[W_1 \leqq c_t]\}^m$, which by (20) and (21) is not less than the limit of $(1 - p\xi/t)^m$, which is $e^{-\xi}$.

To return to the proof of the lemma, we observe that we can restrict attention to values of ξ for which the inequalities defining $c_n(\xi)$ reduce to equalities, these values forming an everywhere dense set. Then (21) is also an equality.

It now follows that $P[Z_t \leqq c_t]$ tends to $e^{-\xi}$ provided $S_2/S_1 \to 0$, by using the fact that

$$(1 - S_1)^m \leqq \{P[W_1 \leqq c_t]\}^m \leqq (1 - S_1 + S_2)^m, \tag{23}$$

and this condition is just that given in the lemma.

We have therefore found that $P[Z_t \leqq c_t]$ converges as t tends to ∞ along the sequence $m(p + q)$. We can deal with values of n not belonging to this sequence by using the fact that any n lies between two consecutive values of t, say $s = t_m$ and $t = t_{m+1}$, and then

$$P[Z_t \leqq c_s] \leqq P[Z_n \leqq c_n] \leqq P[Z_s \leqq c_t]. \tag{24}$$

It is not difficult to show that, since $p_{m+1}/p_m \to 1$, the outer members of this inequality both tend to the limit of $P[Z_t \leqq c_t]$, as $n \to \infty$.

For brevity in the statement of the next results we shall define the *associated independent process* of the process $\{X_n\}$ to be any sequence of mutually independent identically distributed random variables $\{\hat{X}_n\}$ which has the same marginal distribution; i.e., $P[\hat{X}_n \leqq x] = P[X_n \leqq x]$ for all x. Norming the maxima by the constants a_n, b_n means replacing Z_n by $a_n^{-1}(Z_n - b_n)$.

LEMMA 2. *Suppose that the distribution of the maxima of the associated independent process, normed by the constants a_n, b_n, converge to the distribution $\Phi(x)$ If for each x such that $\Phi(x) > 0$ there are sequences of integers $\{p_m\}$, $\{q_m\}$ satisfying*

$mg(q_m) \to 0$, $q_m/p_m \to 0$ and $p_{m+1}/p_m \to 1$, with the property that

$$(25)\quad \sum_{i=1}^{p-1} [(p-i)/p] \cdot \{P[X_1 > a_n x + b_n, X_{i+1} > a_n x + b_n]/P[X_1 > a_n x + b_n]\} \to 0$$

as $m \to \infty$, then the distribution of the normed maxima of $\{X_n\}$, $P[Z_n \leqq a_n x + b_n]$ also converges to $\Phi(x)$.

Because of the monotonic character of $\Phi(x)$ and $P[Z_n \leqq a_n x + b_n]$, we may clearly confine our attention to values of x such that $\Phi(x) > 0$. For these x it follows from the hypothesis on the associated independent process that

$$P(Z_n > a_n x + b_n] \sim -\log \Phi(x)/n.$$

Then the proof of Lemma 1 applies to the present lemma.

It follows from Lemma 2 that if an m-dependent process satisfies Watson's condition, and the normed maxima of its associated independent process converge in distribution, then the process itself has the same property.

In Lemmas 1 and 2 we have obtained sufficient conditions for convergence. In view of Berman's results it is obvious that these are not necessary, but to investigate this aspect would equally obviously be very difficult.

We can give other conditions which ensure that Lemmas 1 and 2 are applicable. They are, however, very strong.

It is in fact sufficient to suppose that

$$P[X_i > c, X_j > c]/P[X_i > c] \to 0$$

as $c \to 0$ for each fixed i and $j (i \neq j)$, and that

$$|P[A \cap B] - P(A)P(B)| < h(k)P(B)$$

whenever A and B are separated by k, where $\sum h(k) < \infty$. This latter condition is satisfied in the m-dependent situation. It is of course a mixing condition, of a type considered by Ibragimov [8].

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