ON ORTHOGONAL ARRAYS[1]

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0. Summary. It was shown in [11] that one can construct orthogonal arrays $(\lambda 2^3, k + 1, 2, 3)$ from arrays $(\lambda 2^2, k, 2, 2)$ with the maximum number of constraints $k + 1$ provided that k is the maximum number of constraints for the arrays of strength two. This result is generalized here to construction of arrays $(\lambda 2^{t+1}, k + 1, 2, t + 1)$ from arrays $(\lambda 2^t, k, 2, t)$.

The structure of arrays $(\lambda 2^t, t + 1, 2, t)$ is analyzed and for $\lambda = q2^n$, q odd, a method of extending any array $(\lambda 2^t, t + 1, 2, t)$ to $t + n + 1$ constraints is described.

Orthogonal arrays $(\lambda 2^4, k, 2, 4)$ are discussed in detail for $\lambda = 1$ through $\lambda = 5$. The maximum value of k is established in each of these cases and arrays assuming these values are effectively constructed.

1. Introduction. Orthogonal arrays were introduced first into statistics by C. R. Rao [9] under the name of hypercubes and then by R. C. Bose and K. A. Bush [1]. Following their definition, a $k \times N$ matrix A with entries from a set Σ of $s \geqq 2$ elements is called an orthogonal array of size N, k constraints, s levels, strength t, if any $t \times N$ submatrix of A contains all possible $t \times 1$ column vectors with the same frequency λ. Such an array is denoted by the symbol (N, k, s, t) and the number λ is called the index of the array. Clearly $N = \lambda s^t$.

Orthogonal arrays with $t = 2$, $s = 2$ were considered as long ago as 1867 by Sylvester, [13] who gives an explicit construction for the case $N = 2^m$. In pure mathematics papers, orthogonal arrays are usually called Hadamard matrices. R. E. A. C. Paley [7] was interested in orthogonal arrays with $t = 2$, $s = 2$ because of their applications to the theory of polytopes. The work of Paley solved the problem of weighing designs suggested by Hotelling [4] and continued by Mood [6]. Placket and Burman [8] applied Paley's work to their research in physics and industry. However, the statistical application of Paley's work is limited to cases in which no interaction between factors under consideration is present. Such a situation prevails indeed in the use of weighing designs but is rare in general. This led Rao, Bose, Bush, Ray Chaudhuri and others to the consideration of arrays of strength greater than two. It is well known that using arrays of strength $2t$, all interactions involving t or fewer factors can be estimated, if one can assume that interactions of more than t factors are negligible. With an array of strength $2t + 1$, interactions of t factors can be estimated, even if interactions of $t + 1$ factors are present.

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More recently R. C. Bose [2] applied orthogonal arrays to information theory and pointed out the analogy between the problems of design of experiments and information theory. Farrell, Kiefer and Walbran used orthogonal arrays in their work on optimum multivariate designs [5].

2. Some algebraic and structural properties of orthogonal arrays. Let n_{ij} denote the number of columns (other than the ith) having j coincidences with the ith column of an orthogonal array, $i = 1, \cdots, N; j = 0, \cdots, k$. Bose and Bush [1] showed that for any integer h such that $0 \leqq h \leqq t$, the following equalities hold

$$\sum_{j=0}^{k} \binom{j}{h} n_{ij} = \binom{k}{h}(\lambda s^{t-h} - 1). \tag{2.1}$$

These equalities are necessary but not sufficient for the existence of an orthogonal array. For given values for λ, s, k and t there may be several sets of solutions for n_{ij}'s. The same array may yield different sets of solutions in respect to different columns. Some properties of a particular array can sometimes be established by examining the relations between the different sets of solutions which it satisfies. Equations 2.1 sometimes enable one to prove the non-existence of an array by showing that there is no set of solutions which satisfies them.

For any given values of s, t and λ one can always construct orthogonal arrays for some values of k. For example all possible t-tuples of s elements repeated λ times will given an orthogonal array $(\lambda s^t, t, s, t)$. The problem of construction of orthogonal arrays reduces to construction of arrays with the maximum values of k for given values of s, t and λ.

C. R. Rao [10] showed that if $t \geqq 2$, k must satisfy the following inequalities:

$$\lambda s^t - 1 \geqq \binom{k}{1}(s-1) + \cdots + \binom{k}{u}(s-1)^u \qquad \text{if} \quad t = 2u, \tag{2.2}$$

$$\lambda s^t - 1 \geqq \binom{k}{1}(s-1) + \cdots + \binom{k}{u}(s-1)^u + \binom{k-1}{u}(s-1)^{u+1} \qquad \text{if} \quad t = 2u+1. \tag{2.3}$$

Bush [3] proved that if $\lambda = 1$,

$$k \leqq s + t - 1 \qquad \text{if} \quad s \quad \text{is even,} \tag{2.4}$$

$$k \leqq s + t - 2 \qquad \text{if} \quad s \quad \text{is odd.} \tag{2.5}$$

He also showed that for $s \leqq t$ the bounds are attained.

We will establish in this section some additional properties of orthogonal arrays for an arbitrary strength t and discuss some methods of constructing them.

PROPOSITION 2.1. *Let k' and k denote the maximum number of constraints of orthogonal arrays $(\lambda s^{t-1}, k', s, t-1)$ and $(\lambda s^t, k, s, t)$ respectively. Then $k \leqq k' + 1$.*

PROOF. Select arbitrarily one row of the array with λs^t columns. Divide the whole array into s sets of λs^{t-1} columns, each set having the same elements in the chosen row. The remaining $k - 1$ rows of each of these sets will clearly form an array of strength $t - 1$. Hence the inequality.

PROPOSITION 2.2. *In an array $(\lambda 2^t, t+1, 2, t)$ of strength t and $t + 1$ constraints,* *any two columns differing in an even number of elements appear the same number of times while any two columns differing in an odd number of elements appear together* λ *times.*

PROOF. Let $(a_1, a_2, \cdots, a_{t+1})$ be any column of the array, where each a_i assumes the value 0 or 1. Let

$$a_i^* = 0 \quad \text{if} \quad a_i = 1,$$

$$a_i^* = 1 \quad \text{if} \quad a_i = 0.$$

Let $x(a_1, \cdots, a_{t+1})$ denote the number of times the column $(a_1, \cdots, a_{t+1})$ appears in the array. Since the array is of strength t and index λ

$$x(a_1, \cdots, a_i, \cdots, a_j, \cdots, a_{t+1}) + x(a_1, \cdots, a_i^*, \cdots, a_j, \cdots, a_{t+1}) = \lambda,$$

$$x(a_1, \cdots, a_i^*, \cdots, a_j, \cdots, a_{t+1}) + x(a_1, \cdots, a_i^*, \cdots, a_j^*, \cdots, a_{t+1}) = \lambda.$$

Hence:

$$x(a_1, \cdots, a_i, \cdots, a_j, \cdots, a_{t+1}) = x(a_1, \cdots, a_i^*, \cdots, a_j^*, \cdots, a_{t+1}).$$

Successive applications of these equalities prove the proposition.

The following theorem was proven in [11]: *Let* S *be an ordered set of* s *elements* $e_0, e_1, \cdots, e_{s-1}$. *For any integer* t *consider the* s^t *different* t*-tuples of the elements of* S. *They can be divided into* s^{t-1} *sets, each consisting of* s t*-tuples and closed under cyclic permutation of the elements of* S. *Denote these sets by* S_i, $i = 1, 2, \cdots, s^{t-1}$. *Suppose that it is possible to find a scheme of* r *rows with elements belonging to* S

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rn} \end{matrix} \qquad (n = \lambda s^{t-1})$$

such that in every t*-rowed sub-matrix the number of columns belonging to each* S_i *is the same, say equal to* λ*; then we can use this scheme in order to construct an orthogonal array* $(\lambda s^t, r, s, t)$. *If in addition this scheme consists of an array of strength* $t - 1$, *then one can construct an orthgonal array* $(\lambda s^t, r + 1, s, t)$.

PROPOSITION 2.3. *If* $t = 2u$ *an array* $(\lambda 2^t, k, 2, t)$ *forms a scheme for construction of an array* $(\lambda 2^{t+1}, k + 1, 2, t + 1)$. *If* k *is the maximum number of constraints of the array of strength* t, *then* $k + 1$ *will be the maximum number of constraints of the resulting array of strength* $t + 1$.

PROOF. Consider any column $(t + 1)$-tuple of the array. Since $t + 1$ is odd:

$$x(a_1, \cdots, a_{t+1}) + x(a_1^*, \cdots, a_{t+1}^*) = \lambda.$$

Thus the array forms a scheme satisfying the above theorem of [11]. The remainder of this proposition follows from Proposition 2.1.

PROPOSITION 2.4. *For any given* t *consider all possible* $2^{t+1}(t + 1)$*-tuple columns. They can be split in a unique way into two orthogonal arrays* $(2^t, t + 1, 2, t)$.

PROOF. For $t \geqq 2$ the existence of an array $(2^t, t + 1, 2, t)$ was established by Bush [3]. Choose any $(t + 1)$-tuple column. There are exactly $2^t - 1$ other columns which differ from it by an even number of elements. By Proposition 2.2 each of them has to appear once. Hence the array is constructed. Since the relation of differing by an even number of elements is symmetric, reflexive, and transitive, an array formed in the described method will be closed under this relation. The remaining 2^t columns will form the second array.

REMARK. It may be noticed that if t is even the $2^t(t+1)$-columns consist of all the columns having an even number of zeros and an odd number of ones or vice-versa. For t odd, the columns of the array consist of either an even or an odd number of both zeros and ones.

Since the construction of arrays with $t = 4$ will be discussed later in more detail, it seems worthwhile to present at this stage the two arrays $(2^4, 5, 2, 4)$ whose construction is established by Proposition 2.4.

D	D'
0 0 0 0 0 0 0 1 1 1 1 0 1 1 1 1	1 1 1 1 1 1 1 0 0 0 0 1 0 0 0 0
0 0 0 0 1 1 1 0 0 0 1 1 0 1 1 1	1 1 1 1 0 0 0 1 1 1 0 0 1 0 0 0
0 0 1 1 0 0 1 0 0 1 0 1 1 0 1 1	1 1 0 0 1 1 0 1 1 0 1 0 0 1 0 0
0 1 0 1 0 1 0 0 1 0 0 1 1 1 0 1	1 0 1 0 1 0 1 1 0 1 1 0 0 0 1 0
0 1 1 0 1 0 0 1 0 0 0 1 1 1 1 0	1 0 0 1 0 1 1 0 1 1 1 0 0 0 0 1

The second array D' can be obtained from D by permuting the elements zero and one. Henceforth D' will be called the complementary array of D.

PROPOSITION 2.5. *Any array* $(\lambda 2^t, t+1, 2, t)$ *is a juxtaposition of* λ *arrays* $(2^t, t+1, 2, t)$ *of index unity.*

PROOF. Consider any $(t+1)$-tuple column of the array. Suppose that it appears x times. Each column which differs from it by an even number of elements will also appear x times, forming x identical arrays of index unity, i.e., $(2^t, t+1, 2, t)$.

If $x < \lambda$ then each column differing from the chosen column in an odd number of elements will appear $\lambda - x$ times, forming $\lambda - x$ arrays of the complementary type.

PROPOSITION 2.6. *If* $\lambda = q2^n$, q *odd, then any array* $(\lambda 2^t, t+1, 2, t)$ *can be extended to an array* $(\lambda 2^t, t+n+1, 2, t)$.

Before proving this proposition we wish to remark that the existence of an array $(\lambda 2^t, t+n+1, 2, t+n)$ when $\lambda = q2^n$ is a consequence of the previously mentioned result of Bush [3]. Using his result one can always construct an array $(2^{n+t}, n+t+1, 2, t+n)$ and repeat q times. The main point of our proposition is to describe a simple method of extending *any* array $(\lambda 2^t, t+1, 2, t)$ to $t+n+1$ constraints which does not, however, insure any increase in the strength of the array beyond the starting strength t.

PROOF. By Proposition 2.5 one can decompose the array $(\lambda 2^t, t+1, 2, t)$ with $\lambda = q2^n$ into $q2^n$ elementary arrays of index unity. Divide the $q2^n$ elementary arrays into 2^n components, each consisting of q arrays. Adjoin to each of these components n rows such that the columns form one of the 2^n possible n-tuples of zeros and ones. Use a different n-tuple for each different component. Consider now any fixed column t-tuple of the array. It may have $t - m$ elements belonging to the original array and m elements in the added n rows. Each of the $q2^n$ elementary arrays will have the same $(t - m)$-tuple 2^m times, and $q2^{n-m}$ of the elementary arrays will have an m-tuple in the added rows coinciding with the corresponding m elements of the considered t-tuple. Hence the fixed t-tuple will appear in the extended array $2^m \times q2^{n-m}$, or $q2^n$, times as required.

REMARK. Propositions 2.5 and 2.6 may prove useful if one tries to enumerate all possible orthogonal arrays for given values of the parameters λ, s, and t. Using Proposition 2.5, one would first enumerate all different arrays with $k = t + 1$. If $\lambda = q2^n$, $n \geqq 1$, then one could examine the number of ways these arrays can be extended to $k = t + n + 1$ using Proposition 2.6. Clearly there is no assurance that this method of extension is exhaustive. In general this will not be the case and further examination of other possibilities will be required.

PROPOSITION 2.7. *If λ is odd and any two columns of the array differ in an even number of elements then the array $(\lambda 2^t, t + 1, 2, t)$ cannot be extended.*

PROOF. Clearly such an array is a juxtaposition of λ identical arrays of index unity. Hence, for any column i

$$n_{i,t+1} = \lambda - 1; \qquad n_{i,t} = 0; \qquad n_{i,t-1} = \binom{t+1}{t-1}\lambda; \qquad n_{i,t-2} = 0.$$

If this array could be extended to $t + 2$ rows the following equations would hold:

$$n_{i,t+2} + n_{i,t+1} = \lambda - 1,$$
$$n_{i,t} + n_{i,t-1} = \binom{t+1}{t-1}\lambda.$$

In addition, the $(t + 2)$-rowed array would have to satisfy Equations 2.1. The last two equations become in this case:

$$n_{i,t} + \binom{t+1}{t}n_{i,t+1} + \binom{t+2}{t}n_{i,t+2} = \binom{t+2}{t}(\lambda - 1,)$$
$$n_{i,t-1} + \binom{t}{t-1}n_{i,t} + \binom{t+1}{t-1}n_{i,t+1} + \binom{t+2}{t-1}n_{i,t+2} = \binom{t+2}{t-1}(2\lambda - 1).$$

The four equations together give unique solutions for the four values of the n_{ij}'s. It is easy to check that one obtains $n_{i,t+2} = \lambda/2 - 1$ and since $n_{i,t+2}$ has to be an integer, λ cannot be odd.

The result of this section will be applied presently to the construction of some orthogonal arrays with $t = 4$, $s = 2$. In addition to this, a well known method of construction using projective geometries will be utilized in some of the cases. The geometrical method of construction shown by Bose involves the following steps: First one finds a set of, say, k points, no t conjoint, in finite projective space with $r - 1$ dimensions $PG(r - 1, s)$. Then one multiplies the $k \times r$ matrix so obtained, say, C, by the matrix B_r consisting of all possible s^r r-tuples, including the r-tuple consisting of all zeros. The $k \times s^r$ product matrix will be an array of strength t. Moreover the columns will form a group in respect to the distinct columns of the matrix. The maximum number of constraints of such a matrix clearly coincides with the maximum number of points which one can find in $PG(r - 1, s)$, no t conjoint. This number is usually denoted by $m_t(r, s)$. Once an array is constructed using geometrical methods, one may try to extend it using other combinatorial methods. In general the known algebraic bounds for the maximum number of constraints exceed the geometric bounds. However, there is no known example in which an array was constructed whose number of constraints exceeded $m_t(r, s)$.

3. Construction of orthogonal arrays for $t = 4$, $s = 2$, and $N = 16, 32, 48, 64$ and 80.

PROPOSITION 3.1. *For an orthogonal array* $(16, k, 2, 4)$ *the maximum value for* k *is* 5.

The construction of $(16, 5, 2, 4)$ follows from (2.4). Take any 5-tuple column and adjoin to it all columns which coincide with it in an odd number of elements, i.e., one and three. The remaining sixteen 5-tuples will form the complementary array. The fact that five is the maximum number of constraints follows from Proposition 2.7. This result follows also from the theorem of Bush [3].

PROPOSITION 3.2. *For an orthogonal array* $(32, k, 2, 4)$ *the maximum value for* k *is* 6.

First we will show that $k < 7$ and then construct arrays with $k = 6$. Using Equations (2.1) we may express n_{i0} through n_{i4} in terms of the remaining n_{ij}'s. This gives

$$n_{i0} = 3 - n_{i5} - 5n_{i6} - 15n_{i7},$$

$$n_{i3} = -35 + 10n_{i5} + 40n_{i6} + 105n_{i7}.$$

The first equation gives $n_{i6} = n_{i7} = 0$; $n_{i5} \leqq 3$. Hence there is no non-negative integer solution for n_{i3}. For $k = 6$ there are only two solutions of the Equations (2.1). They are:

$$n_{i0} = 0, \quad n_{i1} = 5, \quad n_{i2} = 5, \quad n_{i3} = 10, \quad n_{i4} = 10, \quad n_{i5} = 1, \quad n_{i6} = 0,$$

$$n_{i0} = 1, \quad n_{i1} = 0, \quad n_{i2} = 15, \quad n_{i3} = 0, \quad n_{i4} = 15, \quad n_{i5} = 0, \quad n_{i6} = 0.$$

Arrays corresponding to each of these solutions may be constructed using Propositions 2.5 and 2.6. Utilizing for the first five rows all possible compositions of the arrays D and D' one obtains exactly four arrays, two for each of the solutions. They are:

$D\ D$	$D'\ D'$	$D\ D'$	$D\ D'$
0 1	0 1	0 1	1 0

Zero or one below the array means that to each column of the array the same element, either 0 or 1, is added in accordance with Proposition 2.6. The second and the fourth array are obtained from the first and the third respectively by permuting the elements 0 and 1. The first and the second array have the first solution in respect to each column, while the third and fourth admit the second solution in respect to each of their columns. One may use geometrical methods to construct the above arrays. It was pointed out in [12] that there are essentially just two sets of six points no four in one plane in $PG(4, 2)$. They consist of the five points having just one coordinate equal to zero and either the point (1, 1, 1, 1, 0) or (1, 1, 1, 1, 1). Multiplying the first matrix of size 6×5 by the matrix whose columns are all the points of $PG(4, 2)$ augmented by the column of all zeros one obtains the first array if one interchanges the fifth and the sixth row. The matrix of the second set of points multiplied by the same matrix of size 5×32 gives the third array.

The four arrays exhibited exhaust all the possibilities for the construction of arrays (32, 6, 2, 4).

PROPOSITION 3.3. *For the array* (48, k, 2, 4) *the maximum value for* k *is* 5. *Solving again the Equations* (2.1) *for* $k = 6$ *we obtain* $n_{i0} = 4 - 5n_{i6} - n_{i5}$. *This gives* $n_{i6} = 0$ *for all* i, *meaning that an array of six rows could not have two identical columns. Consider the* 6×2^6 *array of all possible distinct* 6*-tuples. In any four rows of this array every* 4*-tuple will appear exactly four times. If* 48 *columns could be chosen for this array to form an array of index* 3, *then in any four rows of the remaining* 16 *columns, each* 4*-tuple would have to appear exactly once. This would yield an array of* 16 *columns with* $k = 6$, *contradicting Proposition* 3.1.

Proposition 2.5 now enables us to give an exhaustive construction of the arrays (48, 5, 2, 4). It is obtainable by enumerating all distinct juxtapositions of arrays of types D and D'. Clearly half of them can be obtained from the others by permuting the elements zero and one.

PROPOSITION 3.4. *The maximum number of constraints for the orthogonal array* (64, k, 2, 4) *is equal to eight.*

This proposition will be proven in several steps. We will enumerate first of all

TABLE 1

Solutions for $k = 6, 7, 8, 9$

	n_{i0}	n_{i1}	n_{i2}	n_{i3}	n_{i4}	n_{i5}	n_{i6}
6.1	0	10	10	20	20	2	1
6.2	1	5	20	10	25	1	1
6.3	2	0	30	0	30	0	1
6.4	0	11	5	30	10	7	0
6.5	1	6	15	20	15	6	0
6.6	2	1	25	10	20	5	0
7.1	0	4	15	5	30	6	3
7.2	1	0	20	5	25	10	2
7.3	0	5	10	15	20	11	2
7.4	0	6	5	25	10	16	1
7.5	1	1	15	15	15	15	1
7.6	0	7	0	35	0	21	0
7.7	1	2	10	25	5	20	0
8.1	0	2	9	12	15	18	7
8.2	0	1	14	2	25	13	8
8.3	0	3	4	22	5	23	6
9.1	0	0	9	6	18	9	21
9.2	0	1	4	16	8	14	20

the solutions of (2.1) for $k = 6, 7, 8, 9$ eliminating at each stage solutions which without further reasoning are found not to represent orthogonal arrays. (See Table 1.) For the solutions which are not eliminated at the final step we will give effective constructions of arrays satisfying them. For $k = 7$ Equations (2.1) give $n_{i7} = 0$ for all possible solutions. Hence $n_{ij} = 0$ for $j \geqq 7$ and $k \geqq 7$ since otherwise there would exist an array with $k = 7$ for which $n_{i7} \neq 0$.

Arrays admitting each of the solutions for $k = 6$ can be constructed. They will not be exhibited separately since each of them will be a sub-array of some larger array to be constructed later. The establishment of Proposition 3.4 and the effective construction of the arrays will be done in several steps.

STEP 1. Solutions 8.2 and 8.3 do not represent arrays.

To make the proofs more readable we will assume without loss of generality that the first column of the array consists of all zeros and that the array admits the solution under investigation in respect to the first column.

First we will show that the solution 8.2 does not represent an array. Solution 8.2 has $n_{i1} = 1$. The single zero of this column cannot coincide with a zero of the columns of type n_{i2} because the two columns would have seven coincidences. Hence if we delete from the array the row which includes the zero element of the column with a single zero we will get a seven-rowed array which satisfies the conditions $n_{i0} = 1$, $n_{i1} = 0$. This means that this sub-array will have to satisfy solution 7.2 in respect to the first column. Thus $n_{i2} = 20$. Out of these columns six would have to have zeros added if the extension has solution 8.2. But 8.2 has $n_{i3} = 2$; hence a contradiction.

Next we will show that no array exists satisfying solution 8.3. In this case $n_{i1} = 3$. Columns having one coincidence with the first column must have this coincidence in different rows, in order to eliminate having seven or more coincidences among themselves. Deleting the row in which one of the columns of the type n_{i1} coincides with the first column we would get a sub-array of seven rows with $n_{i0} = 1$, $n_{i1} \geqq 2$. Hence the seven-rowed sub-array would have to satisfy solution 7.7 with $n_{i0} = 1$, $n_{i1} = 2$. But 8.3 has $n_{i5} + n_{i6} = 29$, while 7.7 has $n_{i4} + n_{i5} + n_{i6} = 25$; hence a contradiction.

STEP 2. The unique array with eight constraints satisfying solution 8.1 cannot be extended to nine rows.

We first show that solution 9.2 does not represent an array. First, $n_{i1} = 1$; i.e., there is one column which has one coincidence with the ith column. Deleting the row in which this coincidence occurs yields an eight-rowed array with $n_{i0} = 1$. But 8.1 has $n_{i0} = 0$.

An array with solution 8.1 cannot be extended to an array with solution 9.1. Solution 8.1 has $n_{i1} = 2$. Two such columns must have at least six coincidences. Since 9.1 has $n_{i1} = 0$ these two columns would also have to coincide in the 9th row which together would give at least 7 coincidences. This is impossible.

STEP 3. An array satisfying solution 8.1 can be constructed. The sets of points no four on one plane in $PG(5, 2)$ were considered in [12]. It was shown there that if one includes in the set the six points forming the identity matrix, then there are just two ways of completing this set to eight points up to interchanging the role of the coordinates. These two sets of eight points can be exhibited as follows:

$$C_1 = \begin{bmatrix} I_6 \\ 111100 \\ 110011 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} I_6 \\ 111100 \\ 111011 \end{bmatrix}.$$

Multiplying either of these 8×6 matrices by the 6×2^6 matrix, say B_6, consisting of all possible six-tuples of the two elements 0 and 1, one obtains the required array satisfying solution 8.1. The first seven rows of either C_1 or C_2 multiplied by B_6 gives an array satisfying solution 7.3. If the seventh row is replaced by the eighth row of C_2 one obtains a matrix satisfying solution 7.5.

The matrix $\begin{bmatrix} I_6 \\ 111111 \end{bmatrix}$ multiplied by B_6 gives the solution 7.6.

PROPOSITION 3.5. *Arrays* (64, 7, 2, 4) *satisfying solutions* 7.1, 7.2, 7.4, 7.6, *and* 7.7. *cannot be extended to eight constraints.*

(i) Suppose that an array has solution 7.1. Since $n_{i6} = 3$, we may assume that there are four columns as in (a):

(a)	(b)
0000	......
0000	......
0000	......
0000	......
0001	000000
0010	000000
0100	000000

Suppose one of the last three rows of (a) is deleted. In each case the remaining six rows have $n_{i6} = 1$ and $n_{i5} = 2$ and hence must form an array of six constraints satisfying solution 6.1. The array of seven constraints with solution 7.1 must now have six columns consisting of five 0's and two 1's since $n_{i5} = 6$. However, deleting one of the last three rows must leave each of these columns with only four 0's. Therefore, in the last three rows, these six columns have only 0's as in (b). Suppose the array could be extended to eight rows. Four of the columns in (b) must coincide with the first column in the eighth row since solution 8.1 has $n_{i6} = 7$. We would then have the 4-tuple $(0, 0, 0, 0)'$ appearing five times in the last four rows. Thus the extension is impossible.

(ii) Suppose an array satisfying solution 7.2 is extended to eight rows. According to 7.2 the extension must have $n_{i0} + n_{i1} = 1$. Since the unique solution for $k = 8$ has $n_{i0} + n_{i1} = 2$, this is impossible.

(iii) Consider an array with solution 7.4. The array must have six columns with exactly one 0. If an eighth row is added, these columns will be followed by two 1's and four 0's as in (c):

(c)	(d)
011111	11111
101111	11111
110111	.....
111011	.....
111101	.....
111110	.....
111111	.....
110000	.....

The eight-rowed array must now have five more columns with exactly two 0's. All these columns must have 1's in the first two rows as in (d); otherwise there would be a column having seven coincidences with at least one of the first two columns.

If any one of the rows 3 through 8 is deleted, the remaining seven-rowed array would have $n_{i0} = 0$ and hence $n_{i1} \geqq 5$. This implies that if any one of rows 3 through 6 is deleted then at least two of the columns of (d) must have only one 0 in the remaining rows, while if row 7 is deleted, at least three of the columns of (d) must have only one 0 in the remaining rows. Therefore the five columns of (d), with two 0's in each column, must have at least $4 \times 2 + 3 = 11$ zeros. This is clearly a contradiction.

(iv) An array with solution 7.6 can be extended only to an array of eight constraints with $n_{i5} + n_{i6} = 21$. Solution 8.1 has $n_{i5} + n_{i6} = 25$.

(v) Suppose an array with solution 7.7 is extended to eight rows. Solution 7.7 has $n_{i0} = 1$, $n_{i1} = 2$. Let $i = 1$. The column with no 0 and one column with one 0, must both have a 0 in the eighth row. This would result in two columns of the eight-rowed array having seven coincidences, while 8.1 has $n_{i7} = 0$.

It remains to be shown that one can in fact construct arrays satisfying solutions 7.1, 7.2, 7.4, 7.7. Here is an array which satisfies solutions 7.2, 7.4 and 7.7 in respect to some columns:

D	D	D	D'
0	0	1	1
0	1	0	1

It is easy to see that this array satisfies solution 7.2 in respect to the first 16 columns, solution 7.4 in respect to the next 32 columns and solution 7.7 in respect to the remaining 16 columns. The proof follows from Propositions 2.5 and 2.6.

An array satisfying solution 7.1 in respect to each of its columns was found by trial and error. The first 32 columns have the form:

D	D
0	1
1	0

The last 32 columns have the form:

(II)	(III)
0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1	0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1
0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1	0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1
0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1	0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1
0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1	1 0 0 1 0 1 1 0 1 0 0 1 0 1 1 0
0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1	1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

It is easy to check that this array does in fact satisfy the condition imposed upon it. Note that the first five rows have the form $D\ D\ D\ D'$. In the first 32 columns the first five rows together with either row 6 or 7 form an array (32, 6, 2, 4).

One could give an exhaustive enumeration of all the arrays (64, 7, 2, 4) but this does not seem worthwhile because of the unique structure of the array (64, 8, 2, 4) which stems from the unique construction of the arrays 7.3 and 7.5. It may be worthwhile to point out that the arrays 7.3, 7.5, and 7.6 have the following structure up to interchanging the rows or the elements 0 and 1:

7.3	7.5	7.6
$D\ D\ D\ D$	$D\ D'\ D\ D'$	$D\ D'\ D'\ D$
0 0 1 1	0 0 1 1	0 0 1 1
0 1 0 1	0 1 0 1	0 1 0 1

PROPOSITION 3.6. *The maximum number of constraints for the orthogonal array* (80, k, 2, 4) *is six.*

We will prove this proposition in several steps. First we will enumerate all possible solutions for $k = 5$ and $k = 6$. (See Table 2.)

TABLE 2

Solutions for $k = 5, 6$

	n_{i0}	n_{i1}	n_{i2}	n_{i3}	n_{i4}	n_{i5}	n_{i6}
5.1	0	25	0	50	0	4	
5.2	1	20	10	40	5	3	
5.3	2	15	20	30	10	2	
5.4	3	10	30	20	15	1	
5.5	4	5	40	10	20	0	
6.1	0	12	15	20	30	0	2
6.2	0	13	10	30	20	5	1
6.3	1	8	20	20	25	4	1
6.4	2	3	30	10	30	3	1
6.5	0	14	5	40	10	10	0
6.6	1	9	15	30	15	9	0
6.7	2	4	25	20	20	8	0

STEP 1. Arrays 5.1, 5.2 and 5.5 cannot be extended to six rows.

PROOF. The fact that array 5.1 cannot be extended follows from Proposition 2.7. λ is equal to 5 and any two columns coincide in an odd number of elements.

Arrays 5.2 and 5.5 are obtainable from each other by permuting the elements 0 and 1. Hence it is enough to show that 5.5 cannot be extended. Assume that the first column consists of all zeros. If the array could be extended then its six-rowed extension would have to have $n_{16} = 0$. Solving 2.1 in terms of n_{15} gives:

$$n_{11} = -36 + 5n_{15}, \quad \text{which implies} \quad n_{15} \geqq 8$$

$$n_{10} = 10 - n_{15}, \quad \text{which implies} \quad n_{15} \leqq 10.$$

Moreover any extension of 5.5 would have to have $n_{10} + n_{11} \leqq 9$. Hence there is only one solution for the extension if possible. It is:

$$n_{10} = 2, \quad n_{11} = 4, \quad n_{12} = 25, \quad n_{13} = 20, \quad n_{14} = 20, \quad n_{15} = 8, \quad n_{16} = 0.$$

An array of type 5.5 has four columns of all 1's, one column of each possible type with one 0, and four columns of each type with four 0's. The first five rows of the following matrix are part of such an array, and the sixth row is the extension if possible:

$$\begin{array}{ccccccccccccc} 1&1&1&1&0&1&1&1&1&0&0&0&0\\ 1&1&1&1&1&0&1&1&1&0&0&0&0\\ 1&1&1&1&1&1&0&1&1&0&0&0&0\\ 1&1&1&1&1&1&1&0&1&0&0&0&0\\ 1&1&1&1&1&1&1&1&0&1&1&1&1\\ \hline 1&1&0&0&0&0&0&1&1&1&1&1&0 \end{array}$$

The elements added to the nine columns are determined by the solution $n_{10} = 2$, $n_{11} = 4$, for the extension. Then if the fifth row is deleted, a five-rowed array remains with $n_{10} = 3$, which satisfies only solution 5.4. In 5.4, each column with four 0's is repeated three times, and $n_{15} = 1$. Therefore the last four columns above must have three 1's and one 0 added. Now suppose the fourth row is deleted. Again the same array with solution 5.4 is obtained. 5.4 is of the form $D\,D\,D'\,D'\,D'$. Hence it should have each column with three 0's repeated twice. Since $(00011)'$ appears three times in this sub-array, we obtain a contradiction.

Note that 5.3 and 5.4 represent the same array with 0 and 1 interchanged.

STEP 2. There are no arrays satisfying solution 6.4 or 6.5.

PROOF. Consider an array having solution 6.4 with respect to a first column of 0's. Since $n_{i6} = 1$, and $n_{i5} = 3$, there must be another column of all 0's and three columns with five 0's and one 1. The columns with five 0's must be distinct, since any five-rowed sub-array has to satisfy either solution 5.3 or 5.4 and cannot have more than three columns consisting of five 0's. Thus the array would have to have the following five columns:

0 0 1 0 0
0 0 0 1 0
0 0 0 0 1
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0

To complete the requirement that the four-tuple $(0000)'$ appears five times in each four-rowed sub-array, the columns $(000011)'$ and $(000101)'$ must each appear three times. However, the four-tuple $(0001)'$ would then appear six times in rows 1, 2, 3, and 6.

Suppose one could construct an array satisfying solution 6.5. With $n_{i5} = 10$, it must have a sub-array satisfying solution 5.4. Consequently it would be necessary that $n_{i0} + n_{i1} \leqq 13$, but 6.5 has $n_{i1} = 14$.

STEP 3. An array admitting solution 6.1 with respect to some column cannot be extended to seven rows.

PROOF. A seven-rowed extension of an array satisfying solution 6.1 must have $n_{i6} + n_{i7} = 2$. There are only two solutions of Equations (2.1) which might correspond to such an extension.

	n_{i0}	n_{i1}	n_{i2}	n_{i3}	n_{i4}	n_{i5}	n_{i6}	n_{i7}
7.1	0	4	23	0	40	10	1	1
7.2	0	8	4	35	10	20	2	0

First consider solution 7.1. An array satisfying solution 7.1 could be constructed only by extending an array satisfying solution 6.1. But an array with solution 7.1 must have a six-rowed sub-array with $n_{i0} > 0$. This gives a contradiction. An array satisfying solution 7.2 has $n_{i1} = 8$. Hence it must have at least two identical columns having one coincidence with the ith column. Crossing out the row in which this coincidence occurs we would get $n_{i0} > 1$. Hence the sub-array obtained would have to satisfy solution 6.7. On the other hand 6.7 has $n_{i0} + n_{i1} = 6$. Hence the extension would have to have $n_{i0} + n_{i1} \leqq 6$, but 7.2 has $n_{i0} + n_{i1} = 8$. Thus a contradiction.

STEP 4. An array satisfying solution 6.1 can be constructed in a unique way.

PROOF. It can be easily checked that in order to construct an array satisfying solution 6.1 in respect to some column one has to distribute each type of the remaining columns symmetrically. Each column having one, four, or six coincidences with the chosen column has to appear twice, and each column having two or three coincidences, once. Moreover, such a construction will in fact result in an orthogonal array. Every five-rowed sub-array will satisfy solution 5.3 with respect to the fixed column used in the process of construction.

One can enumerate the solutions of the constructed array with respect to the remaining columns:

Type of Column	Solution
One coincidence	6.2
Two coincidences	6.7
Three coincidences	6.6
Four coincidences	6.3

STEP 5. No array satisfying solutions 6.2, 6.3, 6.6, 6.7 can be extended to seven rows.

This will be established showing that an array satisfying any of the solutions other than 6.1 must include a column in respect to which it satisfies solution 6.1.

An array satisfying solution 6.2 must satisfy solution 6.1 in respect to at least one of the columns having one coincidence with the ith column, since $n_{i1} = 13$.

Solutions 6.3, 6.6, and 6.7 will be considered with $i = 1$, assuming the first column to be all 0's. Solution 6.3 has $n_{i5} = 4$. Any five-rowed sub-array has to satisfy either solution 5.3 or 5.4. Therefore an array with this solution must have six columns as follows:

0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0
0	0	0	0	0	0

Then since $\lambda = 5$ the column $(000011)'$ must appear three times, yielding solution 6.1. An array satisfying solution 6.7 is obtained from an array 6.3 by permuting the elements 0 and 1.

An array with solution 6.6 must have columns as follows:

0	1	1	0	0	0	0	0	0	0	1	1	1
0	0	0	1	1	0	0	0	0	0	1	1	1
0	0	0	0	0	1	1	0	0	0	1	1	1
0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0

Consider the first 10 columns shown. Deleting the first row leaves a five-rowed array with $n_{15} = 2$. Since there are three columns with five 0's and since columns differing in an even number of elements must appear the same number of times, each column with three zeros must appear three times. In particular, the column $(11000)'$ must appear three times in the last five rows. Further, it must be preceded each time by a 1 in the first row, since the four-tuple $(0000)'$ already appears in rows 1, 4, 5 and 6 five times.

REMARK. It may be noticed that the proof of Proposition 3.5 gave also a method of exhaustive enumeration of the orthogonal arrays (80, 6, 2, 4).

4. Orthogonal arrays (81, 5, 3, 4) and (243, 11, 3, 4). The arrays (81, 5, 3, 4) and (243, 11, 3, 4) can be constructed geometrically. The array (81, 5, 3, 4) can be constructed by adjoining to the matrix I_4 the point (1111) and multiplying the augmented matrix by the 4×81 matrix of all possible four-tuples consisting of the elements 0, 1 and 2. The maximum number of constraints is assured by the theorem of Bush [3]. Tallini [14] gave an example of 11 points in $PG(4, 3)$ such that no 4 are on one plane. Such points can also be obtained easily by simple combinatorial argument. An example of a set of 11 points can be exhibited as follows:

$$\begin{array}{c} I_5 \\ 1\ 1\ 1\ 1\ 1 \\ 1\ 2\ 1\ 2\ 0 \\ 1\ 2\ 2\ 0\ 1 \\ 1\ 1\ 0\ 2\ 2 \\ 1\ 0\ 2\ 1\ 2 \\ 0\ 1\ 2\ 2\ 1 \end{array}$$

No other method of construction can give a larger number of constraints for the orthogonal array (243, k, 3, 4) because eleven is the upper bound for k in the inequality 2.2 with $\lambda = 3$, $s = 3$, $t = 4$.

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