ASYMPTOTIC VARIANCES AND COVARIANCES OF MAXIMUM-LIKELIHOOD ESTIMATORS, FROM CENSORED SAMPLES, OF THE PARAMETERS OF WEIBULL AND GAMMA POPULATIONS

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0. Summary. For ordered samples of size n, with proportions q_1 and q_2 of the sample values censored from below and from above, respectively, from three-parameter Weibull and gamma populations, expressions are given for the elements of the maximum-likelihood information matrices, each element being the negative of the expected value of one of the second partial derivatives of the likelihood function with respect to the parameters. An important result is that, while one or more of the elements become infinite for values of the shape parameter less than or equal to 2 when $q_1 = 0$, this does not happen for $q_1 > 0$. If one lets $n \to \infty$ while holding q_1 and q_2 fixed and then inverts the information matrix and its submatrices, the results are the asymptotic variance-covariance matrices whose elements are the asymptotic variances and covariances of the joint maximum-likelihood estimators of all three parameters and of any one or two parameters, the other(s) being known. Tables are given of the coefficients of $(1/n)$ times powers of the scale parameter θ in the asymptotic variances and covariances for the cases $q_1 = 0.000(0.005)0.25$, $q_2 = 0.00(0.25)0.75$ for both Weibull and gamma populations with shape parameters 1, 2, and 3, omitting cases for which $q_1 = 0$ when the shape parameter is 1 or 2 and the location parameter is one of those being estimated. Results of a limited Monte Carlo study indicate that when at least one of the parameters is known, the variances and covariances of samples of size as small as 50 agree closely with the results given by the asymptotic formulas, but that when all three parameters are unknown, the variances and the absolute values of the covariances, even for samples of size as large as 100, are substantially in excess of the asymptotic values.

1. Introduction. While the method of maximum likelihood was employed by Gauss (1880) in particular cases, its general use was first proposed by Fisher (1912). Halperin (1952) proved under mild regularity conditions that the maximum-likelihood estimator of a single parameter from singly censored samples is consistent, asymptotically normally distributed and of minimum variance for large samples and outlined the proof of an extension of the theorem to the case of several parameters and indicated that his results would hold for more general censoring. Plackett (1958) showed that maximum-likelihood estimators are asymptotically linear and that the best linear unbiased estimators are asymptotically normal and efficient. This method has been applied to the problem of

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estimating the parameters of the gamma (Pearson Type III) population by a number of authors, including Fisher (1922), Masuyama and Kuroiwa (1951), Des Raj (1953), Chapman (1956), Greenwood and Durand (1960), Wilk, Gnanadesikan, and Huyett (1962), Mickey, Mundle, Walker, and Glinski (1963), and Harter and Moore (1965). It has also been used to estimate the parameters of the Weibull population by Kao (1956), (1958), Leone, Rutenberg, and Topp (1960), Lehman (1963), Dubey (1963), Ravenis (1964), and Harter and Moore (1965). Of these authors, only Dubey and Ravenis have given the information matrix; both of them give results for a complete (uncensored) sample from the three-parameter Weibull population, using a slightly different parameterization than the one which will be found in the present paper. More recently, Cohen (1965) has given the information matrix for complete, singly censored, and progressively censored samples from a two-parameter Weibull population.

This paper gives the elements of the maximum-likelihood information matrices for the doubly censored, three-parameter Weibull and gamma populations in Sections 2 and 3, respectively. Two well-known populations are special cases of these. The negative exponential population is a special case (with shape parameter equal to 1) of both the Weibull population and the gamma population. The chi-square population with ν degrees of freedom is a gamma population with shape parameter $\alpha = \nu/2$. Conditions for regular estimation are discussed in Section 4. Numerical inversion of the information matrices to obtain the asymptotic variance-covariance matrices is discussed in Section 5, and the results for some typical values of the shape parameters and proportions censored are given in Tables W1–W3 and G1–G3. Results of a Monte Carlo study are reported in Section 6.

2. Weibull maximum-likelihood information matrix. The probability density function of the random variable X having a Weibull distribution with location parameter $c \geqq 0$, scale parameter θ, and shape parameter K is given by

$$(2.1) \qquad g(x; c, \theta, K) = [K(x - c)^{K-1}/\theta^K] \exp\{-[(x - c)/\theta]^K\}, \qquad \theta, K > 0, x \geqq c \geqq 0.$$

The natural logarithm of the likelihood function for a sample of size n from such a population, the lowest r and the highest $n - m$ sample values having been censored, is given by

$$(2.2) \qquad L_{r+1,m} = \ln n! - \ln(n - m)! - \ln r! + (m - r)(\ln K - \ln \theta) + (K - 1)\textstyle\sum_{i=r+1}^{m} \ln z_i - \sum_{i=r+1}^{m} z_i^K - (n - m)z_m^K + r \ln F(z_{r+1}),$$

where $z_i = (x_i - c)/\theta$, $F(z_i) = \int_0^{z_i} f(t)\,dt = 1 - \exp(-z_i^K)$, and $f(z_i) = Kz_i^{K-1}\exp(-z_i^K)$.

In the case of complete samples, the standard procedure for finding the elements of the information matrix is to take the limits, as $n \to \infty$, of the negatives of the expected values of the second partial derivatives of the likelihood function with respect to the parameters. In the case of censored samples, these limits do not exist in the usual sense, but they can be replaced by limits in probability of conditional expectations [see Halperin (1952)]. Let $q_1 = r/n$, $q_2 = (n - m)/n$, and $p = 1 - q_1 - q_2 = (m - r)/n$. As $n \to \infty$ (q_1 and q_2 fixed), z_{r+1} converges in probability to $\hat{z}_{r+1}$ where $F(\hat{z}_{r+1}) = \int_0^{\hat{z}_{r+1}} f(t)\, dt = q_1$ and z_m converges in probability to $\hat{z}_m$ where $1 - F(\hat{z}_m) = \int_{\hat{z}_m}^{\infty} f(t)\, dt = q_2$. If one denotes by $E[\cdots]$ a conditional expectation given z_{r+1} and z_m, the elements of the information matrix (multiplied by $1/n$) may be written as

$$\begin{aligned} &\lim \text{pr}\, E[-(1/n)(\partial^2 L/\partial\theta^2)] \\ &= \theta^{-2}\{-Kp + K(K+1)[\Gamma(2; \hat{z}_m^K) - \Gamma(2; \hat{z}_{r+1}^K)] + K(K+1)q_2\hat{z}_m^K \\ &\quad + \hat{z}_{r+1}f(\hat{z}_{r+1})[K\hat{z}_{r+1}^K - (K+1)q_1]/q_1\} = v^{11}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\lim \text{pr}\, E[-(1/n)(\partial^2 L/\partial K^2)] \\ &= p/K^2 + [\Gamma''(2; \hat{z}_m^K) - \Gamma''(2; \hat{z}_{r+1}^K)]/K^2 + q_2\hat{z}_m^K \ln^2 \hat{z}_m \\ &\quad + \hat{z}_{r+1}f(\hat{z}_{r+1}) \ln^2 \hat{z}_{r+1}[\hat{z}_{r+1}^K - q_1]/Kq_1 = v^{22}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\lim \text{pr}\, E[-(1/n)(\partial^2 L/\partial c^2)] \\ &= \theta^{-2}\{(K-1)[\Gamma(1 - 2/K; \hat{z}_m^K) - \Gamma(1 - 2/K; \hat{z}_{r+1}^K)] \\ &\quad + K(K-1)[\Gamma(2 - 2/K; \hat{z}_m^K) - \Gamma(2 - 2/K; \hat{z}_{r+1}^K)] \\ &\quad + K(K-1)q_2\hat{z}_m^{K-2} + f(\hat{z}_{r+1})[K\hat{z}_{r+1}^K - (K-1)q_1]/q_1\hat{z}_{r+1}\} = v^{33}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} &\lim \text{pr}\, E[-(1/n)(\partial^2 L/\partial\theta\partial K)] \\ &= \theta^{-1}\{p - [\Gamma'(2; \hat{z}_m^K) - \Gamma'(2; \hat{z}_{r+1}^K)] \\ &\quad - [\Gamma(2; \hat{z}_m^K) - \Gamma(2; \hat{z}_{r+1}^K)] - q_2\hat{z}_m^K(K \ln \hat{z}_m + 1) \\ &\quad - \hat{z}_{r+1}f(\hat{z}_{r+1})[K\hat{z}_{r+1} \ln \hat{z}_{r+1} - (K \ln \hat{z}_{r+1} + 1)q_1]/Kq_1\} = v^{12}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\lim \text{pr}\, E[-(1/n)(\partial^2 L/\partial\theta\partial c)] \\ &= \theta^{-2}\{K^2[\Gamma(2 - 1/K; \hat{z}_m^K) - \Gamma(2 - 1/K; \hat{z}_{r+1}^K)] + K^2q_2\hat{z}_m^{K-1} \\ &\quad + Kf(\hat{z}_{r+1})[\hat{z}_{r+1}^K - q_1]/q_1\} = v^{13}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\lim \text{pr}\, E[-(1/n)(\partial^2 L/\partial K\partial c)] \\ &= \theta^{-1}\{[\Gamma(1 - 1/K; \hat{z}_m^K) - \Gamma(1 - 1/K; \hat{z}_{r+1}^K)] - [\Gamma(2 - 1/K; \hat{z}_m^K) \\ &\quad - \Gamma(2 - 1/K; \hat{z}_{r+1}^K)] - [\Gamma'(2 - 1/K; \hat{z}_m^K) - \Gamma'(2 - 1/K; \hat{z}_{r+1}^K)] \\ &\quad - q_2\hat{z}_m^{K-1}(K \ln \hat{z}_m + 1) \\ &\quad - f(\hat{z}_{r+1})[K\hat{z}_{r+1}^K \ln \hat{z}_{r+1} - (K \ln \hat{z}_{r+1} + 1)q_1]/Kq_1\} = v^{23}, \end{aligned} \tag{2.8}$$

where the definition $\Gamma(s; b) = \int_0^b t^{s-1}e^{-t}\, dt$ of the incomplete gamma function for $s > 0$ is extended to cases in which $s \leqq 0$, the terms involving $q_1(q_2)$ drop out when $q_1(q_2) = 0$, and the primes indicate differentiation.

3. Gamma maximum-likelihood information matrix. The probability density function of the random variable X having a gamma distribution with location parameter $c \geqq 0$, scale parameter θ, and shape parameter α is given by

$$g(x; c, \theta, \alpha) = [1/\Gamma(\alpha)\theta][(x - c)/\theta]^{\alpha-1} \exp[-(x - c)/\theta], \quad \theta, \alpha > 0, x \geqq c \geqq 0. \tag{3.1}$$

The natural logarithm of the likelihood function for a sample of size n from such a population, the lowest r and the highest $n - m$ sample values having been censored, is given by

$$\begin{aligned} L_{r+1,m} = {} & \ln n! - \ln(n - m)! - \ln r! - (m - r)[\ln \Gamma(\alpha) + \ln \theta] \\ & + (\alpha - 1) \textstyle\sum_{i=r+1}^{m} \ln z_i - \sum_{i=r+1}^{m} z_i \\ & + (n - m) \ln [1 - F(z_m)] + r \ln F(z_{r+1}), \end{aligned} \tag{3.2}$$

where $z_i = (x_i - c)/\theta$, $F(z_i) = \int_0^{z_i} f(t)\,dt = \Gamma_{z_i}(\alpha)/\Gamma(\alpha)$, and $f(z_i) = z_i^{\alpha-1} \exp(-z_i)/\Gamma(\alpha)$.

As in the case of the Weibull population, the elements of the information matrix can be found by taking the limits in probability of the conditional expectations of the negatives of the second partial derivatives of the likelihood function with respect to the parameters. Let $q_1 = r/n$, $q_2 = (n - m)/n$, and $p = 1 - q_1 - q_2 = (m - r)/n$. As $n \to \infty$ (q_1 and q_2 fixed), z_{r+1} converges in probability to $\hat{z}_{r+1}$ where $F(\hat{z}_{r+1}) = \int_0^{\hat{z}_{r+1}} f(t)\,dt = q_1$ and z_m converges in probability to $\hat{z}_m$ where $1 - F(\hat{z}_m) = \int_{\hat{z}_m}^{\infty} f(t)\,dt = 1 - q_2$.

The elements of the information matrix (multiplied by $1/n$) may be written as

$$\begin{aligned} & \lim \text{pr } E[-(1/n)(\partial^2 L/\partial\theta^2)] \\ & = \theta^2\{-\alpha p + 2[\Gamma(\alpha + 1; \hat{z}_m) - \Gamma(\alpha + 1; \hat{z}_{r+1})]/\Gamma(\alpha) \\ & \quad - \hat{z}_m f(\hat{z}_m)[q_2(\hat{z}_m - \alpha - 1) - \hat{z}_m f(\hat{z}_m)]/q_2 \\ & \quad + \hat{z}_{r+1} f(\hat{z}_{r+1})[q_1(\hat{z}_{r+1} - \alpha - 1) + \hat{z}_{r+1} f(\hat{z}_{r+1})]/q_1\} = v^{11}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} & \lim \text{pr } E[-(1/n)(\partial^2 L/\partial\alpha^2)] \\ & = \{\Gamma(\alpha)\Gamma''(\alpha) - [\Gamma'(\alpha)]^2\}/[\Gamma(\alpha)]^2 \\ & \quad - \{[\Gamma(\alpha) - \Gamma(\alpha; \hat{z}_m)][\Gamma''(\alpha) - \Gamma''(\alpha; \hat{z}_m)] \\ & \quad - [\Gamma'(\alpha) - \Gamma'(\alpha; \hat{z}_m)]^2\}/q_2[\Gamma(\alpha)]^2 \\ & \quad - \{\Gamma(\alpha; \hat{z}_{r+1})\Gamma''(\alpha; \hat{z}_{r+1}) - [\Gamma'(\alpha; \hat{z}_{r+1})]^2\}/q_1[\Gamma(\alpha)]^2 = v^{22}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} & \lim \text{pr } E[-(1/n)(\partial^2 L/\partial c^2)] \\ & = \theta^{-2}\{(\alpha - 1)[\Gamma(\alpha - 2; \hat{z}_m) - \Gamma(\alpha - 2; \hat{z}_{r+1})]/\Gamma(\alpha) \\ & \quad - f(\hat{z}_m)[q_2(\hat{z}_m - \alpha + 1) - \hat{z}_m f(\hat{z}_m)]/q_2 \hat{z}_m \\ & \quad + f(\hat{z}_{r+1})[q_1(\hat{z}_{r+1} - \alpha + 1) + \hat{z}_{r+1} f(\hat{z}_{r+1})]/q_1 \hat{z}_{r+1}\} = v^{33}, \end{aligned} \tag{3.5}$$

$$\lim \operatorname{pr} E[-(1/n)(\partial^2 L/\partial\theta\partial\alpha)]$$

$$(3.6) \quad = \theta^{-1}\{p - \hat{z}_m f(\hat{z}_m)\{q_2 \ln \hat{z}_m - [\Gamma'(\alpha) - \Gamma'(\alpha; \hat{z}_m)]/\Gamma(\alpha)\}/ q_2 + \hat{z}_{r+1} f(\hat{z}_{r+1})[q_1 \ln \hat{z}_{r+1} - \Gamma'(\alpha; \hat{z}_{r+1})/\Gamma(\alpha)]/q_1\} = v^{12},$$

$$\lim \operatorname{pr} E[-(1/n)(\partial^2 L/\partial\theta\partial c)]$$

$$(3.7) \quad = \theta^{-2}\{p - f(\hat{z}_m)[q_2(\hat{z}_m - \alpha) - \hat{z}_m f(\hat{z}_m)]/q_2 + f(\hat{z}_{r+1})[q_1(\hat{z}_{r+1} - \alpha) + \hat{z}_{r+1} f(\hat{z}_{r+1})]/q_1\} = v^{13},$$

$$\lim \operatorname{pr} E[-(1/n)(\partial^2 L/\partial\alpha\partial c)]$$

$$(3.8) \quad = \theta^{-1}\{[\Gamma(\alpha - 1; \hat{z}_m) - \Gamma(\alpha - 1; \hat{z}_{r+1})]/\Gamma(\alpha) - f(z_m)\{q_2 \ln \hat{z}_m - [\Gamma'(\alpha) - \Gamma'(\alpha; \hat{z}_m)]/\Gamma(\alpha)\}/q_2 + f(\hat{z}_{r+1})[q_1 \ln \hat{z}_{r+1} - \Gamma'(\alpha; \hat{z}_{r+1})/\Gamma(\alpha)]/q_1\} = v^{23},$$

where the notation is the same as in equations (2.3)–(2.8).

4. Regular versus non-regular estimation. The authors have verified that the regularity conditions [see Kendall and Stuart (1961), pp. 43–44], where the parameter θ is interpreted as a vector, and the additional assumptions [see Halperin (1952)] necessary in the case of estimation from censored samples are satisfied if and only if at least one of the following conditions holds: (1) the shape parameter is greater than 2; (2) the location parameter is known; (3) a proportion $q_1 > 0$ of the sample is censored from below. Therefore estimation is regular in all the cases considered here and the estimators are asymptotically of minimum variance, unbiased, and p-variate normal, where p is the number of parameters being estimated.

5. Asymptotic variances and covariances. For all cases in which estimation is regular, the asymptotic variance-covariance matrix for the estimators $\hat{\theta}$, $\hat{K}$ [or $\hat{\alpha}$], and $\hat{c}$ is given by $(1/n)[v_{ij}]$, where $[v_{ij}] = [v^{ij}]^{-1}$ and the v^{ij} are given by equations (2.3)–(2.8) for the Weibull population and by equations (3.3)–(3.8) for the gamma population. The computation of the elements v^{ij} of the information matrix (multiplied by $1/n$) and the inversion of this matrix and its submatrices to obtain the coefficients of $1/n$ in the variance-covariance matrices for simultaneous estimation of all three parameters, or of one or two parameters when the other(s) are known, were performed on the IBM 7094 computer. Computation is quite straightforward when the shape parameter is greater than 2 and/or the location parameter is known; otherwise, one encounters quantities of the form $\Gamma(s; b) - \Gamma(s; a)$, with $s \geqq 0$. These become infinite when $a = 0$ and take the indeterminate form $\infty - \infty$ when $a > 0$. In the latter case, one may use the alternate form $\int_a^b t^{s-1}e^{-t}\,dt$, which is finite and can be evaluated by numerical integration. Since $a = \hat{z}_{r+1}^K$ for the Weibull population and $a = \hat{z}_{r+1}$ for the gamma population, $a = 0$ if and only if $\hat{z}_{r+1} = 0$, which is true if and only if $q_1 = 0$. Hence the asymptotic variances and covariances of the estimators have not been found when $q_1 = 0$, the location parameter is one of those being estimated, and the shape parameter K (Weibull) or α (gamma) is less than or equal to 2. With this exception, the coefficients of $(1/n)$ times a power of the scale parameter θ in the asymptotic variances and covariances were computed for $q_1 = 0.000(0.005)0.025$, $q_2 = 0.00(0.25)0.75$, and shape parameters 1, 2, and 3, accurate to within a unit in the fifth decimal place. To attain this accuracy, it was necessary to perform the computations in double precision, since several decimal places may be lost in inverting the information matrix. The results, rounded to three decimal places, are given for Weibull populations with $K = 1(1)3$ in Tables W1–W3 and for gamma populations with $\alpha = 1(1)3$ in Tables G1–G3. Interpolation in these tables will not always give accurate results, but the authors have a computer program which can readily be used to obtain additional values.

It will be observed that the above procedure breaks down if estimation is non-regular. No systematic attempt has been made to determine the asymptotic variances and covariances of non-regular estimators, though this may be possible in some cases. It is known, for example [see Dubey (1965) and Blischke et al. (1965)], that when the shape parameter is less than 2, the first order statistic is a hyper-efficient estimator of the location parameter; in particular, for the special case of the exponential population (shape parameter 1), this estimator has variance proportional to n^{-2} rather than to n^{-1} as in the case of regular estimators.

6. Monte Carlo study. In order to check the rate of convergence of the variances and covariances to their asymptotic values, a limited Monte Carlo study was conducted in which 500 random samples each of sizes $n = 50$ and $n = 100$ were drawn from a Weibull population with location parameter $c = 20$, scale parameter $\theta = 10$, and shape parameter $K = 3$ and the iterative procedure outlined by Harter and Moore (1965) was used to estimate all three parameters, also every subset of one or two parameters with the other(s) known. The means, variances and covariances of the estimates from the complete samples were then computed, and the results are shown in Table M1, where the asymptotic values of the variances and covariances, found by multiplying coefficients read from Table W3 by the proper powers of θ and dividing by n, are shown in juxtaposition with the sample values for purposes of comparison. The results shown in Table M1 lead to the tentative conclusion that when all three parameters are unknown, the variances and the absolute values of the covariances exceed their asymptotic values, with the excess closely proportional to n^{-2}, a phenomenon previously observed [see Harter and Moore (1966)] in the case of local-maximum-likelihood estimators of the parameters of a three-parameter lognormal population; when at least one of the parameters is known, the sample variances and covariances agree quite well with their asymptotic values, even for sample size n as small as 50.

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TABLE M1

Results of Monte Carlo study of estimates of parameters of Weibull population with scale parameter $\theta = 10$, shape parameter $K = 3$, and location parameter $c = 20$ from 500 random samples of size N

	$N = 50$	$N = 100$		$N = 50$	$N = 100$		$N = 50$	$N = 100$
$M(\hat{\theta})$	9.92	9.83	$M(\hat{K})$	3.11	3.00	$M(\hat{c})$	20.03	20.11
$M(\hat{\theta} \mid K)$	9.76	9.90	$M(\hat{K} \mid \theta)$	3.10	3.04	$M(\hat{c} \mid \theta)$	20.01	19.99
$M(\hat{\theta} \mid c)$	10.01	9.98	$M(\hat{K} \mid c)$	3.11	3.03	$M(\hat{c} \mid K)$	20.25	20.08
$M(\hat{\theta} \mid K, c)$	9.99	9.98	$M(\hat{K} \mid \theta, c)$	3.07	3.02	$M(\hat{c} \mid \theta, K)$	20.04	19.99
$V(\hat{\theta})$	11.15	3.44	$V(\hat{K} \mid \theta)$	0.122	0.051	$V(\hat{\theta} \mid c)$	0.241	0.132
$AV(\hat{\theta})$	2.72	1.36	$AV(\hat{K} \mid \theta)$	0.103	0.051	$AV(\hat{\theta} \mid c)$	0.246	0.123
$V(\hat{K})$	1.69	0.55	$V(\hat{c} \mid \theta)$	0.190	0.108	$V(\hat{K} \mid c)$	0.123	0.056
$AV(\hat{K})$	0.40	0.20	$AV(\hat{c} \mid \theta)$	0.194	0.097	$AV(\hat{K} \mid c)$	0.109	0.055
$V(\hat{c})$	9.66	2.99	$C(\hat{K}, \hat{c} \mid \cdot\theta)$	−0.033	−0.014	$C(\hat{\theta}, \hat{K} \mid c)$	0.045	0.031
$AV(\hat{c})$	2.14	1.07	$AC(\hat{K}, \hat{c} \mid \theta)$	−0.028	−0.014	$AC(\hat{\theta}, \hat{K} \mid c)$	0.051	0.026
$C(\hat{\theta}, \hat{K})$	4.15	1.30	$V(\hat{\theta} \mid K)$	0.787	0.359	$V(\hat{\theta} \mid K, c)$	0.218	0.116
$AC(\hat{\theta}, \hat{K})$	0.89	0.45	$AV(\hat{\theta} \mid K)$	0.704	0.352	$AV(\hat{\theta} \mid K, c)$	0.222	0.111
$C(\hat{\theta}, \hat{c})$	−10.26	−3.15	$V(\hat{c} \mid K)$	0.657	0.328	$V(\hat{K} \mid \theta, c)$	0.112	0.049
$AC(\hat{\theta}, \hat{c})$	−2.30	−1.15	$AV(\hat{c} \mid K)$	0.592	0.296	$AV(\hat{K} \mid \theta, c)$	0.099	0.049
$C(\hat{K}, \hat{c})$	−3.87	−1.21	$C(\hat{\theta}, \hat{c} \mid K)$	−0.611	−0.282	$V(\hat{c} \mid \theta, K)$	0.181	0.104
$AC(\hat{K}, \hat{c})$	−0.78	−0.39	$AC(\hat{\theta}, \hat{c} \mid K)$	−0.534	−0.267	$AV(\hat{c} \mid \theta, K)$	0.187	0.093

M = mean; V = variance; C = covariance; prefix A indicates asymptotic value.

REFERENCES

BLISCHKE, W. R., GLINSKI, A. M., JOHNS, M. V. JR., MUNDLE, P. B. and TRUELOVE, A. J. (1965). On non-regular estimation, minimum variance bounds and the Pearson type III distribution. ARL 65-177. Aerospace Research Laboratories, Wright-Patterson Air Force Base.

CHAPMAN, DOUGLAS G. (1956). Estimating the parameters of a truncated gamma distribution. *Ann. Math. Statist.* **27** 498–506.

COHEN, A. CLIFFORD (1965). Maximum likelihood estimation in the Weibull distribution based on complete and on censored samples. *Technometrics* **7** 579–588.

DUBEY, SATYA D. (1963). On some statistical inferences for Weibull laws (abstract). *J. Amer. Statist. Assoc.* **58** 549.

DUBEY, SATYA D. (1965). Hyper-efficient estimator of the location parameter of the Weibull law (abstract). *Ann. Math. Statist.* **36** 734.

FISHER, R. A. (1912). On an absolute criterion for fitting frequency curves. *Messenger Math.* **41** 155–160.

FISHER, R. A. (1922). On the mathematical foundations of theoretical statistics. *Phil. Trans. Roy. Soc. London Ser A* **222** 309–368.

GAUSS, CARL FRIEDRICH (1880). *Werke Band IV*. Königlichen Gesellschaft der Wissenschaften, Göttingen.

GREENWOOD, J. ARTHUR and DURAND, DAVID (1960). Aids for fitting the gamma distribution by maximum likelihood. *Technometrics* **2** 55–65.

HALPERIN, MAX (1952). Maximum likelihood estimation in truncated samples. *Ann. Math. Statist.* **23** 226–238.

HARTER, H. LEON and MOORE, ALBERT H. (1965). Maximum-likelihood estimation of the parameters of gamma and Weibull populations from complete and from censored samples. *Technometrics* **7** 639–643.

Harter, H. Leon and Moore, Albert H. (1966). Local-maximum-likelihood estimation of the parameters of three-parameter lognormal populations from complete and censored samples. *J. Amer. Statist. Assoc.* **61** 842–851.

Kao, John H. K. (1956). A new life-quality measure for electron tubes. *IRE Transactions* **PGRQC-7** 1–11.

Kao, John H. K. (1958). Computer methods for estimating Weibull parameters in reliability studies. *IRE Transactions* **PGRQC-13** 15–22.

Kendall, Maurice G. and Stuart, Alan (1961). *The Advanced Theory of Statistics*, **2.** Hafner, New York.

Lehman, Eugene H., Jr. (1963). Shapes, moments and estimators of the Weibull distribution. *IEEE Trans. Reliab.* **R-12** 3 32–38.

Leone, F. C., Rutenberg, Y. H. and Topp, C. W. (1960). Order statistics and estimators for the Weibull distribution. Case Statistical Laboratory Publication No. 1026 (AFOSR Report No. TN 60-389). Statistical Laboratory, Case Institute of Technology.

Masuyama, M. and Kuroiwa, Y. (1951). Table for the likelihood solutions of gamma distributions and its medical applications. *Rep. Statist. Appl. Res. Un. Japan. Sci. Engrs.* **1** 18–23.

Mickey, M. R., Mundle, P. B., Walker, D. N. and Glinski, A. M. (1963). Test criteria for Pearson type III distributions. ARL 63-100. Aeronautical Research Laboratories, Wright-Patterson Air Force Base.

Plackett, R. L. (1958). Linear estimation from censored data. *Ann. Math. Statist.* **29** 131–142.

Raj, Des (1953). Estimating the parameters of type III populations from truncated samples. *J. Amer. Statist. Assoc.* **48** 336–349.

Ravenis, Joseph, V. J., II (1964). Estimating Weibull-distribution parameters. *Electro-Technology* March 46-54.

Wilk, M. B., Gnanadesikan, R. and Huyett, Marilyn J. (1962). Estimation of parameters of the gamma distribution using order statistics. *Biometrika* **49** 525–545.

Tables of coefficients in asymptotic variances and covariances of maximum-likelihood estimators of parameters of Weibull and gamma populations from samples of size N with proportions Q1 censored from below and Q2 from above

W1. *Weibull—Location parameter c, scale parameter θ, and shape parameter $K = 1$*

Q1	Q2	$\frac{V(\hat\theta)}{\theta^2/N}$	$\frac{V(\hat K)}{1/N}$	$\frac{V(\hat c)}{\theta^2/N}$	$\frac{C(\hat\theta, \hat K)}{\theta/N}$	$\frac{C(\hat\theta, \hat c)}{\theta^2/N}$	$\frac{C(\hat K, \hat c)}{\theta/N}$	$\frac{V(\hat K \mid \theta)}{1/N}$	$\frac{V(\hat c \mid \theta)}{\theta^2/N}$	$\frac{C(\hat K, \hat c \mid \theta)}{\theta/N}$
.005	.00	1.143	0.678	0.006	0.306	−0.014	−0.020	0.596	0.005	−0.016
.010	.00	1.173	0.725	0.012	0.343	−0.028	−0.037	0.624	0.011	−0.029
.015	.00	1.202	0.766	0.019	0.378	−0.042	−0.054	0.647	0.018	−0.041
.020	.00	1.231	0.805	0.027	0.411	−0.057	−0.072	0.667	0.025	−0.053
.025	.00	1.260	0.842	0.036	0.445	−0.073	−0.090	0.685	0.032	−0.064
.005	.25	1.342	1.196	0.006	−0.002	−0.007	−0.032	1.196	0.006	−0.032
.010	.25	1.352	1.317	0.013	0.033	−0.015	−0.061	1.316	0.013	−0.061
.015	.25	1.364	1.427	0.021	0.069	−0.025	−0.091	1.424	0.021	−0.090
.020	.25	1.377	1.532	0.031	0.106	−0.036	−0.123	1.524	0.030	−0.120
.025	.25	1.392	1.635	0.041	0.146	−0.049	−0.156	1.620	0.040	−0.151
.005	.50	2.554	2.157	0.006	−1.073	0.016	−0.052	1.706	0.006	−0.045
.010	.50	2.569	2.471	0.015	−1.143	0.027	−0.103	1.963	0.014	−0.091
.015	.50	2.579	2.766	0.025	−1.196	0.037	−0.157	2.211	0.024	−0.140
.020	.50	2.585	3.057	0.037	−1.239	0.045	−0.216	2.463	0.036	−0.194
.025	.50	2.589	3.350	0.050	−1.273	0.053	−0.280	2.724	0.049	−0.254
.005	.75	12.501	5.496	0.007	−6.803	0.118	−0.112	1.794	0.006	−0.048
.010	.75	13.780	6.880	0.019	−8.133	0.238	−0.236	2.080	0.015	−0.096
.015	.75	14.946	8.295	0.034	−9.417	0.371	−0.383	2.362	0.025	−0.150
.020	.75	16.080	9.806	0.054	−10.726	0.521	−0.556	2.651	0.037	−0.209
.025	.75	17.218	11.449	0.079	−12.093	0.690	−0.759	2.955	0.051	−0.275

Q1	Q2	$\frac{V(\hat\theta \mid K)}{\theta^2/N}$	$\frac{V(\hat c \mid K)}{\theta^2/N}$	$\frac{C(\hat\theta, \hat c \mid K)}{\theta^2/N}$	$\frac{V(\hat\theta \mid c)}{\theta^2/N}$	$\frac{V(\hat K \mid c)}{1/N}$	$\frac{C(\ddot\theta, \hat K \mid c)}{\theta/N}$	$\frac{V(\theta \mid K, c)}{\theta^2/N}$	$\frac{V(\hat K \mid \theta, c)}{1/N}$	$\frac{V(\hat c \mid \theta, K)}{\theta^2/N}$
.000	.00				1.109	0.608	0.257	1.000	0.548	
.005	.00	1.005	0.005	−0.005	1.109	0.610	0.258	1.000	0.550	0.005
.010	.00	1.010	0.010	−0.010	1.109	0.612	0.259	1.000	0.551	0.010
.015	.00	1.015	0.015	−0.015	1.110	0.614	0.260	1.000	0.553	0.015
.020	.00	1.020	0.021	−0.021	1.110	0.616	0.260	1.000	0.555	0.020
.025	.00	1.026	0.026	−0.026	1.111	0.618	0.261	1.000	0.556	0.026
.000	.25				1.335	1.020	−0.037	1.333	1.019	
.005	.25	1.342	0.005	−0.007	1.335	1.025	−0.038	1.333	1.024	0.005
.010	.25	1.351	0.010	−0.014	1.335	1.031	−0.038	1.333	1.030	0.010
.015	.25	1.361	0.016	−0.021	1.335	1.036	−0.038	1.333	1.035	0.015
.020	.25	1.370	0.021	−0.028	1.335	1.042	−0.038	1.333	1.041	0.020
.025	.25	1.379	0.027	−0.035	1.335	1.048	−0.038	1.333	1.047	0.026
.000	.50				2.510	1.716	−0.936	2.000	1.367	
.005	.50	2.020	0.005	−0.010	2.515	1.731	−0.944	2.000	1.377	0.005
.010	.50	2.041	0.010	−0.021	2.519	1.747	−0.952	2.000	1.387	0.010
.015	.50	2.062	0.016	−0.031	2.524	1.763	−0.961	2.000	1.397	0.015
.020	.50	2.083	0.021	−0.042	2.529	1.780	−0.970	2.000	1.408	0.020
.025	.50	2.105	0.027	−0.053	2.534	1.797	−0.979	2.000	1.418	0.026
.000	.75				10.498	3.736	−4.927	4.000	1.423	
.005	.75	4.082	0.005	−0.020	10.623	3.807	−5.022	4.000	1.434	0.005
.010	.75	4.167	0.011	−0.042	10.754	3.883	−5.121	4.000	1.444	0.010
.015	.75	4.255	0.016	−0.064	10.892	3.963	−5.226	4.000	1.456	0.015
.020	.75	4.348	0.022	−0.088	11.036	4.047	−5.336	4.000	1.467	0.020
.025	.75	4.444	0.028	−0.113	11.187	4.136	−5.452	4.000	1.479	0.026

W2. *Weibull—Location parameter c, scale parameter θ, and shape parameter K = 2*

$Q1$	$Q2$	$V(\hat{\theta})$ θ^2/N	$V(\hat{K})$ $1/N$	$V(\hat{c})$ θ^2/N	$C(\hat{\theta}, \hat{K})$ θ/N	$C(\hat{\theta}, \hat{c})$ θ^2/N	$C(\hat{K}\ \hat{c},)$ θ/N	$V(\hat{K} \mid \theta)$ $1/N$	$V(\hat{c} \mid \theta)$ θ^2/N	$C(\hat{K}, \hat{c} \mid \theta)$ θ/N
.005	.00	0.886	5.897	0.434	1.709	−0.514	−1.225	2.602	0.136	−0.234
.010	.00	1.064	6.765	0.567	2.102	−0.668	−1.565	2.613	0.148	−0.246
.015	.00	1.214	7.459	0.681	2.424	−0.799	−1.846	2.618	0.156	−0.251
.020	.00	1.351	8.072	0.787	2.714	−0.919	−2.101	2.619	0.162	−0.254
.025	.00	1.481	8.639	0.888	2.986	−1.034	−2.340	2.620	0.166	−0.256
.005	.25	0.916	11.535	0.544	2.042	−0.563	−2.011	6.980	0.198	−0.756
.010	.25	1.133	13.758	0.746	2.738	−0.773	−2.682	7.144	0.220	−0.815
.015	.25	1.330	15.628	0.929	3.344	−0.962	−3.265	7.219	0.233	−0.847
.020	.25	1.520	17.347	1.105	3.916	−1.145	−3.816	7.259	0.243	−0.866
.025	.25	1.710	18.992	1.281	4.474	−1.327	−4.354	7.282	0.250	−0.880
.005	.50	0.969	22.484	0.691	1.358	−0.485	−3.279	20.581	0.448	−2.600
.010	.50	1.156	27.993	1.002	2.371	−0.726	−4.588	23.128	0.546	−3.099
.015	.50	1.344	32.883	1.300	3.332	−0.963	−5.795	24.626	0.610	−3.409
.020	.50	1.543	37.580	1.603	4.298	−1.209	−6.989	25.609	0.657	−3.622
.025	.50	1.756	42.248	1.918	5.295	−1.468	−8.201	26.283	0.692	−3.776
.005	.75	2.736	63.723	1.059	−6.992	0.291	−7.167	45.853	1.028	−6.422
.010	.75	2.736	86.664	1.716	−6.967	0.287	−11.048	68.923	1.686	−10.318
.015	.75	2.743	109.409	2.429	−6.593	0.220	−15.076	93.562	2.412	−14.547
.020	.75	2.762	133.401	3.234	−5.921	0.097	−19.469	120.707	3.230	−19.261
.025	.75	2.799	159.352	4.151	−4.948	−0.086	−24.349	150.605	4.149	−24.501

$Q1$	$Q2$	$V(\hat{\theta} \mid K)$ θ^2/N	$V(\hat{c} \mid K)$ θ^2/N	$C(\hat{\theta}, \hat{c} \mid K)$ θ^2/N	$V(\hat{\theta} \mid c)$ θ^2/N	$V(\hat{K} \mid \hat{c})$ $1/N$	$C(\hat{\theta}, \hat{K} \mid c)$ θ/N	$V(\hat{\theta} \mid K, c)$ θ^2/N	$V(\hat{K} \mid \theta, c)$ $1/N$	$V(\hat{c} \mid \theta, K)$ θ^2/N
.000	.00				0.277	2.432	0.257	0.250	2.193	
.005	.00	0.391	0.179	−0.159	0.277	2.439	0.258	0.250	2.199	0.115
.010	.00	0.411	0.205	−0.182	0.277	2.447	0.259	0.250	2.206	0.125
.015	.00	0.426	0.224	−0.199	0.277	2.455	0.260	0.250	2.212	0.132
.020	.00	0.438	0.240	−0.213	0.278	2.463	0.260	0.250	2.219	0.137
.025	.00	0.449	0.254	−0.225	0.278	2.471	0.261	0.250	2.226	0.141
.000	.25				0.334	4.080	−0.037	0.333	4.075	
.005	.25	0.554	0.193	−0.207	0.334	4.101	−0.038	0.333	4.097	0.116
.010	.25	0.588	0.224	−0.239	0.334	4.123	−0.038	0.333	4.119	0.127
.015	.25	0.614	0.246	−0.263	0.334	4.145	−0.038	0.333	4.141	0.134
.020	.25	0.636	0.266	−0.284	0.334	4.169	−0.038	0.333	4.164	0.139
.025	.25	0.655	0.283	−0.302	0.334	4.192	−0.038	0.333	4.188	0.144
.000	.50				0.628	6.865	−0.936	0.500	5.469	
.005	.50	0.887	0.213	−0.287	0.629	6.925	−0.944	0.500	5.508	0.120
.010	.50	0.955	0.250	−0.337	0.630	6.987	−0.952	0.500	5.547	0.131
.015	.50	1.007	0.279	−0.376	0.631	7.052	−0.961	0.500	5.588	0.138
.020	.50	1.052	0.304	−0.409	0.632	7.119	−0.970	0.500	5.630	0.144
.025	.50	1.093	0.326	−0.440	0.633	7.187	−0.979	0.500	5.674	0.149
.000	.75				2.624	14.942	−4.927	1.000	5.693	
.005	.75	1.969	0.253	−0.495	2.656	15.230	−5.022	1.000	5.735	0.129
.010	.75	2.176	0.308	−0.601	2.688	15.534	−5.121	1.000	5.778	0.141
.015	.75	2.346	0.352	−0.688	2.723	15.853	−5.226	1.000	5.822	0.150
.020	.75	2.499	0.392	−0.767	2.759	16.190	−5.336	1.000	5.868	0.157
.025	.75	2.645	0.431	−0.842	2.797	16.544	−5.452	1.000	5.915	0.163

W3. *Weibull—Location parameter c, scale parameter θ, and shape parameter K = 3*

Q1	Q2	$V(\hat\theta)$ θ^2/N	$V(\hat K)$ $1/N$	$V(\hat c)$ θ^2/N	$C(\hat\theta, \hat K)$ θ/N	$C(\hat\theta, \hat c)$ θ^2/N	$C(\hat K, \hat c)$ θ/N	$V(\hat K \mid \theta)$ $1/N$	$V(\hat c \mid \theta)$ θ^2/N	$C(\hat K, \hat c \mid \theta)$ θ/N
.000	.00	1.361	19.832	1.072	4.474	−1.152	−3.924	5.130	0.097	−0.138
.005	.00	2.340	30.383	1.931	7.686	−2.069	−6.933	5.134	0.102	−0.137
.010	.00	2.735	34.326	2.283	8.934	−2.442	−8.111	5.139	0.103	−0.135
.015	.00	3.064	37.522	2.578	9.960	−2.753	−9.082	5.146	0.104	−0.132
.020	.00	3.364	40.367	2.847	10.884	−3.038	−9.957	5.154	0.104	−0.130
.025	.00	3.648	43.007	3.103	11.749	−3.307	−10.779	5.163	0.105	−0.128
.000	.25	1.503	35.308	1.317	5.913	−1.336	−5.867	12.052	0.130	−0.612
.005	.25	2.981	61.075	2.752	12.082	−2.792	−11.944	12.112	0.137	−0.629
.010	.25	3.660	71.795	3.409	14.779	−3.460	−14.599	12.115	0.138	−0.627
.015	.25	4.258	80.907	3.989	17.115	−4.049	−16.899	12.125	0.139	−0.625
.020	.25	4.828	89.336	4.541	19.305	−4.610	−19.055	12.140	0.139	−0.622
.025	.25	5.387	97.425	5.083	21.432	−5.160	−21.149	12.160	0.140	−0.619
.000	.50	1.528	60.538	1.581	6.570	−1.406	−8.444	32.294	0.289	−2.402
.005	.50	3.429	119.035	3.890	17.108	−3.500	−20.060	33.689	0.317	−2.598
.010	.50	4.447	146.176	5.084	22.364	−4.603	−25.754	33.715	0.320	−2.607
.015	.50	5.410	170.457	6.199	27.198	−5.639	−30.956	33.716	0.321	−2.607
.020	.50	6.376	193.877	7.308	31.955	−6.674	−36.052	33.726	0.322	−2.604
.025	.50	7.370	217.202	8.441	36.771	−7.735	−41.194	33.751	0.322	−2.601
.000	.75	1.768	133.333	2.061	2.815	−1.113	−14.335	128.849	1.360	−12.562
.005	.75	3.625	334.411	6.921	22.065	−4.113	−45.576	200.095	2.254	−20.539
.010	.75	5.069	449.338	10.083	34.945	−6.250	−64.640	208.423	2.377	−21.552
.015	.75	6.684	563.786	13.398	48.539	−8.564	−84.118	211.287	2.426	−21.927
.020	.75	8.539	684.737	17.038	63.516	−11.162	−105.098	212.260	2.447	−22.069
.025	.75	10.684	815.705	21.102	80.279	−14.115	−128.169	212.509	2.455	−22.114

Q1	Q2	$V(\hat\theta \mid K)$ θ^2/N	$V(\hat c \mid K)$ θ^2/N	$C(\hat\theta, \hat c \mid K)$ θ^2/N	$V(\hat\theta \mid c)$ θ^2/N	$V(\hat K \mid c)$ $1/N$	$C(\hat\theta, \hat K \mid c)$ θ/N	$V(\hat\theta \mid K, c)$ θ^2/N	$V(\hat K \mid \theta, c)$ $1/N$	$V(\hat c \mid \theta, K)$ θ^2/N
.000	.00	0.352	0.296	−0.267	0.123	5.471	0.257	0.111	4.935	0.093
.005	.00	0.395	0.349	−0.315	0.123	5.488	0.258	0.111	4.949	0.098
.010	.00	0.409	0.366	−0.330	0.123	5.506	0.259	0.111	4.963	0.099
.015	.00	0.420	0.379	−0.343	0.123	5.524	0.260	0.111	4.977	0.100
.020	.00	0.430	0.391	−0.353	0.123	5.542	0.260	0.111	4.992	0.101
.025	.00	0.438	0.401	−0.362	0.123	5.561	0.261	0.111	5.007	0.102
.000	.25	0.513	0.342	−0.354	0.148	9.179	−0.037	0.148	9.170	0.099
.005	.25	0.591	0.416	−0.429	0.148	9.227	−0.038	0.148	9.217	0.104
.010	.25	0.618	0.440	−0.455	0.148	9.276	−0.038	0.148	9.267	0.106
.015	.25	0.638	0.460	−0.475	0.148	9.327	−0.038	0.148	9.317	0.107
.020	.25	0.656	0.477	−0.492	0.148	9.379	−0.038	0.148	9.369	0.108
.025	.25	0.672	0.492	−0.508	0.148	9.433	−0.038	0.148	9.423	0.108
.000	.50	0.815	0.403	−0.489	0.279	15.446	−0.936	0.222	12.306	0.110
.005	.50	0.971	0.509	−0.617	0.279	15.581	−0.944	0.222	12.392	0.117
.010	.50	1.026	0.547	−0.663	0.280	15.722	−0.952	0.222	12.481	0.118
.015	.50	1.070	0.577	−0.699	0.280	15.867	−0.961	0.222	12.573	0.120
.020	.50	1.109	0.604	−0.732	0.281	16.017	−0.970	0.222	12.668	0.121
.025	.50	1.145	0.628	−0.762	0.282	16.172	−0.979	0.222	12.766	0.122
.000	.75	1.708	0.520	−0.810	1.166	33.620	−4.927	0.444	12.810	0.135
.005	.75	2.169	0.709	−1.106	1.180	34.267	−5.022	0.444	12.903	0.145
.010	.75	2.351	0.784	−1.223	1.195	34.951	−5.121	0.444	13.000	0.148
.015	.75	2.505	0.848	−1.322	1.210	35.670	−5.226	0.444	13.100	0.150
.020	.75	2.647	0.906	−1.413	1.226	36.427	−5.336	0.444	13.203	0.152
.025	.75	2.783	0.963	−1.501	1.243	37.223	−5.452	0.444	13.309	0.154

G1. *Gamma—Location parameter c, scale parameter θ, and shape parameter α = 1*

Q1	Q2	$\frac{V(\hat\theta)}{\theta^2/N}$	$\frac{V(\hat\alpha)}{1/N}$	$\frac{V(\hat c)}{\theta^2/N}$	$\frac{C(\hat\theta, \hat\alpha)}{\theta/N}$	$\frac{C(\hat\theta, \hat c)}{\theta^2/N}$	$\frac{C(\hat\alpha, \hat c)}{\theta/N}$	$\frac{V(\hat\alpha \mid \theta)}{1/N}$	$\frac{V(\hat c \mid \theta)}{\theta^2/N}$	$\frac{C(\hat\alpha, \hat c \mid \theta)}{\theta/N}$
.005	.00	2.812	1.896	0.006	−1.851	0.039	−0.045	0.678	0.006	−0.019
.010	.00	2.970	2.131	0.014	−2.044	0.074	−0.088	0.725	0.012	−0.037
.015	.00	3.105	2.344	0.023	−2.213	0.109	−0.131	0.766	0.019	−0.054
.020	.00	3.228	2.546	0.033	−2.371	0.144	−0.177	0.805	0.027	−0.071
.025	.00	3.343	2.744	0.045	−2.522	0.181	−0.225	0.842	0.035	−0.089
.005	.25	4.956	2.556	0.006	−3.039	0.063	−0.058	0.692	0.006	−0.020
.010	.25	5.360	2.941	0.015	−3.433	0.120	−0.115	0.741	0.012	−0.038
.015	.25	5.714	3.297	0.025	−3.788	0.180	−0.175	0.785	0.019	−0.055
.020	.25	6.045	3.643	0.037	−4.127	0.243	−0.239	0.826	0.027	−0.073
.025	.25	6.363	3.989	0.050	−4.459	0.308	−0.307	0.865	0.035	−0.091
.005	.50	10.389	3.808	0.007	−5.645	0.110	−0.081	0.741	0.006	−0.021
.010	.50	11.625	4.540	0.016	−6.596	0.218	−0.164	0.797	0.012	−0.040
.015	.50	12.751	5.242	0.028	−7.485	0.334	−0.256	0.848	0.019	−0.060
.020	.50	13.839	5.946	0.043	−8.361	0.459	−0.357	0.895	0.027	−0.079
.025	.50	14.918	6.669	0.060	−9.244	0.595	−0.468	0.941	0.036	−0.099
.005	.75	35.945	7.938	0.008	−15.904	0.277	−0.148	0.901	0.006	−0.026
.010	.75	43.463	10.290	0.020	−20.109	0.584	−0.320	0.986	0.013	−0.050
.015	.75	50.925	12.746	0.038	−24.390	0.946	−0.528	1.065	0.020	−0.075
.020	.75	58.722	15.415	0.061	−28.952	1.372	−0.777	1.141	0.029	−0.101
.025	.75	67.055	18.365	0.091	−33.910	1.872	−1.075	1.217	0.039	−0.128

Q1	Q2	$\frac{V(\hat\theta \mid \hat\alpha)}{\theta^2/N}$	$\frac{V(\hat c \mid \alpha)}{\theta^2/N}$	$\frac{C(\hat\theta, \hat c \mid \alpha)}{\theta^2/N}$	$\frac{V(\hat\theta \mid c)}{\theta^2/N}$	$\frac{V(\hat\alpha \mid c)}{1/N}$	$\frac{C(\hat\theta, \hat\alpha \mid c)}{\theta/N}$	$\frac{V(\hat\theta \mid \alpha, c)}{\theta^2/N}$	$\frac{V(\hat\alpha \mid \theta, c)}{1/N}$	$\frac{V(\hat c \mid \theta, \alpha)}{\theta^2/N}$
.000	.00				2.551	1.551	−1.551	1.000	0.608	
.005	.00	1.005	0.005	−0.005	2.563	1.563	−1.563	1.000	0.610	0.005
.010	.00	1.010	0.010	−0.010	2.575	1.575	−1.575	1.000	0.612	0.010
.015	.00	1.015	0.015	−0.015	2.587	1.587	−1.587	1.000	0.614	0.015
.020	.00	1.020	0.021	−0.021	2.600	1.600	−1.600	1.000	0.615	0.020
.025	.00	1.026	0.026	−0.026	2.612	1.613	−1.612	1.000	0.617	0.026
.000	.25				4.314	2.005	−2.445	1.333	0.620	
.005	.25	1.342	0.005	−0.007	4.344	2.025	−2.469	1.333	0.622	0.005
.010	.25	1.351	0.010	−0.014	4.375	2.046	−2.494	1.333	0.623	0.010
.015	.25	1.361	0.016	−0.021	4.406	2.067	−2.520	1.333	0.625	0.015
.020	.25	1.370	0.021	−0.028	4.437	2.088	−2.546	1.333	0.627	0.020
.025	.25	1.379	0.027	−0.035	4.469	2.110	−2.572	1.333	0.629	0.026
.000	.50				8.513	2.801	−4.271	2.000	0.658	
.005	.50	2.020	0.005	−0.010	8.605	2.840	−4.331	2.000	0.660	0.005
.010	.50	2.041	0.010	−0.021	8.699	2.881	−4.393	2.000	0.662	0.010
.015	.50	2.062	0.016	−0.031	8.795	2.922	−4.456	2.000	0.665	0.015
.020	.50	2.083	0.021	−0.042	8.893	2.965	−4.521	2.000	0.667	0.020
.025	.50	2.105	0.027	−0.053	8.993	3.009	−4.587	2.000	0.669	0.026
.000	.75				25.685	5.021	−10.435	4.000	0.782	
.005	.75	4.082	0.005	−0.020	26.241	5.150	−10.702	4.000	0.785	0.005
.010	.75	4.167	0.011	−0.042	26.820	5.284	−10.981	4.000	0.788	0.010
.015	.75	4.255	0.016	−0.064	27.424	5.425	−11.273	4.000	0.791	0.015
.020	.75	4.348	0.022	−0.088	28.054	5.572	−11.577	4.000	0.794	0.020
.025	.75	4.444	0.028	−0.113	28.713	5.726	−11.895	4.000	0.798	0.026

G2. *Gamma—Location parameter c, scale parameter θ, and shape parameter α = 2*

$Q1$	$Q2$	$\frac{V(\hat{\theta})}{\theta^2/N}$	$\frac{V(\hat{\alpha})}{1/N}$	$\frac{V(\hat{c})}{\theta^2/N}$	$\frac{C(\hat{\theta}, \hat{\alpha})}{\theta/N}$	$\frac{C(\hat{\theta}, \hat{c})}{\theta^2/N}$	$\frac{C(\hat{\alpha}, \hat{c})}{\theta/N}$	$\frac{V(\ \mid \theta)}{1/N}$	$\frac{V(\hat{c} \mid \theta)}{\theta^2/N}$	$\frac{C(\hat{\alpha}, \hat{c} \mid \theta)}{\theta/N}$
.005	.00	3.637	18.996	1.226	−7.580	1.309	−3.841	3.196	0.755	−1.113
.010	.00	4.000	22.592	1.710	−8.722	1.728	−5.161	3.569	0.964	−1.393
.015	.00	4.286	25.571	2.149	−9.646	2.082	−6.305	3.861	1.138	−1.618
.020	.00	4.536	28.272	2.574	−10.468	2.408	−7.376	4.115	1.295	−1.818
.025	.00	4.765	30.820	2.996	−11.231	2.719	−8.413	4.346	1.445	−2.004
.005	.25	6.743	28.967	1.446	−13.141	2.134	−5.324	3.355	0.771	−1.164
.010	.25	7.687	35.606	2.096	−15.645	2.917	−7.400	3.764	0.989	−1.463
.015	.25	8.467	41.334	2.710	−17.759	3.609	−9.275	4.087	1.171	−1.705
.020	.25	9.175	46.702	3.323	−19.708	4.268	−11.089	4.369	1.338	−1.922
.025	.25	9.843	51.911	3.951	−21.573	4.915	−12.898	4.627	1.496	−2.124
.005	.50	14.483	48.691	1.788	−25.492	3.755	−7.916	3.824	0.814	−1.306
.010	.50	17.245	62.453	2.725	−31.657	5.364	−11.507	4.342	1.057	−1.660
.015	.50	19.652	74.952	3.661	−37.141	6.865	−14.927	4.757	1.263	−1.953
.020	.50	21.930	87.162	4.640	−42.415	8.358	−18.384	5.127	1.454	−2.219
.025	.50	24.165	99.456	5.680	−47.657	9.883	−21.960	5.470	1.638	−2.470
.005	.75	50.409	118.410	2.668	−75.497	9.356	−15.741	5.339	0.931	−1.728
.010	.75	65.498	166.105	4.525	−102.323	14.648	−25.150	6.254	1.249	−2.267
.015	.75	80.141	214.395	6.609	−128.914	20.171	−35.182	7.026	1.532	−2.734
.020	.75	95.331	266.111	9.013	−156.941	26.213	−46.330	7.741	1.805	−3.176
.025	.75	111.532	322.739	11.803	−187.230	32.936	−58.899	8.432	2.077	−3.609

$Q1$	$Q2$	$\frac{V(\hat{\theta} \mid \alpha)}{\theta^2/N}$	$\frac{V(\hat{c} \mid \alpha)}{\theta^2/N}$	$\frac{C(\hat{\theta}, \hat{c} \mid \alpha)}{\theta^2/N}$	$\frac{V(\theta \mid c)}{\theta^2/N}$	$\frac{V(\hat{\alpha} \mid c)}{1/N}$	$\frac{C(\hat{\theta}, \hat{\alpha} \mid c)}{\theta/N}$	$\frac{V(\hat{\theta} \mid \alpha, c)}{\theta^2/N}$	$\frac{V(\hat{\alpha} \mid \theta, c)}{1/N}$	$\frac{V(\hat{c} \mid \theta, \alpha)}{\theta^2/N}$
.000	.00				2.225	6.900	−3.450	0.500	1.551	
.005	.00	0.612	0.449	−0.224	2.239	6.959	−3.479	0.500	1.554	0.367
.010	.00	0.632	0.531	−0.265	2.254	7.018	−3.508	0.500	1.557	0.420
.015	.00	0.647	0.595	−0.296	2.268	7.078	−3.538	0.500	1.560	0.460
.020	.00	0.660	0.650	−0.323	2.283	7.138	−3.567	0.500	1.563	0.492
.025	.00	0.672	0.700	−0.347	2.297	7.199	−3.597	0.500	1.567	0.521
.000	.25				3.561	9.269	−5.227	0.612	1.594	
.005	.25	0.781	0.468	−0.281	3.594	9.375	−5.287	0.612	1.597	0.367
.010	.25	0.813	0.558	−0.334	3.627	9.482	−5.347	0.612	1.601	0.421
.015	.25	0.837	0.629	−0.376	3.661	9.589	−5.406	0.612	1.604	0.460
.020	.25	0.858	0.690	−0.412	3.694	9.698	−5.467	0.612	1.608	0.492
.025	.25	0.877	0.747	−0.445	3.728	9.808	−5.528	0.612	1.611	0.521
.000	.50				6.500	13.417	−8.718	0.836	1.725	
.005	.50	1.138	0.501	−0.389	6.593	13.639	−8.861	0.836	1.729	0.368
.010	.50	1.199	0.605	−0.469	6.686	13.862	−9.005	0.836	1.733	0.422
.015	.50	1.247	0.688	−0.532	6.780	14.089	−9.151	0.836	1.737	0.461
.020	.50	1.290	0.762	−0.588	6.874	14.319	−9.299	0.836	1.741	0.494
.025	.50	1.329	0.831	−0.640	6.970	14.554	−9.449	0.836	1.745	0.523
.000	.75				17.124	24.788	−19.698	1.470	2.128	
.005	.75	2.273	0.576	−0.680	17.599	25.546	−20.298	1.470	2.134	0.372
.010	.75	2.466	0.717	−0.845	18.082	26.320	−20.910	1.470	2.140	0.427
.015	.75	2.626	0.836	−0.983	18.578	27.120	−21.540	1.470	2.146	0.468
.020	.75	2.773	0.947	−1.111	19.091	27.950	−22.193	1.470	2.153	0.502
.025	.75	2.914	1.054	−1.233	19.624	28.814	−22.871	1.470	2.159	0.532

G3. *Gamma—Location parameter c, scale parameter θ, and shape parameter α = 3*

$Q1$	$Q2$	$\frac{V(\hat\theta)}{\theta^2/N}$	$\frac{V(\hat\alpha)}{1/N}$	$\frac{V(\hat c)}{\theta^2/N}$	$\frac{C(\hat\theta,\hat\alpha)}{\theta/N}$	$\frac{C(\hat\theta,\hat c)}{\theta^2/N}$	$\frac{C(\hat\alpha,\hat c)}{\theta/N}$	$\frac{V(\hat\alpha \mid \theta)}{1/N}$	$\frac{V(\hat c \mid \theta)}{\theta^2/N}$	$\frac{C(\hat\alpha,\hat c \mid \theta)}{\theta/N}$
.000	.00	3.635	50.165	4.635	−12.541	2.635	−12.541	6.900	2.725	−3.450
.005	.00	5.022	84.475	10.073	−19.437	5.376	−26.191	9.247	4.318	−5.383
.010	.00	5.517	97.990	12.554	−22.024	6.485	−31.981	10.073	4.933	−6.096
.015	.00	5.911	109.157	14.721	−24.121	7.408	−36.901	10.727	5.436	−6.669
.020	.00	6.256	119.241	16.761	−25.987	8.248	−41.435	11.298	5.888	−7.177
.025	.00	6.573	128.710	18.741	−27.719	9.039	−45.766	11.819	6.310	−7.646
.000	.25	6.376	73.928	5.328	−20.605	4.008	−16.594	7.336	2.809	−3.641
.005	.25	9.865	138.790	13.140	−35.646	9.219	−39.087	9.990	4.523	−5.774
.010	.25	11.249	166.799	17.040	−41.871	11.542	−49.538	10.942	5.196	−6.574
.015	.25	12.402	190.929	20.590	−47.146	13.566	−58.794	11.702	5.751	−7.224
.020	.25	13.452	213.467	24.043	−52.012	15.470	−67.615	12.370	6.253	−7.802
.025	.25	14.447	235.264	27.494	−56.668	17.323	−76.288	12.985	6.723	−8.340
.000	.50	12.355	115.244	6.203	−36.316	6.288	−22.598	8.501	3.003	−4.116
.005	.50	21.756	246.979	17.975	−71.499	16.789	−61.947	12.001	5.019	−6.772
.010	.50	25.920	310.255	24.544	−87.732	22.018	−82.334	13.310	5.840	−7.809
.015	.50	29.573	367.583	30.845	−102.203	26.816	−101.340	14.376	6.530	−8.666
.020	.50	33.044	423.385	37.241	−116.121	31.528	−120.232	15.328	7.161	−9.442
.025	.50	36.458	479.369	43.880	−129.944	36.288	−139.509	16.216	7.761	−10.172
.000	.75	34.127	230.795	7.847	−86.430	12.237	−36.352	11.901	3.459	−5.362
.005	.75	76.378	638.600	31.199	−217.665	43.580	−133.842	18.288	6.333	−9.645
.010	.75	99.404	879.632	47.419	−292.163	62.903	−196.364	20.925	7.614	−11.483
.015	.75	121.867	1122.696	64.835	−366.053	82.681	−261.425	23.178	8.740	−13.075
.020	.75	145.232	1381.914	84.276	−443.877	103.993	−332.414	25.278	9.812	−14.576
.025	.75	170.198	1664.652	106.274	−527.894	127.428	−411.278	27.313	10.869	−16.042

$Q1$	$Q2$	$\frac{V(\hat\theta \mid \alpha)}{\theta^2/N}$	$\frac{V(\hat c \mid \alpha)}{\theta^2/N}$	$\frac{C(\hat\theta,\hat c \mid \alpha)}{\theta^2/N}$	$\frac{V(\hat\theta \mid c)}{\theta^2/N}$	$\frac{V(\hat\alpha \mid c)}{1/N}$	$\frac{C(\hat\theta,\hat\alpha \mid c)}{\theta/N}$	$\frac{V(\hat\theta \mid \alpha, c)}{\theta^2/N}$	$\frac{V(\hat\alpha \mid \theta, c)}{1/N}$	$\frac{V(\hat c \mid \theta, \alpha)}{\theta^2/N}$
.000	.00	0.500	1.500	−0.500	2.137	16.234	−5.411	0.333	2.532	1.000
.005	.00	0.550	1.953	−0.650	2.153	16.376	−5.458	0.333	2.536	1.184
.010	.00	0.567	2.116	−0.703	2.168	16.519	−5.505	0.333	2.540	1.244
.015	.00	0.581	2.247	−0.746	2.183	16.661	−5.551	0.333	2.544	1.290
.020	.00	0.593	2.362	−0.783	2.198	16.805	−5.598	0.333	2.549	1.328
.025	.00	0.604	2.468	−0.817	2.213	16.950	−5.644	0.333	2.553	1.363
.000	.25	0.633	1.604	−0.617	3.361	22.251	−8.124	0.395	2.616	1.002
.005	.25	0.710	2.132	−0.819	3.396	22.516	−8.220	0.395	2.620	1.186
.010	.25	0.738	2.327	−0.893	3.430	22.781	−8.315	0.395	2.624	1.246
.015	.25	0.760	2.485	−0.952	3.464	23.046	−8.410	0.395	2.629	1.292
.020	.25	0.780	2.626	−1.005	3.498	23.314	−8.506	0.395	2.634	1.331
.025	.25	0.797	2.756	−1.053	3.533	23.585	−8.602	0.395	2.638	1.366
.000	.50	0.911	1.771	−0.833	5.981	32.914	−13.408	0.519	2.858	1.010
.005	.50	1.057	2.437	−1.145	6.075	33.489	−13.640	0.519	2.863	1.198
.010	.50	1.112	2.695	−1.264	6.168	34.061	−13.870	0.519	2.869	1.259
.015	.50	1.157	2.907	−1.361	6.260	34.638	−14.102	0.519	2.874	1.306
.020	.50	1.196	3.098	−1.448	6.354	35.223	−14.336	0.520	2.880	1.345
.025	.50	1.233	3.279	−1.530	6.449	35.817	−14.573	0.520	2.886	1.381
.000	.75	1.760	2.121	−1.377	15.045	62.390	−29.743	0.866	3.591	1.044
.005	.75	2.187	3.148	−2.039	15.503	64.428	−30.709	0.866	3.599	1.246
.010	.75	2.365	3.584	−2.317	15.960	66.480	−31.677	0.866	3.608	1.313
.015	.75	2.516	3.961	−2.556	16.428	68.586	−32.669	0.866	3.616	1.364
.020	.75	2.657	4.315	−2.780	16.909	70.762	−33.693	0.866	3.625	1.407
.025	.75	2.793	4.662	−2.997	17.406	73.021	−34.753	0.866	3.634	1.446