RANDOM HYDRODYNAMIC FORCES ON OBJECTS[1]

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1. Summary. The force exerted on an object immersed in a flowing turbulent fluid is considered as a zero-memory, nonlinear transformation of the bivariate Gaussian process whose components are the fluid particle velocities and local accelerations that would be present at the location of the object if the object were not disturbing the fluid. A model often used in applications is assumed and the probability density and the moment generating function are derived and investigated. The covariance between the forces at two space-time points in the presence of a space-varying mean flow is developed and a series approximation outlined. Under suitable restrictions the partial sums of the series are used to obtain an easily computed approximation to the spectral density of the force at a fixed space location.

2. Introduction. Let $V(x, y, z, t)$ be the velocity of fluid flow at a specified point (x, y, z) within a fluid at time t. It will be assumed that the velocities are always directed along the x-coordinate axis. The force that would be exerted by the fluid on an object placed at the point has been approximated in many applications (Morrison, O'Brien, Johnson, and Schaaf [8], Reid and Bretschneider [14], Wiegel, Beebe, and Moon [17], Wiegel [16], pp. 256 ff.) by the formula

$$\Phi(x, y, z, t) = cV|V| + k\partial V/\partial t. \tag{2.1}$$

The two terms are called the drag and inertial forces respectively. The constants c and k are determined by the shape and dimensions of the object, the physical properties of the fluid, and the general regime of flow as evidenced by flow parameters like the Reynolds number or the Froude number.

It is often reasonable to consider $V(x, y, z, t)$ a Gaussian stochastic process in (x, y, z, t). On this premise, we will investigate the probability properties of the random quantity Φ, and, in particular, will consider (1) the probability law and moments of $\Phi(x_0, y_0, z_0, t_0)$, (2) the covariance of $\Phi(x_1, y_1, z_1, t_1)$ and $\Phi(x_2, y_2, z_2, t_2)$, and (3) the spectrum of $\Phi(t) = \Phi(x_0, y_0, z_0, t)$. (The subscripts denote fixed values of the coordinates, while a coordinate without a subscript indicates a stochastic process parameter.)

3. Assumptions. It will be assumed that

(1) $V(x, y, z, t)$ and the sample-function derivative,

$$A(x, y, z, t) = \partial V(x, y, z, t)/\partial t \tag{3.1}$$

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are a two-component Gaussian stochastic process over the parameter space of all vectors of the form (x, y, z, t).

(2) the random variables, $V(x_0, y_0, z_0, t_0)$ and $A(x_0, y_0, z_0, t_0)$, are independent, and have means $m(x_0, y_0, z_0, t_0)$ and 0 respectively, and variances $\sigma^2(x_0, y_0, z_0, t_0)$ and $\rho^2(x_0, y_0, z_0, t_0)$ respectively.

(3)

$$\Phi(x, y, z, t) = cV(x, y, z, t)\, |V(x, y, z, t)| + kA(x, y, z, t). \tag{3.2}$$

An additional property which often holds, but which will be taken as a basic assumption only in Section 6, is

(4)

$$-\operatorname{Cov}[V(x_0, y_0, z_0, t), A(x_0, y_0, z_0, t+\tau)] = \operatorname{Cov}[V(x_0, y_0, z_0, t+\tau), A(x_0, y_0, z_0, t)]. \tag{3.3}$$

Assumptions (1) and (2), as well as (4) arise quite naturally in applications, in several ways.

1. In the linear theory of waves (Wiegel [16], pp. 13–17) velocity and acceleration at position (x, y, t) in a wave with phase ϕ, traveling in the x-direction, is

$$V(x, y, t) = a\omega(\cosh ky/\sinh kd) \cos (kx - \omega t + \phi),$$
$$A(x, y, t) = a\omega^2(\cosh ky/\sinh kd) \sin (kx - \omega t + \phi)$$

where x and y are the horizontal and vertical coordinates, the origin is on the sea floor, ω is angular frequency, d is water depth, and k is determined by the relation $\omega^2 = gk \tanh kd$. If ϕ is assumed to be a uniform random variable on $(0, 2\pi)$, then $V(x_0, y_0, t_0)$ and $A(x_0, y_0, t_0)$ are independent by the orthogonality of the trigonometric functions and have zero expectations. This independence is preserved if the wave train is taken to be the sum of a large number of wavelets of this form, each with its own frequency, amplitude, and independent random phase. An application of the central limit theorem, under certain reasonable regularity conditions, gives that V and A, in the limit, are jointly normal (Pierson [10]) and the above properties continue to hold.

Equation (3.3) also follows directly from the elementary properties of trigonometric functions and holds in the limit.

2. Suppose $V(t)$ is taken as a separable, stationary, Gaussian process with an absolutely continuous spectrum whose density, $p(\lambda)$, satisfies

$$\int_{-\infty}^{\infty} \lambda^2 p(\lambda)\, d\lambda < \infty. \tag{3.4}$$

Then the local acceleration, $A(t)$, may be considered as the derivative in quadratic mean of $V(t)$ since (Doob [2], pp. 535–536) the derivative in quadratic mean exists, almost all sample functions are absolutely continuous, and the sample derivative and the quadratic mean derivatives are equal with probability 1.0.

Let

$$A_h(t) = [V(t+h) - V(t)]/h.$$

The mean vector, $\boldsymbol{\mu}$, and covariance matrix, $\mathbf{M}$, of $V(0)$, $A_h(0)$, $V(t)$, and $A_h(t)$ in the limit as h tends to zero are

$$\boldsymbol{\mu} = (m, o, m, o)$$

and

$$\lim_{h\to 0} \mathbf{M} = \begin{pmatrix} R(0) & 0 & R(t) & R'(t) \\ 0 & R''(0) & -R'(t) & -R''(t) \\ R(t) & -R'(t) & R(0) & 0 \\ R'(t) & -R''(t) & 0 & -R''(0) \end{pmatrix}.$$

in the above relation $R(h) = \mathrm{Cov}\,[V(t+h), V(t)]$ and it is assumed that $R(h)$ is everywhere differentiable and is twice differentiable at $h = 0$. (The latter condition is equivalent to (3.4)).

The jointly normal characteristic function of $V(0)$, $A_h(0)$, $V(t)$, and $A_h(t)$ remains normal as h tends to infinity and it follows from the multivariate continuity theorem (Takano [15], p. 58) and from the relation

$$A_h(t) \to_{\mathrm{q.m.}} A(t)$$

that $V(0)$, $A(0)$, $V(t)$, and $A(t)$ are jointly normally distributed, $V(0)$ and $A(0)$ are independent, and

$$\mathrm{Cov}\,[V(0), A(t)] = -\,\mathrm{Cov}\,[A(0), V(t)].$$

4. The probability law of $\Phi(x_0, y_0, z_0, t_0)$. In the following section, $V(x_0, y_0, z_0, t_0)$ and $A(x_0, y_0, z_0, t_0)$ will be written V and A for simplicity. Similarly, Var V, Var A, and EV will be denoted by σ^2, ρ^2, and m.

THEOREM 4.1. *The probability density of*

$$Y = \Phi/\rho k = [c|V|V + kA]/\rho k \tag{4.1}$$

is given by

$$f_Y(y) = (\alpha/2\pi^2)^{\frac{1}{2}} \exp[-(\gamma^2 + y^2)/2] \int_0^\infty t^{-\frac{1}{2}} \cdot \exp(-\alpha t - t^2/2) \cosh(\gamma(2\alpha t)^{\frac{1}{2}} + yt)\,dt \tag{4.2}$$

where

$$\alpha = \rho k/2c\sigma^2, \tag{4.3}$$

$$\gamma = m/\sigma. \tag{4.4}$$

PROOF. The distribution function of $W = V|V|$ is

$$\begin{aligned} F_W(w) = P[W \leqq w] &= P[V \leqq -(-w)^{\frac{1}{2}}], && \text{if } w < 0, \\ &= P[V \leqq w^{\frac{1}{2}}], && \text{if } w > 0, \\ &= \int_{-\infty}^{\pm|w|^{\frac{1}{2}}} ((2\pi)^{\frac{1}{2}}\sigma)^{-1} \exp[-(x-m)^2/2\sigma^2]\,dx \end{aligned}$$

where the + or − sign holds according as w is positive or negative, respectively.

The density of W is obtained by differentiation, for both $w > 0$ and $w < 0$, as

$$f_W(w) = (8\pi|w|\sigma^2)^{-\frac{1}{2}} \exp[-(\pm |w|^{\frac{1}{2}} - m)^2/2\sigma^2]. \tag{4.5}$$

The joint density of W and A is then

$$f_{W,A}(w, a) = [\exp(-(\pm |w|^{\frac{1}{2}} - m)^2/2\sigma^2 - a^2/2\rho^2)]/4\pi\sigma\rho|w|^{\frac{1}{2}}.$$

The density of the transformed variables

$$Y = [cW + kA]/\rho k,$$

$$T = cW/\rho k,$$

is

$$f_{Y,T}(y, t) = ((2\alpha)^{\frac{1}{2}}/4\pi)\, |t|^{-\frac{1}{2}} \exp(-(\pm(2\alpha|t|)^{\frac{1}{2}} - \gamma)^2/2 - (y - t)^2/2).$$

The marginal density of Y is obtained by integrating t over $(-\infty, 0)$ and $(0, \infty)$, transforming the $\int_{-\infty}^{0}$ to $\int_0^\infty$ and collecting terms

$$f_Y(y) = ((2\alpha)^{\frac{1}{2}}/4\pi) \exp(-(y^2 + \gamma^2)/2) \int_0^\infty [t^{-\frac{1}{2}} \exp(-\alpha t - t^2/2)] \cdot [\exp(-\gamma(2\alpha t)^{\frac{1}{2}} - yt) + \exp(\gamma(2\alpha t)^{\frac{1}{2}} + yt)]\, dt. \tag{4.6}$$

The asserted result follows immediately.

Equation (4.2) can be integrated numerically for particular values of γ and α. This is essentially the procedure used by Pierson and Holmes [12]. If $m = 0$ (i.e., $\gamma = 0$), then (4.2) can be stated in terms of parabolic cylinder functions in a more useful form.

THEOREM 4.2. *If* $m = 0$, (i.e., $\gamma = 0$), *then*

$$\begin{aligned} f_Y(y) &= (\alpha/8\pi)^{\frac{1}{2}} e^{-y^2/2}[\exp((\alpha + y)^2/4) U(0, \alpha + y) \\ &\qquad + \exp((\alpha - y)^2/4)\pi^{\frac{1}{2}} V(0, y - \alpha)], \quad \text{if } y \geqq \alpha \\ &= (\alpha/8\pi)^{\frac{1}{2}} e^{-y^2/2}[\exp((\alpha + y)^2/4) U(0, \alpha + y) \\ &\qquad + \exp((\alpha - y)^2/4) U(0, \alpha - y)], \quad \text{if } -\alpha \leqq y \leqq \alpha \\ &= (\alpha/8\pi)^{\frac{1}{2}} e^{-y^2/2}[\exp((\alpha + y)^2/4)\pi^{\frac{1}{2}} V(0, -\alpha - y) \\ &\qquad + \exp((\alpha - y)^2/4) U(0, \alpha - y)], \quad \text{if } y \leqq -\alpha \end{aligned} \tag{4.7}$$

where $U(a, x)$ *and* $V(a, x)$ *are the parabolic cylinder functions tabled by Miller* [7].

PROOF. In (4.6), let $m = 0$ (i.e., $\gamma = 0$). Then

$$f_Y(y) = ((2\alpha)^{\frac{1}{2}}/4\pi) e^{-y^2/2} [\int_0^\infty t^{-\frac{1}{2}} \exp(-(\alpha + y)t - t^2/2)\, dt + \int_0^\infty t^{-\frac{1}{2}} \exp(-(\alpha - y)t - t^2/2)\, dt]. \tag{4.8}$$

By equation (3), p. 119 of [3]

$$D_\nu(z) = (\Gamma(-\nu))^{-1} \exp(-z^2/4) \int_0^\infty t^{-(\nu+1)} \exp(-zt - t^2/2)\, dt.$$

If $\nu = -\frac{1}{2}$, this reduces to

$$\int_0^\infty t^{-\frac{1}{2}} \exp(-zt - t^2/2)\, dt = \pi^{\frac{1}{2}} e^{z^2/4} D_{-\frac{1}{2}}(z)$$

and hence can be applied to (4.8) to give

$$(4.9)\quad f_Y(y) = (\alpha/8\pi)^{\frac{1}{2}}e^{-y^2/2}[\exp((\alpha + y)^2/4)D_{-\frac{1}{2}}(\alpha + y) + \exp((\alpha - y)^2/4)D_{-\frac{1}{2}}(\alpha - y)].$$

But (Miller [7], p. 687, equations 19.3.7 and 19.3.8)

$$D_{-\frac{1}{2}}(m) = U(0, m), \qquad \text{if } m > 0,$$
$$= \pi^{\frac{1}{2}}V(0, -m), \quad \text{if } m < 0.$$

If this is applied to (4.9), then (4.7) is obtained.

THEOREM 4.3. *The moment generating function for Y is*

$$(4.10)\quad M_Y(s) = e^{(s^2-\gamma^2)/2}[(1 + s/\alpha)^{-\frac{1}{2}} \exp(\gamma^2(1 + s/\alpha)^{-1}/2)Q((1 + s/\alpha)^{-\frac{1}{2}}\gamma) + (1 - s/\alpha)^{-\frac{1}{2}} \exp(\gamma^2(1 - s/\alpha)^{-1}/2)Q(-(1 - s/\alpha)^{-\frac{1}{2}}\gamma)]$$

where

$$(4.11)\qquad Q(x) = \int_x^\infty (2\pi)^{-\frac{1}{2}}e^{-z^2/2}\,dz.$$

If $m = 0$ *(i.e.,* $\gamma = 0$*), then*

$$(4.12)\qquad M_Y(s) = \tfrac{1}{2}e^{s^2/2}[(1 + s/\alpha)^{-\frac{1}{2}} + (1 - s/\alpha)^{-\frac{1}{2}}].$$

PROOF. From (4.1) and (4.3)

$$(4.13)\qquad Y = (c/\rho k)V|V| + A/\rho = V|V|/2\alpha\sigma^2 + A/\rho$$

where, by assumption, V and A are independent. Hence

$$M_Y(s) = M_1(s)M_2(s)$$

where

$$M_1(s) = E[\exp(V|V|s/2\alpha\sigma^2)]$$
$$M_2(s) = E[\exp(sA/\rho)].$$

The random variable A/ρ is $N(0, 1)$, so

$$(4.14)\qquad M_2(s) = e^{s^2/2}.$$

The other generating function is obtained as follows.

$$M_1(s) = \int_{-\infty}^\infty \exp(sv|v|/2\alpha\sigma^2)(2\pi\sigma^2)^{-\frac{1}{2}} \exp(-(v - m)^2/2\sigma^2)\,dv$$
$$= \int_0^\infty (2\pi\sigma^2)^{-\frac{1}{2}} \exp(-sv^2/2\alpha\sigma^2 - (v + m)^2/2\sigma^2)\,dv + \int_0^\infty (2\pi\sigma^2)^{-\frac{1}{2}} \exp(sv^2/2\alpha\sigma^2 - (v - m)^2/2\sigma^2)\,dv.$$

The left integral comes from the integral over $(-\infty, 0)$ in the original expression after a transformation. Let $w = v/\sigma$, expand the quadratic in the exponent, and introduce γ from (4.4). Then

$$M_1(s) = e^{-\gamma^2/2}[\int_0^\infty \exp(-(1 + s/\alpha)w^2/2 - \gamma w)\,dw + \int_0^\infty \exp(-(1 - s/\alpha)w^2/2 + \gamma w)\,dw].$$

If the squares are completed in the two expressions and

$$z = (1 + s/\alpha)^{\frac{1}{2}}(w + \gamma/(1 + s/\alpha)), \qquad \text{in the first integral,}$$
$$= (1 - s/\alpha)^{\frac{1}{2}}(w - \gamma/(1 - s/\alpha)), \qquad \text{in the second integral,}$$

the expression becomes

$$M_1(s) = e^{-\gamma^2/2}[(1 + s/\alpha)^{-\frac{1}{2}} \exp(\gamma^2(1 + s/\alpha)^{-1}/2)Q((1 + s/\alpha)^{-\frac{1}{2}}\gamma) + (1 - s/\alpha)^{-\frac{1}{2}} \exp(\gamma^2(1 - s/\alpha)^{-1}/2)Q((1 - s/\alpha)^{-\frac{1}{2}}\gamma)]. \tag{4.15}$$

The required result follows from the product of (4.14) and (4.15).

Let $Z(x)$ and $P(x)$ denote the functions

$$Z(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2), \qquad P(x) = \int_0^x Z(y)\,dy. \tag{4.16}$$

THEOREM 4.4. *The first four moments of Y about the origin are*

$$EY = [\gamma Z(\gamma) + (\gamma^2 + 1)P(\gamma)]/\alpha; \tag{4.17}$$

$$E(Y^2) = (\gamma^4 + 6\gamma^2 + 3)/4\alpha^2 + 1; \tag{4.18}$$

$$E(Y^3) = (\gamma Z(\gamma)/4)[(\gamma^4 + 14\gamma^2 + 33)/\alpha^3 + 12/\alpha] + [(\gamma^6 + 15\gamma^4 + 45\gamma^2 + 15)/4\alpha^3 + 3(\gamma^2 + 1)/\alpha]P(\gamma); \tag{4.19}$$

$$E(Y^4) = (\gamma^8 + 28\gamma^6 + 210\gamma^4 + 420\gamma^2 + 105)/16\alpha^4 + 3(\gamma^4 + 6\gamma^2 + 3)/2\alpha^2 + 3 \tag{4.20}$$

and if $m = 0$ (*i.e.*, $\gamma = 0$) *these reduce to*

$$E(Y) = E(Y^3) = 0; \tag{4.21}$$

$$E(Y^2) = 3/4\alpha^2 + 1; \tag{4.22}$$

$$E(Y^4) = 105/16\alpha^4 + 9/2\alpha^2 + 3. \tag{4.23}$$

PROOF. These results are obtained by expanding each of the expressions in (4.10) as a power series in s, and combining the power series as indicated. As a check against these computations, the moments were also obtained directly by integration.

In the case where $m = 0$, (4.22) and (4.23) have been used by Pierson and Holmes [12] to estimate ρk and $c\sigma^2$ by the method of moments for measured ocean wave forces on a segment of a circular piling. The variances σ^2 and ρ^2 were then estimated from the spectral density of the sea surface elevation. The two sets of estimates were combined to yield estimates of c and k for ocean wave forces. Pierson and Holmes report that this procedure yielded reasonably consistent estimates although certain legal restrictions on their data prevented the publication of the numerical results. They do, however, show plots of the data histograms versus the theoretical densities which exhibit very satisfactory agreement.

The theoretical densities fitted to the data by Pierson and Holmes were evaluated by numerical integration of an equation much like (4.2) except that γ was zero.

In an appendix to their paper, Seymour Kaplan [5] obtained a Bessel function representation for (4.7) which was formally capable of being used to evaluate the density, but which presented certain difficulties in practice. The representation, (4.7), avoids most of these difficulties and with the tables of $U(0, x)$ and $V(0, x)$, permits the probability density for $\gamma = 0$ to be computed directly. The probability density and distribution function of the standardized variable obtained by dividing Y be the square root of (4.22),

$$Z = Y(1 + 3/4\alpha^2)^{-\frac{1}{2}} \tag{4.24}$$

which has mean zero and variance 1.0 has been extensively tabled by Brown and Borgman [20].

THEOREM 4.5. *Let Z be defined by* (4.24). *Then, taking limits in quadratic mean,*

$$\lim_{\alpha\to 0} Z = V|V|/\sigma^2 3^{\frac{1}{2}}, \tag{4.25}$$

$$\lim_{\alpha\to\infty} Z = A/\rho. \tag{4.26}$$

This implies that the limiting forms of the probability density are

$$\lim_{\alpha\to 0} f_Z(z) = 3^{\frac{1}{4}}(8\pi)^{-\frac{1}{2}}|Z|^{-\frac{1}{2}} \exp[-(\pm 3^{\frac{1}{4}}|Z|^{\frac{1}{2}} - \gamma)^2/2], \tag{4.27}$$

$$\lim_{\alpha\to\infty} f_Z(z) = e^{-z^2/2}/(2\pi)^{\frac{1}{2}} \tag{4.28}$$

where the symbol $\pm$ denotes a plus sign for $z > 0$ and a negative sign for $z < 0$.

PROOF. Combining (4.24) and (4.13) gives

$$\begin{aligned} Z &= (V|V|/2\alpha\sigma^2 + A/\rho)(1 + 3/4\alpha^2)^{-\frac{1}{2}} \\ &= V|V|/\sigma^2(4\alpha^2 + 3)^{\frac{1}{2}} + A/\rho(1 + (3/4\alpha^2))^{\frac{1}{2}}. \end{aligned}$$

Since

$$\begin{aligned} EA &= 0, \\ E(V^4) &= \sigma^4(\gamma^4 + 6\gamma^2 + 3), \\ E(A^2) &= \rho^2, \end{aligned}$$

it follows that

$$E|Z - V|V|/\sigma^2 3^{\frac{1}{2}}|^2 = (1 + 3/4\alpha^2)^{-2} + (\gamma^4 + 6\gamma^2 + 3)((4\alpha^2 + 3)^{-\frac{1}{2}} - 3^{-\frac{1}{2}})^2 \to 0 \quad \text{as } \alpha \to 0$$

and

$$E|Z - A/\rho|^2 = (\gamma^4 + 6\gamma^2 + 3)/(4\alpha^2 + 3) + ((4\alpha^2 + 3)^{-\frac{1}{2}} 2\alpha - 1)^2 \to 0 \quad \text{as } \alpha \to \infty.$$

This demonstrates (4.25) and (4.26). Convergence in quadratic mean implies convergence in law, so (4.28) follows immediately and (4.27) is obtained from (4.5) after making the transformation, $Z = W/\sigma^2 3^{\frac{1}{2}}$.

THEOREM 4.6. *For* $|y| \gg |\alpha|$, (4.7) *can be expressed in series form as*

$$f_Y(y) \sim (\alpha/8\pi)^{\frac{1}{2}} e^{-y^2/2}[(y+\alpha)^{-\frac{1}{2}}\{1 - 3/8(y+\alpha)^2 + 105/128(y+\alpha)^4 + \cdot\} + (2/(y-\alpha))^{\frac{1}{2}}\{1 + 3/8(y-\alpha)^2 + 105/128(y-\alpha)^4 + \cdots\}].$$

PROOF. This follows immediately from the corresponding expansions for $U(0, x)$ and $V(0, x)$ given by [7], p. 689, equations 19.8.1 and 19.8.2.

5. The covariance of $\Phi(x_1, y_1, z_1, t_1)$ and $\Phi(x_2, y_2, z_2, t_2)$. Since the covariance relations for $\Phi(x, y, z, t)$ can be obtained by a simple direct method if $m(x, y, z, t) \equiv 0$, this special case will be considered before the general formulation. Let Φ_1, V_1, and A_1 be the force, velocity and acceleration at (x_1, y_1, z_1, t_1). Similarly let Φ_2, V_2, and A_2 be corresponding properties at (x_2, y_2, z_2, t_2). The symbols r_{vv}, r_{aa}, r_{va}, and r_{av} will be used to specify the correlation coefficients for the pairs of random variables (V_1, V_2), (A_1, A_2), (V_1, A_2), and (A_1, V_2) respectively. The variances of V_1, V_2, A_1, and A_2 will be denoted by σ_1^2, σ_2^2, ρ_1^2, and ρ_2^2 respectively.

THEOREM 5.1. *The covariance of Φ_1 and Φ_2 is given by*

$$(5.1) \quad E[\Phi_1\Phi_2] = c^2\sigma_1^2\sigma_2^2 G(r_{vv}) + ck(8/\pi)^{\frac{1}{2}}(\rho_2\sigma_1^2 r_{va} + \rho_1\sigma_2^2 r_{av}) + k^2\rho_1\rho_2 r_{aa}$$

where

$$(5.2) \quad G(r) = [(2 + 4r^2) \arcsin r + 6r(1 - r^2)^{\frac{1}{2}}]/\pi.$$

PROOF. Since $m = 0$, $E\Phi_1 = E\Phi_2 = 0$. It follows that the covariance of Φ_1 and Φ_2 is

$$(5.3) \quad E[\Phi_1\Phi_2] = c^2E[(V_1V_2)|V_1V_2|] + ck\{E[V_1 |V_1| A_2] + E[A_1V_2 |V_2|]\} + k^2E[A_1A_2].$$

Now

$$E[V_1 |V_1| A_2] = E\{V_1 |V_1| E[A_2 \mid V_1]\} = (\rho_2/\sigma_1)r_{va}E\{V_1^2 |V_1|\}$$

since

$$E\{A_2 \mid V_1\} = (\rho_2/\sigma_1)r_{va}V_1.$$

But

$$E\{V_1^2 V_1|/\sigma_1^3\} = 2\int_0^\infty x^3(2\pi)^{-\frac{1}{2}} \exp[-x^2/2]\,dx = (8/\pi)^{\frac{1}{2}}.$$

So $E[V_1 |V_1| A_2] = (8/\pi)^{\frac{1}{2}}\rho_2\sigma_1^2 r_{va}$. By analogy $E[A_1V_2 |V_2|] = (8/\pi)^{\frac{1}{2}}\rho_1\sigma_2^2 r_{av}$. The last two equations yield the middle term in (5.1). Since the expectation in the last term is, by definition of r_{aa},

$$E[A_1A_2] = \rho_1\rho_2 r_{aa}, \tag{5.4}$$

only the expectation in the first term remains to be computed. Let

$$W(t) = V(t)/\sigma. \tag{5.5}$$

Then

$$\begin{aligned} G &= E[(V_1V_2)|V_1V_2|]/(\sigma_1^2\sigma_2^2) \\ &= E[W_1W_2\,|W_1W_2|] \end{aligned} \tag{5.6}$$

is the covariance function of the stochastic process obtained by making the zero-memory, non-linear transformation

$$h(w) = w\,|w| \tag{5.7}$$

on the Gaussian process $W(x, y, z, t)$. The covariance function for $W(x, y, z, t)$, evaluated at the same two space-time points is, by definition, r_{vv}.

A theorem due to Price [13] (see also Deutsch [1], pp. 15–26) can be used to obtain G. For simplicity in notation, let $r_{vv} = r$ in the following development, and consider G as a function of r. The theorem referred to gives integral representations for the various derivatives of G with respect to r as

$$\partial^n G/\partial r = E[h^{(n)}(W_1)h^{(n)}(W_2)] \tag{5.8}$$

with

$$h^{(n)}(w) = \partial^n h(w)/\partial w^n. \tag{5.9}$$

Differentiating (5.7) gives

$$\begin{aligned} h'(w) &= 2\,|w| \\ h''(w) &= 2 \operatorname{sgn} w \\ h'''(w) &= 4\delta(w) \end{aligned}$$

where $\delta(w)$ is the Dirac delta function. Hence

$$\begin{aligned} G(0) &= 0; \\ G'(0) &= 8/\pi; \\ G''(0) &= 0; \end{aligned} \tag{5.10}$$

and

$$G'''(r) = 16/[2\pi(1 - r^2)^{\frac{1}{2}}]. \tag{5.11}$$

The solution to (5.11) using (5.10) as boundary conditions is

$$G(r) = [(4r^2 + 2) \arcsin r + 6r(1 - r^2)^{\frac{1}{2}}]/\pi. \tag{5.12}$$

It follows that $E[(V_1V_2)|V_1V_2|] = \sigma_1^2\sigma_2^2G(r_{vv})$ which provides the first term in (5.1).

COROLLARY 5.1. *The series expansion of* $G(r)$ *about* $r = 0$ *is given by*

$$(5.13)\qquad G(r) = \pi^{-1}[8r + 4r^3/3 + r^5/15 + r^7/70 + 5r^9/1008 + \cdots]$$

PROOF. The expansion follows immediately from the binomial expansion of $(1 - r^2)^{\frac{1}{2}}$ and the elementary series for arc sin r.

The function $G(r)$ over the interval, $-1 < r < +1$, is quite close to a straight line. At $r = 1$, the straight line, $g_1(r) = 8r/\pi$ differs from $G(r)$ at $r = 1$ by only 15%. The cubic approximation

$$(5.14)\qquad g_3(r) = [8r + 4r^3/3]/\pi$$

is remarkably accurate, differing from $G(r)$ at $r = 1$ by only 1.1%.

THEOREM 5.2. *Let* $EV_1 = m_1$ *and* $EV_2 = m_2$. *The expected product of* Φ_1 *and* Φ_2 *is given by*

$$(5.15)\qquad \begin{aligned} E[\Phi_1\Phi_2] &= c^2\sigma_1^2\sigma_2^2G(\gamma_1, \gamma_2, r_{vv}) \\ &+ 4ck\sigma_1^2\rho_2r_{va}[Z(\gamma_1) + \gamma_1P(\gamma_1)] \\ &+ 4ck\sigma_2^2\rho_1r_{av}[Z(\gamma_2) + \gamma_2P(\gamma_2)] \\ &+ k^2\rho_1\rho_2r_{aa} \end{aligned}$$

with

$$(5.16)\qquad \begin{aligned} G(\gamma_1, \gamma_2, r) &= [(1 + \gamma_1^2)(1 + \gamma_2^2) + 4\gamma_1\gamma_2r + 2r^2] \\ &\quad\cdot[1 - 2Q(\gamma_1) - 2Q(\gamma_2) + 4L(\gamma_1, \gamma_2, r)] \\ &+ (2/\pi)(1 - r^2)^{\frac{1}{2}}(\gamma_1\gamma_2 + 3r)\exp[-(\gamma_1^2 - 2\gamma_1\gamma_2r + \gamma_2^2)/2(1 - r^2)] \\ &+ (2/\pi)^{\frac{1}{2}}[\gamma_1(1+\gamma_2^2) + 4\gamma_2r]\exp(-\gamma_1^2/2)P(\gamma_2 - \gamma_1r)/(1 - r^2)^{\frac{1}{2}} \\ &+ (2/\pi)^{\frac{1}{2}}[\gamma_2(1 + \gamma_1^2) + 4\gamma_1r]\exp(-\gamma_2^2/2)P((\gamma_1 - \gamma_2r)/(1 - r^2)^{\frac{1}{2}}) \end{aligned}$$

where

$$(5.17)\qquad \begin{aligned} \gamma_1 &= m_1/\sigma_1, \\ \gamma_2 &= m_2/\sigma_2, \end{aligned}$$

$$(5.18)\qquad \begin{aligned} &L(a, b, R) \\ &= \int_a^\infty\int_b^\infty [2\pi(1 - R^2)^{\frac{1}{2}}]^{-1}\exp[-(x^2 - 2xyR + y^2)/2(1 - R^2)]\,dx\,dy \end{aligned}$$

and $Q(x)$, $Z(x)$, *and* $P(x)$ *are defined* (4.11) *and* (4.16).

PROOF. Equation (5.3) still holds except that it must be interpreted as the expected product instead of the covariance. Also (5.4) is still valid. Proceeding as before

$$(5.19)\qquad \begin{aligned} E[V_1|V_1|A_2] &= E\{V_1|V_1|E[A_2 \mid V_1]\} \\ &= (\rho_2/\sigma_1)r_{va}E\{V_1|V_1|(V_1 - m_1)\} \\ &= \sigma_1^2\rho_2r_{va}[4Z(\gamma_1) + 4\gamma_1P(\gamma_1)]. \end{aligned}$$

By analogy

$$E[A_1 V_2 \,|V_2|] = \sigma_2^{\,2}\rho_1 r_{av}[4Z(\gamma_2) + 4\gamma_2 P(\gamma_2)].$$

The last two equations supply the middle terms in (5.15). This leaves only the first term to be evaluated. The differential equation approach by means of the Price theorem can still be formulated, but the integration is very difficult. It is better to evaluate (5.6) directly. The following lemma is useful in the evaluation.

LEMMA 5.1. *Let*

$$\begin{aligned} &J(a, b, R) \\ (5.20)\quad &= \int_a^\infty \int_b^\infty (x - b)^2 (y - a)^2 \exp[-(x^2 - 2Rxy + y^2)/2(1 - R^2)] \\ &\qquad \cdot [2\pi(1 - R^2)^{\frac{1}{2}}]^{-1}\, dx\, dy. \end{aligned}$$

Then

$$\begin{aligned} &J(a, b, R) = [(1 + a^2)(1 + b^2) + 4abR + 2R^2]L(a, b, R) \\ (5.21)\quad &+ [(ab + 3R)/2\pi](1 - R^2)^{\frac{1}{2}} \exp[-(a^2 - 2abR + b^2)/2(1 - R^2)] \\ &- [(a + ab^2 + 4bR)/(2\pi)^{\frac{1}{2}}]Q((b - aR)/(1 - R^2)^{\frac{1}{2}}) \exp[-a^2/2] \\ &- [(b + a^2 b + 4aR)/(2\pi)^{\frac{1}{2}}]Q((a - bR)/(1 - R^2)^{\frac{1}{2}}) \exp[-b^2/2]. \end{aligned}$$

PROOF OF LEMMA. This identity may be obtained as follows. The variables are transformed to

$$z = (x - Ry)/(1 - R^2)^{\frac{1}{2}},$$

$$y = y,$$

and z is integrated by parts. This gives

$$(5.22)\quad J(a, b, R) = (1 - R^2)\int_a^\infty (y - a)^2\{-\delta e^{-\delta^2/2}/(2\pi)^{\frac{1}{2}} + Q(\delta)(1 + \delta^2)\} \cdot e^{-y^2/2}/(2\pi)^{\frac{1}{2}}\, dy$$

where $\delta = (b - Ry)/(1 - R^2)^{\frac{1}{2}}$. This can be separated into two integrals in the obvious way.

Since

$$\delta^2 + y^2 = b^2 + [(y - Rb)/(1 - R^2)^{\frac{1}{2}}]^2,$$

the first integral, after a change of variable to $w = (y - Rb)/(1 - R^2)^{\frac{1}{2}}$, becomes

$$\sum_{n=0}^{3} (A_n/(2\pi)^{\frac{1}{2}})\int_\theta^\infty W^n \exp[-w^2/2]\, dw = \sum_{n=0}^{3} A_n S_n$$

where $\theta = (a - bR)/(1 - R^2)^{\frac{1}{2}}$ and A_n is a function of a, b, and R.

The second integral arising from (5.22) can be expressed as

$$\sum_{m=0}^{4} (B_m/(2\pi)^{\frac{1}{2}})\int_a^\infty y^m Q(\delta) \exp[-y^2/2]\, dy = \sum_{m=0}^{4} B_m T_m\,.$$

Both sets of integrals, S_n and T_m, can be integrated by parts using the integrability of $x \exp[-x^2/2]$. This reduces $J(a, b, R)$ to a function of a, b, and R involving the functions $Q(x)$ and

$$(2\pi)^{-\frac{1}{2}}\int_a^\infty Q(\delta)\exp[-y^2/2]\,dy = L(a, b, R).$$

This completes the proof of the lemma.

The function $G(r)$ can be evaluated immediately.

By definition, (see (5.6)),

$$G(r) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(w_1w_2)|w_1w_2|\,[2\pi(1 - r^2)^{\frac{1}{2}}]^{-1}$$
$$\cdot\exp\{-[(w_1-\gamma_1)^2 - 2r(w_1-\gamma_1)(w_2-\gamma_2) + (w_2-\gamma_2)^2][2(1-r^2)]^{-1}\}\,dw_1\,dw_2.$$

If $x = w_1 - \gamma_1$, $y = w_2 - \gamma_2$, this becomes after some manipulation

$$G(r) = J(-\gamma_1, -\gamma_2, r) - J(-\gamma_1, \gamma_2, -r) + J(\gamma_1, \gamma_2, r) - J(\gamma_1 - \gamma_2, -r).$$

The collection of like terms after substituting in the lemma identity gives (5.16), provided the elementary identity (Zelen and Severo [18], p. 936)

$$L(-\gamma_1, -\gamma_2, r) - L(-\gamma_1, \gamma_2, -r) + L(\gamma_1, \gamma_2, r) - L(\gamma_1, -\gamma_2, -r)$$
$$= 1 - 2Q(\gamma_1) - 2Q(\gamma_2) + 4L(\gamma_1, \gamma_2, r)$$

is introduced.

If $m = 0$, (i.e., $\gamma = 0$), then (5.16) reduces to (5.12) after substituting the identity ([18], pp. 937, equation 26.3.19)

$$L(0, 0, r) = \tfrac{1}{4} + \arcsin r/2\pi.$$

The function $L(a, b, r)$ is tabled by Pearson [9] and a graphical procedure for its evaluation is given by [18].

THEOREM 5.3. *The series expansion of* $G(\gamma_1, \gamma_2, r)$ *about* $r = 0$ *is*

$$\begin{aligned}
G(\gamma_1, \gamma_2, r)& \\
&= 4\{\gamma_1 Z(\gamma_1) + (1+\gamma_1^2)P(\gamma_1)\}\{\gamma_2 Z(\gamma_2) + (1+\gamma_2^2)P(\gamma_2)\}\\
&+ 16[\{Z(\gamma_1) + \gamma_1 P(\gamma_1)\}\{Z(\gamma_2) + \gamma_2 P(\gamma_2)\}r\\
&+ P(\gamma_1)P(\gamma_2)r^2/2! + Z(\gamma_1)Z(\gamma_2)r^3/3! \qquad (5.23)\\
&+ \gamma_1\gamma_2 Z(\gamma_1)Z(\gamma_2)r^4/4!\\
&+ (\gamma_1^2 - 1)(\gamma_2^2 - 1)\,Z(\gamma_1)\,Z(\gamma_2)r^5/5!\\
&+ \cdots + He_{n-3}(\gamma_1)He_{n-3}(\gamma_2)Z(\gamma_1)Z(\gamma_2)r^n/n! + \cdots]
\end{aligned}$$

where He_n, $n = 0, 1, 2 \cdots$ *are the Hermite polynomials* (*Kendall and Stuart* [6] *p.* 155, *Equation* 6.23) *and* $Z(x)$ *and* $P(x)$ *are defined by* (4.16).

PROOF. The function, $G(\gamma_1, \gamma_2, r)$, is defined by the second half of (5.6) as being

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(w_1w_2)|w_1w_2|$$
$$\cdot\exp\{-[(w_1-\gamma_1)^2 - 2r(w_1-\gamma_1)(w_2-\gamma_2) + (w_2-\gamma_2)^2][2(1-r^2)]^{-1}\}\,dw_1\,dw_2.$$

With the transformation $u_i = w_i - \gamma_i$ for $i = 1, 2$, this becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (u_1 + \gamma_1)(u_2 + \gamma_2) \, |(u_1 + \gamma_1)(u_2 + \gamma_2)| \, f(u_1, u_2, r) \, du_1 \, du_2$$

where $f(u_1, u_2, r)$ is the bivariate normal density with zero means and unit variances. Now (5.24) can be interpreted as the covariance of the output of a nonlinear, zero-memory device characterized by $h(u) = (u + \gamma) \, |u + \gamma|$, which is acting on the Gaussian stochastic process

$$u(x, y, z, t) = [V(x, y, z, t) - m(x, y, z, t)/\sigma(x, y, z, t).$$

Since

$$\begin{aligned} h'(u) &= 2 \, |u + \gamma| \, ; \\ h''(u) &= 2 \operatorname{sgn} (u + \gamma); \\ h'''(u) &= 4\delta(u + \gamma); \\ h^{iv}(u) &= 4\delta'(u + \gamma); \\ &\vdots \\ h^{(n)}(u) &= 4\delta^{(n-3)}(u + \gamma). \end{aligned}$$

The integrals from the Price theorem (see (5.8) and (5.9)) evaluated at $r = 0$ gives the coefficients in (5.23). In the first three coefficients, the integrals are separated into an integral over $(-\infty, -\gamma_i)$ and another over $(-\gamma_i, \infty)$, for $i = 1, 2$. Within these ranges the absolute values and sgn $(u_i + \gamma_i)$ can be eliminated and the integrals evaluated by elementary methods.

The fourth and higher-order coefficients involve the Dirac delta function and its derivatives. Hence setting $r = 0$,

$$\begin{aligned} G^{iv}(\gamma_1, \gamma_2, 0) &= \prod_{i=1}^{2} \int_{-\infty}^{\infty} 4\delta(u_i + \gamma_i) Z(u_i) \, du_i \\ &= 16 Z(\gamma_1) Z(\gamma_2). \end{aligned}$$

The higher-order coefficients lead to the Hermite polynomials, through the relation

$$\begin{aligned} G^{(n)}(\gamma_1, \gamma_2, 0) &= \prod_{i=1}^{2} \int_{-\infty}^{\infty} 4\delta^{(n-3)}(u_i + \gamma_i) Z(u_i) \, du_i \\ &= 16 \prod_{i=1}^{2} \{(-1)^{n-3} Z^{(n-3)}(\gamma_i)\} \\ &= 16 \prod_{i=1}^{2} \{He_{n-3}(\gamma_i) Z(\gamma_i)\}. \end{aligned}$$

(See Friedman [4] pp. 136–143).

6. The spectral density of $\Phi(x_0, y_0, z_0, t)$. Suppose that $V(t) = V(x_0, y_0, z_0, t)$ and $A(t) = A(x_0, y_0, z_0, t)$ are stationary Gaussian processes with constant means m and 0 respectively. Let the covariance function matrix be

$$E\left[\begin{pmatrix} V(t) - m \\ A(t) \end{pmatrix} (V(t + h) - m, A(t + h))\right] = \begin{pmatrix} R_{vv}(h) & R_{va}(h) \\ R_{av}(h) & R_{vv}(h) \end{pmatrix}.$$

It will be assumed that the spectrum of $\Phi(t)$ is absolutely continuous.

The covariance function of $\Phi(t)$, denoted by $R_{\phi\phi}(h)$, can be obtained from (5.15) and (4.1) as

$$\begin{aligned} R_{\phi\phi}(h) &= c^2\sigma^4 G(\gamma, \gamma, R_{vv}(h)/\sigma^2) \\ &\quad + 4ck\sigma^2\rho[Z(\gamma) + \gamma P(\gamma)]\,[r_{va} + r_{av}] \\ &\quad\quad + k^2 R_{aa}(h) - \rho^2 k^2 (EY)^2. \end{aligned} \tag{6.1}$$

The relation, (3.3), will be assumed to hold in this section, so the middle term in (6.1) is zero. Hence, substituting in (4.17),

$$\begin{aligned} R_{\phi\phi}(h) &= c^2\sigma^4 G(\gamma, \gamma, R_{vv}(h)/\sigma^2) + k^2 R_{aa}(h) \\ &\quad - 4c^2\sigma^4[\gamma Z(\gamma) + (\gamma^2 + 1)P(\gamma)]^2. \end{aligned} \tag{6.2}$$

If $R_{vv}(h)$ and $R_{aa}(h)$ are known, the spectral density for $\Phi(t)$ may be obtained by making the Fourier transform of (6.2). In principle this is possible, although numerical integration may be necessary. An approximation that appears quite accurate and greatly reduces the work is provided by (5.14) for the case of $m = 0$, and by analogous relations from (5.23) if $m \neq 0$.

Let $\{S(f)\}^{2*}$ denote

$$\{S(f)\}^{2*} = \int_{-\infty}^{\infty} S(f - g)S(g)\, dg$$

and for $n > 2$

$$\{S(f)\}^{n*} = \int_{-\infty}^{\infty} S(f - g)\{S(g)\}^{(n-1)*}\, dg. \tag{6.3}$$

Thus $\{S(f)\}^{n*}$ is the n-fold convolution of $S(f)$ with itself. The expression $\{S(f)\}^{1*}$ is understood to mean $S(f)$.

If $S_{vv}(f)$ and $S_{aa}(f)$ are the spectral densities of $V(t)$ and $A(t)$, and (5.23) is represented, with $\gamma_1 = \gamma_2 = \gamma$, as

$$G(\gamma, \gamma, r) = \sum_{n=0}^{\infty} C_n r^n/n!$$

then at least formally, the spectral density of $\Phi(t)$ is given by

$$S_{\phi\phi}(f) = c^2\sigma^4 \sum_{n=1}^{\infty} C_n\{S_{vv}(f)\}^{n*}/n!\sigma^{2n} + k^2 S_{aa}(f). \tag{6.4}$$

The partial sums of (6.4) are relatively easy to compute and appear to give reasonable approximations to $S_{\phi\phi}(f)$ if γ is close to zero. Questions of convergence of (6.4) and bounds on accuracy of approximation have not been investigated. An application of the preceeding theory to engineering problems is given by Borgman [19].

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