A SEQUENTIAL SEARCH PROCEDURE[1]

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1. Introduction and summary. Many optimization problems in mathematics and statistics are concerned with a quantity, or procedure, that yields some optimum value rather than the value itself. Frequently, they are referred to as searching problems. In this paper we consider a problem of this nature, which is described as follows.

An object to be found is located in one of R locations. A prior probability p_i that the object is in i is given ($\sum_i p_i = 1$), along with an overlook probability α_i that if the object is in i, it is not found there on a given inspection of $i\,(0 < \alpha_i < 1, i = 1, \cdots, R)$. The R locations—or boxes—can be searched one at a time, and it is assumed that all outcomes are independent, conditional on the location of the object and the inspection procedure used. Consequently, if the object is in i, the realization of the 'first success'—the detection of the object—follows a geometric distribution with parameter $1 - \alpha_i$. A search procedure $\delta = (\delta_1, \delta_2, \cdots)$ is a sequence indicating the location to be inspected at each stage of the search, and one is interested in a δ that in some sense is optimal.

In his notes on dynamic programming, Blackwell (see [3]) has shown that if, in addition, an inspection of i costs c_i, the procedure minimizing the expected searching cost is a one-stage procedure, which instructs the searcher to inspect at each stage that box for which the ratio of the current detection probability, and the cost of an inspection there, is greatest. It is assumed here that $c_i = 1$, $i = 1, \cdots, R$. If $\delta^* = (\delta_1^*, \delta_2^*, \cdots)$ denotes an optimal one-stage procedure and $\delta_n^* = j$, then the detection probability for j on the nth inspection is

$$p_j \alpha_j^{m(j,n-1,\delta^*)}(1 - \alpha_j) = \max_i \{p_i \alpha_i^{m(i,n-1,\delta^*)}(1 - \alpha_i)\}, \tag{1.1}$$

where $m(i, n, \delta)$ is the number of inspections of i among the first n inspections made using δ.

In the next section δ^* is shown to be *strongly optimal* in the sense that

$$P[N > n \mid \delta^*] \leqq P[N > n \mid \delta], \tag{1.2}$$

for each $n = 0, 1, 2, \cdots$, and every search procedure δ. (Here N denotes the (random) number of inspections required to find the object.) The use of procedure δ^* thus ensures the greatest chance of finding the object within any fixed number of inspections, or, according to B. O. Koopman [2], it "optimally allocates

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the available search effort Φ" for any number Φ of inspections. As a consequence of (1.2), δ^* also minimizes the expected 'cost' of the search (since $E[N \mid \delta] = \sum_{n=0}^{\infty} P[N > n \mid \delta]$), which is Blackwell's result for this case.

Among the many interesting problems which arise in connection with this searching problem is the question of periodic features of the optimal procedure. Staroverov [5] and Matula [3] concerned themselves with this question, which was virtually answered by Matula who found necessary and sufficient conditions ensuring ultimate periodicity. We attempt to answer a different question in Section 3, where the same search problem is considered with the modification that $\sum_i p_i = 1 - q < 1$. That is, the object is in one of $R + 1$ locations, but searching is permitted only among the first R. (For instance, the $(R + 1)$st location could be the rest of the world.) In this problem every search procedure has positive probability of never terminating, making it necessary to couple a stopping rule s (integer-valued) with any procedure δ. A loss function is defined by imposing on the searcher a penalty cost $c(>0)$, payable when searching stops if the object has not been found. Thus, one either pays the cost of unsuccessful inspection plus the penalty cost, or simply the cost of inspection if the object is found prior to stopping. A procedure (δ, s) is sought which minimizes the expected cost to the searcher, i.e. which yields the Bayes risk.

Such a procedure exists and is shown to be (δ^*, s^*), where δ^* is a strongly optimal procedure satisfying (1.2). The determination of s^* is the more difficult problem, however, and the main part of Section 3 is devoted to it. Instead of s^* itself we consider the problem of finding the Bayes stopping region S_B, that set of posterior probabilities $p = (p_1, \cdots, p_R)$ for which the Bayes risk equals the penalty cost c. Regions S_L and S_U are given which bound S_B in the sense that $S_L \subset S_B \subset S_U$. In Section 4, it is shown that for sufficiently large c, S_L and S_U 'differ' by at most R inspections taken according to δ^*, and no more than one additional inspection is required in each of the R boxes. If (δ^*, s_U) is a procedure which searches according to δ^*, but stops once the posterior distribution $p \in S_U$, a constant independent of c, is exhibited which bounds the difference between the Bayes risk and the expected risk using (δ^*, s_U).

2. Strong optimality of δ^*. By virtue of the assumption that outcomes of different inspections are independent, given the location i of the object and the search procedure δ, we have

$$P[N > n \mid \delta] = \sum_i p_i P[N > n \mid \delta, i] = \sum_i p_i \alpha_i^{m(i,n,\delta)}. \tag{2.1}$$

Thus, in order to establish (1.2) for each n and every procedure δ, it is necessary to show for each n and every R-tuple in $I_n = \{(n(1), \cdots, n(R)) : n(i) \geqq 0,$ integer; $\sum_i n(i) = n\}$ that

$$\sum_i p_i \alpha_i^{n(i)} \geqq \sum_i p_i \alpha_i^{n^*(i)}, \tag{2.2}$$

where $n^*(i) = m(i, n, \delta^*)$. This holds trivially for $n = 0$, since both sides of (2.2) equal one. If (2.2) holds for some $n \geqq 0$, and all $(n(1), \cdots, n(R)) \in I_n$, then we must show it holds for $n + 1$ and all $(n(1), \cdots, n(R)) \varepsilon I_{n+1}$. It is sufficient to establish this for each $(n(1), \cdots, n(u) + 1, \cdots, n(R))$, where $(n(1), \cdots, n(u), \cdots, n(R)) \varepsilon I_n$. Indeed, if $(n(1), \cdots, n(R)) \varepsilon I_{n+1}$ then $n(u) > 0$ for some u, and $(n(1), \cdots, n(u) - 1, \cdots, n(R)) \varepsilon I_n$. Furthermore, it is sufficient to consider $(n(1), \cdots, n(u) + 1, \cdots, n(R))$ where $(n(1), \cdots, n(R)) \varepsilon I_n$ and also $n(u) \geqq n^*(u)$. For, if $n(u) < n^*(u)$ for some u, there is a v such that $n(v) > n^*(v)$ since $\sum_i n(i) = \sum_i n^*(i) = n$. Hence $(n(1), \cdots, n(v) - 1, \cdots, n(u) + 1, \cdots, n(R)) \varepsilon I_n$ and $n(v) - 1 \geqq n^*(v)$.

Therefore, let $(n(1), \cdots, n(R)) \varepsilon I_n$ so that by assumption (2.2) holds. If $n(u) \geqq n^*(u)$ (this must hold for some $u = 1, \cdots, R$), then since $\alpha_u < 1$ and $\alpha_u^{n(u)} \leqq \alpha_u^{n^*(u)}$, (2.2) for n gives

$$\sum_{i \neq u} p_i[\alpha_i^{n(i)} - \alpha_i^{n^*(i)}] \geqq p_u[\alpha_u^{n^*(u)} - \alpha_u^{n(u)}] \geqq p_u[\alpha_u^{n^*(u)+1} - \alpha_u^{n(u)+1}],$$

which becomes

$$p_u\alpha_u^{n(u)+1} + \sum_{i \neq u} p_i\alpha_i^{n(i)} \geqq p_u\alpha_u^{n^*(u)+1} + \sum_{i \neq u} p_i\alpha_i^{n^*(i)}. \tag{2.3}$$

If $\delta^*_{n+1} = u$, then (2.3) is (2.2) for $n + 1$. If $\delta^*_{n+1} = v \neq u$, then $p_v\alpha_v^{n^*(v)}(1 - \alpha_v) \geqq p_u\alpha_u^{n^*(u)}(1 - \alpha_u)$ from (1.1), and hence

$$p_u\alpha_u^{n^*(u)+1} + \sum_{i \neq u} p_i\alpha_i^{n^*(i)} \geqq p_v\alpha_v^{n^*(v)+1} + \sum_{i \neq v} p_i\alpha_i^{n^*(i)}. \tag{2.4}$$

Combining (2.3) and (2.4), we have (2.2) for $n + 1$ and

$$(n(1), \cdots, n(u) + 1, \cdots, n(R)) \varepsilon I_{n+1}.$$

Thus, by mathematical induction, (1.2) is true for all positive integers n and every procedure δ, establishing the following:

THEOREM 2.1. *A one-stage optimal search procedure* δ^*, *as defined by* (1.1), *is strongly optimal.*

3. Bounds for the Bayes stopping region. In this section the prior probabilities p_i satisfy $\sum_i p_i = 1 - q < 1$, and a penalty cost c is charged to the searcher if the object is not found prior to termination of the search. Since the choice of a procedure δ dictates initially in what order we will examine the locations and thus what the successive posterior distributions will be until the object is found, $s = s(p)$ is known in advance for each stopping rule s, once a prior distribution p is given. Consequently, if the truncated procedure (δ, s) is used then the expected risk, or cost, to the searcher is

$$E[C \mid (\delta, s)] = \sum_{n=1}^{s} nP[N = n \mid \delta] + (s + c)P[N > s \mid \delta], \tag{3.1}$$

where C is the (random) cost of the search. We seek a procedure (δ^*, s^*) which minimizes (3.1).

The expected risk may be expressed in the following form:

$$E[C \mid (\delta, s)] = \sum_{n=0}^{s-1} P[N > n \mid \delta] + cP[N > s \mid \delta]. \tag{3.2}$$

Now, if Q denotes the event that the object is in the $(R + 1)$st location, $P[N > n \mid \delta] = (1 - q)P[N > n \mid \delta, Q^c] + qP[N > n \mid \delta, Q]$, for any n. But

$P[N > n \mid \delta, Q^c] = P'[N > n \mid \delta]$, the probability that $N > n$, given procedure δ, in the search problem with prior probabilities $p_i' = p_i/1-q$. Since $\sum_i p_i' = 1$, a strongly optimal procedure δ^* implies that $P'[N > n \mid \delta] \geqq P'[N > n \mid \delta^*]$ for all n. Moreover, $P[N > n \mid \delta, Q] = 1$ for all n, so that (3.2) becomes

$$E[C \mid (\delta, s)] = (1 - q)\{\textstyle\sum_{n=0}^{s-1} P'[N > n \mid \delta] + cP'[N > s \mid \delta]\} + q(c + s) \geqq E[C \mid (\delta^*, s)]$$

for every s, establishing the first part of the following theorem.

THEOREM 3.1. *For every stopping rule s, $E[C \mid (\delta^*, s)] \leqq E[C \mid (\delta, s)]$ where δ^* is a strongly optimal procedure, and δ is any other search procedure. Moreover, there exists a stopping rule s^* such that $E[C \mid (\delta^*, s^*)] \leqq E[C \mid (\delta, s)]$, for all procedures (δ, s).*

It remains to establish the last statement of the theorem. Since $P[N > s \mid \delta] \geqq q$, for all s and δ, then from (3.1), $E[C \mid (\delta, s)] \geqq (s + c)q$. Hence, for arbitrary δ, the expected cost increases without bound as s increases. Since it is finite for finite s it must attain a minimum for some $s = s^*(p, \delta)$, where $s^*(p, \delta)$ is a known function of (p, δ). Thus, $E[C \mid (\delta^*, s^*(\delta^*))] \leqq E[C \mid (\delta^*, s^*(\delta))] \leqq E[C \mid (\delta, s^*(\delta))] \leqq E[C \mid (\delta, s)]$ for all procedures (δ, s), implying the existence of a Bayes procedure.

To determine s^*, we consider the equivalent problem of determining the Bayes stopping region S_B, the set of posterior probabilities $p = (p_1, \cdots, p_R)$ for which the expected risk equals the penalty cost c. Equivalently, we are determining those p for which $s^*(p) = 0$. If $T_i p = (p_1', \cdots, p_R')$ is the posterior probability vector, after an inspection in i, then by Bayes' rule,

$$(3.3) \qquad p_j' = p_j/[p_i\alpha_i + (1 - p_i)], \quad j \neq i; \qquad p_i' = p_i\alpha_i/[p_i\alpha_i + (1 - p_i)],$$

so that at any given stage of the search the probabilities of detection are proportional to the posterior detection probabilities. Thus, the strongly optimal procedure δ^* could have been defined as well in terms of these latter probabilities. Now, prior to a first inspection there are $R + 1$ choices available to the searcher. He may take no inspection at all, paying the penalty c, or he may inspect any of the R locations. If he decides to inspect i, continuing after that according to the optimal procedure, his expected risk would be $1 + (1 - p_i(1 - \alpha_i))E'[C \mid (\delta^*, s^*)]$, where E' is expectation with respect to the posterior distribution $T_i p$. The optimal choice at the outset of the search is that which yields the smallest of these $R + 1$ quantities. Hence, by Bellman's Principle of Optimality ([1], p. 83), the minimal expected risk when $p = (p_1, \cdots, p_R)$ is the prior probability distribution, satisfies the functional equation

$$(3.4) \qquad f(p) = \min \{c, \min_i [1 + (1 - p_i(1 - \alpha_i))f(T_i p)]\}.$$

Thus, the Bayes stopping region $S_B = \{p : p = (p_1, \cdots, p_R), \sum_i p_i < 1; f(p) = c\}$.

In order to estimate S_B, we define the following p-sets. Let

$$S_U = \{p : p = (p_1, \cdots, p_R), \max_i p_i(1 - \alpha_i) \leqq 1/c\} \quad \text{and} \quad S_L = \textstyle\bigcap_{i=1}^{R} S_i,$$

where

$$S_i = \{p : p = (p_1, \cdots, p_R), p_i(1 - \alpha_i) + (c/c) \textstyle\sum_{j \neq i} p_j(1 - \alpha_j) \leqq 1/c\},$$

and $c_0 = 1/\min_i (1 - \alpha_i)$.

THEOREM 3.2. *The regions S_U, S_L and S_B satisfy the inclusion relation $S_L \subset S_B \subset S_U$, the first of which holds when $c \geqq c_0$. Moreover, the second inclusion is proper.*

PROOF. (i) $S_B \subset S_U$.

Let $p \,\varepsilon\, S_B$. Then,

$$c = f(p) \leqq \min_i [1 + (1 - p_i(1 - \alpha_i))f(T_i p)] \leqq \min_i [1 + (1 - p_i(1 - \alpha_i))c]$$
$$= 1 + [1 - \max_i p_i(1 - \alpha_i)]c.$$

Hence, $\max_i p_i(1 - \alpha_i) \leqq 1/c$, so that $p \,\varepsilon\, S_U$. The inclusion is proper as the following example shows. Let $p \,\varepsilon\, S_U$ such that $p_1(1 - \alpha_1) = p_2(1 - \alpha_2) = 1/c$, and let $s = 2$, $\delta = (1, 2, \delta_3, \cdots)$, where δ_i, $i \geqq 3$, are arbitrary. Then from (3.1)

$$E[C \mid (\delta, s)] = 1/c + 2/c + (2 + c)(1 - 2/c) = c - 1/c < c,$$

so that $f(p) \leqq E[C \mid (\delta, s)] < c$ and $p \,\varepsilon\, S_B^c$.

(ii) $S_L \subset S_B$, when $c \geqq c_0$.

Suppose there is a $p \,\varepsilon\, S_L \cap S_B^c$. Then, by (3.4),

$$c > f(p) = \min_i [1 + (1 - p_i(1 - \alpha_i))f(T_i p)]$$
$$= 1 + (1 - p_j(1 - \alpha_j))f(T_j p) \quad \text{for some} \quad j = j(1)$$
$$\geqq 1 + (1 - 1/c)f(T_j p), \quad \text{since clearly} \quad p \,\varepsilon\, S_L$$
$$\text{implies} \quad p_j(1 - \alpha_j) \leqq 1/c.$$

Thus, $f(T_{j(1)}p) < c$. Now, if $p \,\varepsilon\, S_L$ implies $T_i p \,\varepsilon\, S_L$, $i = 1, \cdots, R$, the preceding argument shows that for some $j = j(2)$, $f(T_{j(2)}T_{j(1)}p) < c$, and $T_{j(2)}T_{j(1)}p \,\varepsilon\, S_L$. Hence, by applying the above argument repeatedly, we would have $f(T_{j(n)} \cdots T_{j(1)}p) < c$ for $n = 1, 2, \cdots$. This implies that the search procedure yielding the minimal expected risk never terminates, contradicting the conclusion of Theorem 3.1. It suffices then, to prove that $p \,\varepsilon\, S_L \Rightarrow T_i p \,\varepsilon\, S_L$, $i = 1, \cdots, R$.

Let $T_i p = (p_1', \cdots, p_R')$. Then, since $p = (p_1, \cdots, p_R) \,\varepsilon\, S_L$,

$$p_i\alpha_i(1 - \alpha_i) + (c_0/c) \textstyle\sum_{k \neq i} p_k(1 - \alpha_k)$$
$$= p_i(1 - \alpha_i) + (c_0/c) \textstyle\sum_{k \neq i} p_k(1 - \alpha_k) - p_i(1 - \alpha_i)^2 \leqq 1/c - p_i(1 - \alpha_i)^2$$
$$= 1/c[1 - p_i(1 - \alpha_i)] + [1/c - (1 - \alpha_i)]p_i(1 - \alpha_i) \leqq 1/c[1 - p_i(1 - \alpha_i)],$$
$$\text{since } c \geqq c_0 \geqq 1/(1 - \alpha_i).$$

Dividing both sides of the inequality by $1 - p_i(1 - \alpha_i) = p_i\alpha_i + (1 - p_i)$, we obtain, by (3.3),

$$p_i'(1 - \alpha_i) + (c_0/c) \textstyle\sum_{k \neq i} p_k'(1 - \alpha_k) \leqq 1/c \quad \text{and hence} \quad T_i p \,\varepsilon\, S_i.$$

For

$$j \neq i,\ p_j(1 - \alpha_j) + (c_0/c)[p_i\alpha_i(1 - \alpha_i) + \textstyle\sum_{k\neq j,i} p_k(1 - \alpha_k)]$$
$$= p_j(1 - \alpha_j) + (c_0/c)\textstyle\sum_{k\neq j} p_k(1 - \alpha_k) - (c_0/c)p_i(1 - \alpha_i)^2$$
$$\leqq 1/c[1 - c_0p_i(1 - \alpha_i)^2] \leqq 1/c[1 - p_i(1 - \alpha_i)],$$

since $c_0 \geqq 1/(1 - \alpha_i), j = 1, \cdots, R$. Therefore, by (3.3) again, $p_j'(1 - \alpha_j) + (c_0/c)\sum_{k\neq j} p_k'(1 - \alpha_k) \leqq 1/c$ so that $T_ip \in S_j$, $j = 1, \cdots, R$. Hence, $T_ip \in \bigcap_{j=1}^R S_j = S_L$, and the proof is complete.

The assumption that $c \geqq c_0$ is crucial in the above argument in establishing that $S_L \subset S_B$. It can be shown in a similar manner that S_B is the set of all probability vectors $p = (p_1, \cdots, p_R)$, $\sum_i p_i < 1$, whenever $c \leqq c_1 = 1/\max(1 - \alpha_i)$. That is, for such small penalty costs c it is not worthwhile taking any inspections. If at least two α_i are distinct, however, the interval (c_1, c_0) is non-degenerate, and presumably there are occasions when searching should be done for $c \in (c_1, c_0)$. In this case S_L is not an inner-bounding region of S_B (possibly some restriction of S_L works), and we pursue it no further at present.

4. An approximation to (δ^*, s^*). The 'closeness' of S_L and S_U, and hence the 'closeness' of either to the Bayes stopping region S_B, is demonstrated in the following

THEOREM 4.1. *Suppose the prior probability distribution* $p \in S_U$ *and*

$$c \geqq c_0^2(R - 1) + c_0.$$

Then, if $\delta_i^* = j(i), i = 1, \cdots, R, T_{j(k)} \cdots T_{j(1)}p \in S_L$ *for some* $k \leqq R$. *Moreover,* $j(1), \cdots, j(k)$ *are distinct integers among* $1, \cdots, R$.

The following lemma will be needed in the proof of the above theorem:

LEMMA 4.1. *For each* $u, v = 1, \cdots, R$, *define*

$$c_u(1) = [c_0(R - 1) + 1]/(1 - \alpha_u),$$
$$c_u(v) = [c_0(R - v + \textstyle\sum_{i=1,i\neq u}^{v} \alpha_i) + v]/(1 - \alpha_u), \quad if \quad u < v$$
$$= [c_0(R - v + \textstyle\sum_{i=1}^{v-1} \alpha_i) + v]/(1 - \alpha_u) \quad if \quad u \geqq v.$$

Then, $c_u(v) \leqq c_0^2(R - 1) + c_0$, *for all* u *and* v.

PROOF. Clearly, $c_u(1) \leqq c_0^2(R - 1) + c_0$ for all u, since $c_0 = 1/\min_i(1 - \alpha_i)$. The remainder of the lemma will be proved if it is shown that $c_u(v)$ are decreasing in v for each u. For $u \geqq 2$,

$$c_u(2) = [c_0(R - 2 + \alpha_1) + 2]/(1 - \alpha_u) = [c_0(R - 1) + 1 + (1 - c_0(1 - \alpha_1))]/(1 - \alpha_u) \leqq c_u(1),$$

and

$$c_1(2) = [c_0(R - 2 + \alpha_2) + 2]/(1 - \alpha_1)$$
$$= [c_0(R - 1) + 1 + (1 - c_0(1 - \alpha_2))]/(1 - \alpha_1) \leqq c_1(1).$$

For $v > 1$, consider two cases:

(i) $u \geqq v + 1$. Then,

$$c_u(v+1) = [c_0(R - v - 1 + \textstyle\sum_{i=1}^{v} \alpha_i) + v + 1]/(1 - \alpha_u)$$
$$= [c_0(R - v + \textstyle\sum_{i=1}^{v-1} \alpha_i) + v + (1 - c_0(1 - \alpha_v))]/(1 - \alpha_u) \leqq c_u(v).$$

(ii) $u < v + 1$. Then,

$$c_u(v+1) = [c_0(R - v - 1 + \textstyle\sum_{i=1, i\neq u}^{v+1} \alpha_i) + v + 1]/(1 - \alpha_u).$$

So that

$$c_u(v+1) = [c_0(R - v + \textstyle\sum_{i=1}^{v-1} \alpha_i) + v + (1 - c_0(1 - \alpha_{v+1}))]/(1 - \alpha_u),$$

if $u = v$, and

$$c_u(v+1) = [c_0(R - v + \textstyle\sum_{i=1, i\neq u}^{v} \alpha_i) + v + (1 - c_0(1 - \alpha_{v+1}))]/(1 - \alpha_u),$$

if $u < v$. In either case, $c_u(v + 1) \leqq c_u(v)$, completing the proof of the lemma.

Proof of Theorem. Let $p \varepsilon S_U$ and suppose the locations are renumbered so that $\pi_1 \geqq \cdots \geqq \pi_R$, where $\pi_i = p_i(1 - \alpha_i)$. Further, let $p(k) = (p_1(k), \cdots, p_R(k))$ and $\pi_i(k) = p_i(k)(1 - \alpha_i)$, where $p(k) = T_{j(k)} \cdots T_{j(1)}p$, and $j(i) = \delta_i^*$, the ith location inspected according to the optimal procedure δ^*. Since $\delta_1^* = 1$ (i.e., $\pi_1 = \max_i \pi_i$), and the fact that $\pi_i \leqq 1/c$, then (3.3) gives $\pi_1(1) = \pi_1\alpha_1/(1 - \pi_1) \leqq \alpha_1/(c - 1)$ and $\pi_i(1) = \pi_i/(1 - \pi_1) \leqq 1/(c - 1)$, $i = 2, \cdots, R$. Therefore, $\pi_1(1) + (c_0/c)\sum_{i=2}^{R} \pi_i(1) \leqq \alpha_1/(c - 1) + c_0(R - 1)/c(c - 1)$. The quantity on the right is $\leqq 1/c$ if and only if $c \geqq c_1(1)$, by simple computa-computation. However, $c_1(1) \leqq c_0^2(R - 1) + c_0$ by the above lemma. Hence, $c \geqq c_0^2(R - 1) + c_0$ implies that $p(1) \varepsilon S_1$. Now, if $p(1) \varepsilon S_2$, then since $\pi_2(1) \geqq \cdots \geqq \pi_R(1)$, it is easily seen that $p(1) \varepsilon S_i$, $i = 3, \cdots, R$, and hence $p(1) \varepsilon S_L$. If $p(1) \not\varepsilon S_2$, then

$$\pi_1(1) + (c_0/c)\textstyle\sum_{i=2}^{R} \pi_i(1) \leqq 1/c < \pi_2(1) + (c_0/c)\sum_{i\neq 2} \pi_i(1)$$

which gives $(1 - c_0/c)(\pi_2(1) - \pi_1(1)) > 0$. Thus, $\pi_2(1) > \pi_1(1)$ since $c \geqq c_0^2(R - 1) + c_0 > c_0$, and so $\delta_2^* = 2$. (Here, and in what follows, we assume that if $\pi_i(k) = \pi_j(k)$, $i < j$, then $\delta_{k+1}^* = i$, the smaller index. If this is not the case, we simply interchange the locations.)

Now, suppose that for some $1 \leqq k < R$, inspections have been made in locations $1, \cdots, k$ in that order according to δ^* and $p(k) \varepsilon S_i$, $i = 1, \cdots, k$. If $p(k) \varepsilon S_{k+1}$, then as above, $p(k) \varepsilon S_i$, $i = k + 2, \cdots, R$, and so $p(k) \varepsilon S_L$. If $p(k) \not\varepsilon S_{k+1}$, another inspection is required. By the same argument used for $p(1)$, remembering that $\pi_{k+1}(k) \geqq \cdots \geqq \pi_R(k)$, it is seen that $\pi_i(k) \leqq \pi_{k+1}(k)$, $i = 1, \cdots, R$, with strict inequality holding for $i \leqq k$, so that $\delta_{k+1}^* = k + 1$. Therefore, by (3.3), and the fact that we have searched $1, \cdots, k + 1$ once each, $p \varepsilon S_U$,

$$\pi_i(k+1) = \pi_i\alpha_i/(1 - \textstyle\sum_{j=1}^{k+1} \pi_j) \leqq \alpha_i/(c - k - 1), \quad i = 1, \cdots, k + 1,$$

and

$$\pi_i(k+1) = \pi_i/(1 - \textstyle\sum_{j=1}^{k+1} \pi_j) \leqq 1/(c-k-1), \quad i = k+2, \cdots, R,$$

so that

$$\pi_i(k+1) + (c_0/c) \textstyle\sum_{j\neq i} \pi_j(k+1)$$
$$\leqq \alpha_i/(c-k-1) + (c_0/c)[(R-k-1+\textstyle\sum_{j=1, j\neq i}^{k} \alpha_j)/(c-k-1)]$$
$$\text{for } i = 1, \cdots, k+1$$

The quantity on the right is $\leqq 1/c$ if and only if $c_i(k+1) \leqq c$, which is true by Lemma 4.1 since $c \geqq c_0^2(R-1) + c_0$. Thus, $p(k+1) \varepsilon S_i$, $i = 1, \cdots, k+1$. Therefore, by induction, $p(R) \varepsilon S_i$, $i = 1, \cdots, R$, which means that $p(R) \varepsilon S_L$, completing the proof.

Consider now the search procedure (δ^*, s_U), where δ^* is a strongly optimal procedure, and s_U is the smallest integer k such that $T_{j(k)} \cdots T_{j(1)}p \varepsilon S_U$, where $j(i) = \delta_i^*$. Theorem 4.1 guarantees that if $c \geqq c_0^2(R-1) + c_0$, s_U will not differ from s^* by more than R. That is, if the searcher always inspects the location most likely leading to discovery of the object, and makes at most one additional inspection in each location whenever all detection probabilities are less than $1/c$, then he shall have taken at most R more than required to minimize his expected risk. How much more one might expect to 'pay' than necessary, using (δ^*, s_U), is answered in the following

THEOREM 4.2. *If* $c \geqq c_0^2(R-1) + c_0$, *then*

$$E[C \mid (\delta^*, s_U)] - E[C \mid (\delta^*, s^*)] \leqq Rq/2c_0(c_0 - 1),$$

where $q = 1 - \sum_i p_i$.

PROOF. By (3.2) for $s_U < s^*$ (otherwise $s_U = s^*$),

$$E[C \mid (\delta^*, s_U)] - E[C \mid (\delta^*, s^*)]$$

$$(4.1) \qquad = -\textstyle\sum_{n=s_U}^{s^*-1} P[N > n \mid \delta^*] + cP[s^* \geqq N > s_U \mid \delta^*]$$
$$= -\textstyle\sum_{n=0}^{s^*-s_U-1} P[N > s_U + n \mid \delta^*] + cP[s^* \geqq N > s_U \mid \delta^*].$$

Now if $s_U = n(1) + \cdots + n(R)$, $n(i) = m(i, s_U, \delta^*)$, then for $k = s_U$ and $j(i) = \delta_i^*$, $T_{j(k)} \cdots T_{j(1)}p \varepsilon S_U$ implies that

$$p_i\alpha_i^{n(i)}(1 - \alpha_i)/(\textstyle\sum_j p_j\alpha_j^{n(j)} + q) \leqq 1/c,$$

by (3.3). Since the denominator on the left is $P[N > s_U \mid \delta^*]$, we have

$$(4.2) \qquad p_i\alpha_i^{n(i)}(1 - \alpha_i) \leqq (1/c)P[N > s_U \mid \delta^*], \quad i = 1, \cdots, R.$$

If the locations had been rearranged so that $\pi_1 \geqq \cdots \geqq \pi_R$ (as in the proof of Theorem 4.1), then $\delta_{k+j}^* = j$, $j = 1, \cdots, R$, by Theorem 4.1, where again $k = s_U$. Thus, adding the inequalities in (4.2) for $i = 1, \cdots, n (n \leqq R)$,

$$(4.3) \qquad P[N > s_U \mid \delta^*] - P[N > s_U + n \mid \delta^*] \leqq (n/c)P[N > s_U \mid \delta^*],$$

so that

$$(4.3') \quad P[N > s_U + n \mid \delta^*] \geqq (1 - n/c)P[N > s_U \mid \delta^*], \quad n = 1, \cdots, R.$$

Further, by dividing the ith inequality in (4.2) by $1 - \alpha_i$ and adding all of them, we obtain

$$P[N > s_U \mid \delta^*] - q \leqq P[N > s_U \mid \delta^*] \textstyle\sum_i 1/c(1 - \alpha_i),$$

yielding

$$(4.4) \qquad P[N > s_U \mid \delta^*] \leqq q/[1 - \textstyle\sum_i 1/c(1 - \alpha_i)].$$

By (4.3) and (4.3′), the right side of (4.1) is smaller than

$$-\textstyle\sum_{n=0}^{s^*-s_U-1} P[N > s_U \mid \delta^*](1 - n/c) + (s^* - s_U)P[N > s_U \mid \delta^*]$$
$$= P[N > s_U \mid \delta^*] \textstyle\sum_{n=1}^{s^*-s_U-1} n/c = P[N > s_U \mid \delta^*](s^* - s_U)(s^* - s_U - 1)/2c$$
$$\leqq R(R - 1)q/2[c - \textstyle\sum_i 1/(1 - \alpha_i)],$$

by (4.4) and the fact that $s^* - s_U \leqq R$. Since $c \geqq c_0^2(R - 1) + c_0$ and $c_0 \geqq 1/(1 - \alpha_i)$, $i = 1, \cdots, R$, the quantity on the right is smaller than $Rq/2c_0(c_0 - 1)$, as was to be proved.

The bound is independent of c, and appears from the proof above to be attainable. The relative error will not vary much with R, since the penalty cost c must grow linearly with R. Accuracy is lost for large q, but in this case few inspections will be made, if any, and greater error is expected. It is interesting to note, however, that if $\max_i \alpha_i$ is close to one, then c_0 is large and greater accuracy is achieved. To illustrate the size of the maximum error, let $R = 20$, $q = \frac{1}{2}$, and $c_0 = 2$ (i.e., $\max_i \alpha_i = \frac{1}{2}$). Then, for $c \geqq 78$, the bound is 2.5. Also, if some $p_i \doteq \frac{1}{2}$ and $\alpha_i \doteq 0$, then $E[C \mid (\delta^*, s_U)] \geqq P[N = 1 \mid \delta^*] + cP[N > s_U \mid \delta^*] \geqq \frac{1}{2} + 78q = 39.5$, yielding a relative error less than 6.3 per cent.

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