SOME DISTRIBUTION PROBLEMS CONNECTED WITH THE CHARACTERISTIC ROOTS OF $S_1S_2^{-1}$

BY C. G. KHATRI

Gujarat University, Ahmedabad

1. Introduction and summary. Let $\mathbf{S}_i: p \times p$ $(i = 1, 2)$ be independently distributed as Wishart $(n_i, p, \mathbf{\Sigma}_i)$. Let the characteristic (ch) roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$ and $\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1}$ be denoted by f_i $(i = 1, 2, \cdots, p)$ and λ_i $(i = 1, 2, \cdots, p)$ respectively such that $0 < f_1 < f_2 < \cdots < f_p < \infty$ and $0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_p < \infty$. The distribution of $f_1, f_2, \cdots, f_p$ as stated by James [5] is not convenient for further development and is slowly convergent for higher values of f_i's. The distribution of $(f_1, \cdots, f_p)$ mentioned by James [5] can be written as

$$(1) \qquad c\,|\mathbf{\Lambda}|^{-\frac{1}{2}n_1}\,|\mathbf{F}|^{\frac{1}{2}(n_1-p-1)}\,\alpha_p(\mathbf{F}) \int_{O(p)} |\mathbf{I}_p + \mathbf{\Lambda}^{-1}\mathbf{HFH}'|^{-\frac{1}{2}(n_1+n_2)}\,d\mathbf{H}$$

where

$$(2) \qquad c = \pi^{\frac{1}{2}p^2}\Gamma_p(\tfrac{1}{2}n_1 + \tfrac{1}{2}n_2)\{\Gamma_p(\tfrac{1}{2}p)\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2)\}^{-1},$$

$$\Gamma_p(t) = \pi^{\frac{1}{4}p(p-1)} \textstyle\prod_{j=1}^{p} \Gamma(t - \tfrac{1}{2}j + \tfrac{1}{2}),$$

$$(3) \qquad \alpha_p(\mathbf{F}) = \textstyle\prod_{i=1}^{p-1} \prod_{j=i+1}^{p} (f_j - f_i), \qquad \mathbf{F} = \operatorname{diag}(f_1, f_2, \cdots, f_p),$$

$$\mathbf{\Lambda} = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_p)$$

and the integral is over an orthogonal group $O(p)$ with $\int_{O(p)} d\mathbf{H} = 1$. For testing the null hypothesis $H_0(\lambda\mathbf{\Lambda} = \mathbf{I}_p), \lambda > 0$ being given, we have two statistics given by

$$(4) \qquad \text{(i) } l = |\lambda\mathbf{F}|^{n_1}/|\mathbf{I}_p + \lambda F|^{n_1+n_2} \quad \text{and} \quad \text{(ii) } \lambda f_p \quad \text{or} \quad \lambda f_p/(1 + \lambda f_p).$$

l is considered by Anderson [1] and λf_p is obtained by Roy [7]. (1) is rewritten in such a way that the joint density function of $(\lambda f_1, \lambda f_2, \cdots, \lambda f_p)$ has non-central parameters $\mathbf{I}_p - (\lambda\mathbf{\Lambda})^{-1}$ and it is given by

$$(5) \qquad c\,|\lambda\mathbf{\Lambda}|^{-\frac{1}{2}n_1}\,|\lambda\mathbf{F}|^{\frac{1}{2}(n_1-p-1)}\,\alpha_p(\lambda\mathbf{F})|\mathbf{I}_p + \lambda\mathbf{F}|^{-\frac{1}{2}(n_1+n_2)}\,{}_1F_0^{(p)}(\tfrac{1}{2}n_1 + \tfrac{1}{2}n_2;$$

$$\mathbf{I}_p - (\lambda\mathbf{\Lambda})^{-1}, \lambda\mathbf{F}(\mathbf{I}_p + \lambda\mathbf{F})^{-1}).$$

Hence, similar to testing of means, we propose the statistics $T = \operatorname{tr}(\lambda\mathbf{F})$ and $V = \operatorname{tr}(\lambda\mathbf{F})(\mathbf{I}_p + \lambda\mathbf{F})^{-1}$ for testing the hypothesis H_0 and obtain their distribution of T only while the moment generating function of V is given. Moreover, if in the null hypothesis $H_0'(\lambda\mathbf{\Lambda} = \mathbf{I}_p), \lambda > 0$ is unknown, then the test procedure will depend on the ratios of roots, and hence, we consider the joint distribution of $(x_1, x_2, \cdots, x_{p-1})$, where $x_i = f_i/f_p$ for $i = 1, 2, \cdots, p-1$, and f_p. Under null hypothesis H_0', we obtain the density functions of x_1 (or x_{p-1}) and of $(y_2, y_3, \cdots, y_{p-1})$ for $y_i = (f_i - f_1)/(f_p - f_1), i = 2, 3, \cdots, p-1$.

Received 5 December 1966; revised 8 February 1967.

2. Some results on integration. We shall use the notations of Khatri [6], James [5] and Constantine [2] [3]. Wherever the order of the identity matrix $\mathbf{I}$ is not clear, we shall write $\mathbf{I}_p$, p being the order of $\mathbf{I}$.

LEMMA 1.

$$\int_{O(p)} |\mathbf{I} + \mathbf{\Lambda}^{-1}\mathbf{HFH}'|^{-t}\, d\mathbf{H} = |\mathbf{I} + \lambda\mathbf{F}|^{-1}\, {}_1F_0^{(p)}(t; \mathbf{I} - (\lambda\mathbf{\Lambda})^{-1}, \lambda\mathbf{F}(\mathbf{I} + \lambda\mathbf{F})^{-1})$$

where λ is any non-negative real number such that the hypergeometric series is convergent for all $\mathbf{F}$.

Proof is immediate by using the following results:

$$|\mathbf{I} + \mathbf{AHFH}'| = |\mathbf{I} + \lambda\mathbf{F}|\, |\mathbf{I} - (\mathbf{I} - \lambda^{-1}\mathbf{A})\mathbf{H}(\lambda\mathbf{F})(\mathbf{I} + \lambda\mathbf{F})^{-1}\mathbf{H}'|,$$

and

$$\int_{O(p)} |\mathbf{I} - \mathbf{PHQH}'|^{-t}\, d\mathbf{H} = {}_1F_0^{(p)}(t; \mathbf{P}, \mathbf{Q}) \quad \text{(see James [5])}.$$

LEMMA 2.

$$\int_{O(p)} |\mathbf{I} + \mathbf{\Lambda}^{-1}\mathbf{HFH}'|^{-t}\, d\mathbf{H} = |\mathbf{I} + \mathbf{\Lambda}^{-1}g(\mathbf{F})|^{-t}\, {}_1F_0^{(p)}(t; (\mathbf{\Lambda} + g(\mathbf{F})\mathbf{I})^{-1}, g(\mathbf{F})\mathbf{I} - \mathbf{F})$$

where $g(\mathbf{F})$ is any function of the elements of $\mathbf{F}$ such that $g(\mathbf{F})$ is non-negative.

Proof is similar to that of Lemma 1. We shall take in our development $g(\mathbf{F}) = \operatorname{tr}(\mathbf{F})/p$ or f_p, the maximum ch root of $\mathbf{F}$.

LEMMA 3. *Let* $\mathbf{R} = \operatorname{diag}(r_1, r_2, \cdots, r_m)$ *such that* $0 < r_1 < r_2 < \cdots r_m < 1$. *Then*

$$\int |\mathbf{R}|^{t-\frac{1}{2}m-\frac{1}{2}}\, |\mathbf{I} - \mathbf{R}|^{u-\frac{1}{2}m-\frac{1}{2}}\, \alpha_m(\mathbf{R}) C_\kappa(\mathbf{R}) d\mathbf{R} = \Gamma_m(t, \kappa)\Gamma_m(u)\Gamma_m(\tfrac{1}{2}m) C_\kappa(\mathbf{I}) / \Gamma_m(t + u, \kappa)\pi^{\frac{1}{2}m^2}.$$

This follows from the Theorem 3 of the Constantine [2].

3. Non-central distributions.

3.1. *Density function of* $(f_1, \cdots, f_p)$. Using Lemma 1 in (1), we get the density function of $(f_1, f_2, \cdots, f_p)$ as

$$(5') \quad c\, |\mathbf{\Lambda}|^{-\frac{1}{2}n_1}\, |\mathbf{F}|^{\frac{1}{2}(n_1-p-1)}\, \alpha_p(\mathbf{F}) |\mathbf{I} + \lambda\mathbf{F}|^{-\frac{1}{2}(n_1+n_2)} \cdot {}_1F_0^{(p)}(\tfrac{1}{2}n_1 + \tfrac{1}{2}n_2; \lambda\mathbf{I} - \mathbf{\Lambda}^{-1}, \mathbf{F}(\mathbf{I} + \lambda\mathbf{F})^{-1})$$

where λ is any non-negative real number. Note that $\lambda = 0$ gives the result as mentioned by James [5]. It is easy to see that the joint density function of $w_i = \lambda f_i/(1 + \lambda f_i)$, $i = 1, 2, \cdots, p$. is given by

$$(6) \quad c\, |\lambda\mathbf{\Lambda}|^{-\frac{1}{2}n_1}\, |\mathbf{W}|^{\frac{1}{2}(n_1-p-1)} |\mathbf{I} - \mathbf{W}|^{\frac{1}{2}(n_2-p-1)}\, \alpha_p(\mathbf{W}) \cdot {}_1F_0^{(p)}(\tfrac{1}{2}n_1 + \tfrac{1}{2}n_2; \mathbf{I} - (\lambda\mathbf{\Lambda})^{-1}, \mathbf{W})$$

where $\mathbf{W} = \operatorname{diag}(w_1, w_2, \cdots, w_p)$.

3.2. *Distribution of* $T = \lambda \operatorname{tr} \mathbf{F}$. Let f_i's be the ch roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$. Then, (1) can be obtained from the joint density function of $\mathbf{S}_1$ and $\mathbf{S}_2$, which is given by

(7) $\{|\mathbf{\Lambda}|^{\frac{1}{2}n_1}\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2)\}^{-1}|\mathbf{S}_1|^{\frac{1}{2}(n_1-p-1)}|\mathbf{S}_2|^{\frac{1}{2}(n_2-p-1)}\exp[-\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{S}_1+\mathbf{S}_2)]$,

and $\mathrm{tr}\,\mathbf{F} = \mathrm{tr}\,\mathbf{S}_1\mathbf{S}_2^{-1}$. The Laplace transform of $T = \lambda\,\mathrm{tr}\,\mathbf{S}_1\mathbf{S}_2^{-1}$ is $E\exp(-t\lambda\,\mathrm{tr}\,\mathbf{S}_1\mathbf{S}_2^{-1})$. Multiplying (7) by $\exp(-t\lambda\,\mathrm{tr}\,\mathbf{S}_1\mathbf{S}_2^{-1})$ and integrating over $\mathbf{S}_1 > 0$, we get the Laplace transform of $\mathbf{T}$ as

(8) $$\{|\mathbf{\Lambda}|^{\frac{1}{2}n_1}\Gamma_p(\tfrac{1}{2}n_2)t^{\frac{1}{2}pn_1}\}^{-1} \cdot\int_{\mathbf{S}_2>0}|\mathbf{S}_2|^{\frac{1}{2}(n_1+n_2-p-1)}|\mathbf{I}+(t\lambda\mathbf{\Lambda})^{-1}\mathbf{S}_2|^{-\frac{1}{2}n_1}\exp(-\mathrm{tr}\,\mathbf{S}_2)\,d\mathbf{S}_2\,.$$

Writing

$$|\mathbf{I}+(t\lambda\mathbf{\Lambda})^{-1}\mathbf{S}_2|^{-\frac{1}{2}n_1} = \textstyle\sum_{k=0}^{\infty}\sum_{\kappa}(\tfrac{1}{2}n_1)_\kappa t^{-k}C_\kappa((\lambda\mathbf{\Lambda})^{-1}\mathbf{S}_2)/k!$$

in (8), we note that (8) can be integrated term-by-term with respect to t for $R(t)$ sufficiently large (see [3], p. 222). Hence, taking the inverse Laplace transform of (8), we get finally the density function of T as

(9) $$\Gamma_p(\tfrac{1}{2}n_1+\tfrac{1}{2}n_2)\{|\lambda\mathbf{\Lambda}|^{\frac{1}{2}n_1}\Gamma_p(\tfrac{1}{2}n_2)\}^{-1}T^{\frac{1}{2}pn_1-1} \cdot\textstyle\sum_{k=0}^{\infty}(-T)^k\{k!\,\Gamma(\tfrac{1}{2}pn_1+k)\}^{-1}\sum_\kappa(\tfrac{1}{2}n_1+\tfrac{1}{2}n_2)_\kappa(\tfrac{1}{2}n_1)_\kappa C_\kappa(\lambda\mathbf{\Lambda})^{-1}.$$

Note that (9) is convergent for $|T/\lambda\lambda_1| < 1$, λ_1 being the minimum ch root of $\mathbf{\Lambda}$. Now, if the jth moment of T exists, then it is given by

(10) $$E(T^j) = \lambda^j\textstyle\sum_J EC_J(\mathbf{S}_1\mathbf{S}_2^{-1}) = \sum_J(\tfrac{1}{2}n_1)_J C_J(\lambda\mathbf{\Lambda})(-1)^j/(\tfrac{1}{2}p-\tfrac{1}{2}n_2-\tfrac{1}{2})_J\,.$$

Now, let us consider an alternative form for the distribution of T. Using the Lemma 2 in (1) by taking $g(\mathbf{F}) = \mathrm{tr}(\mathbf{F})/q = T/\lambda q$ for $0 < q < \infty$ and integrating $\mathbf{F}$ over the surface $T = \sum_{i=1}^{p}\lambda f_i$ with the condition $0 < f_1 < \cdots < f_p$ and making the necessary changes, we get the density function of T as

(11) $$c\,|\lambda\mathbf{\Lambda}|^{-\frac{1}{2}n_1}(T/q)^{\frac{1}{2}pn_1-1}|\mathbf{I}+(\lambda\mathbf{\Lambda})^{-1}T/q|^{-\frac{1}{2}(n_1+n_2)}\textstyle\sum_{k=0}^{\infty}\sum_\kappa(\tfrac{1}{2}n_1+\tfrac{1}{2}n_2)_\kappa \cdot\{k!\,C_\kappa(\mathbf{I}_p)\}^{-1}C_\kappa(T(q\lambda\mathbf{\Lambda}+T\mathbf{I})^{-1})A_\kappa^{(q)}(n_1, p)$$

where

(12) $$A_\kappa^{(q)}(n_1, p) = \int_{\mathfrak{D}}|\mathbf{Y}|^{\frac{1}{2}(n_1-p-1)}\alpha_p(\mathbf{Y})C_\kappa(\mathbf{I}-\mathbf{Y})\,dy_1\cdots dy_{p-1}\,,$$

with $\mathbf{Y} = \mathrm{diag}(y_1, y_2, \cdots, y_p)$, $y_p = q - y_1 - y_2 \cdots - y_{p-1}$ and

(13) $$\mathfrak{D}:\{0 \leqq y_1 \leqq q/p,\ y_1 \leqq y_2 \leqq (q-y_1)/(p-1), \cdots, y_{p-2} \leqq y_{p-1} \leqq (q-y_1-\cdots-y_{p-2})/2\}.$$

Note that (11) is the type of the form which is conjectured by Constantine [3] under the null hypothesis when $q = p$. When $q = p = 2$, $A_\kappa^{(2)}(n_1, 2)$ can be obtained by using the zonal polynomials given by James [5] and since $2 - y_1 - y_2 = 0$, it is easy to see that $A_\kappa^{(2)}(n_1, 2)$ vanishes when k is odd. In the null hypothesis H_0, we obtain Hotelling's result [4].

3.3. *Moment generating function of* $V = \mathrm{tr}\,\mathbf{W}$. We shall obtain the moment generating function of V, which is given by

$$g_1(t) = E(e^{tV}) = \{\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2)|\lambda\mathbf{\Lambda}|^{\frac{1}{2}n_1}\}^{-1}$$
$$\cdot \int_{\mathbf{S}>0} \int_{\mathbf{W}>0}^{\mathbf{I}} |\mathbf{W}|^{\frac{1}{2}(n_1-p-1)}|\mathbf{I} - \mathbf{W}|^{\frac{1}{2}(n_2-p-1)}|\mathbf{S}|^{\frac{1}{2}(n_1+n_2-p-1)}$$
$$\cdot \exp[-\operatorname{tr}\mathbf{S} + \operatorname{tr}\{\mathbf{I}t + \mathbf{S}(\mathbf{I} - (\lambda\mathbf{\Lambda})^{-1})\}\mathbf{W}]\, d\mathbf{W}\, d\mathbf{S}$$
$$= \{|\lambda\mathbf{\Lambda}|^{\frac{1}{2}n_1}\, \Gamma_p(\tfrac{1}{2}n_1 + \tfrac{1}{2}n_2)\}^{-1} \int_{\mathbf{S}>0} |\mathbf{S}|^{\frac{1}{2}(n_1+n_2-p-1)}\, e^{-\operatorname{tr}\mathbf{S}}$$
$$\cdot {}_1F_1(\tfrac{1}{2}n_1\,;\, \tfrac{1}{2}n_1 + \tfrac{1}{2}n_2\,;\, \mathbf{I}t + \mathbf{S}(\mathbf{I} - (\lambda\mathbf{\Lambda})^{-1}))\, d\mathbf{S}.$$

Now, using

$$C_\kappa(\mathbf{I} + \mathbf{A})/C_\kappa(\mathbf{I}) = \textstyle\sum_{n=0}^{k} \sum_{\eta} a_{\kappa,\eta} C_\eta(\mathbf{A})/C_\eta(\mathbf{I}) \qquad \text{(see Constantine [3])}$$

in the expression, we get the moment generating function of V as

$$(14) \quad g_1(t) = |\lambda\mathbf{\Lambda}|^{-\frac{1}{2}n_1} \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{n=0}^{k} \sum_{\eta} \{k!\, (\tfrac{1}{2}n_1 + \tfrac{1}{2}n_2)_\kappa\}^{-1}$$
$$\cdot (\tfrac{1}{2}n_1)_\kappa a_{\kappa,\eta} t^{k-n} C_\eta(\mathbf{I} - (\lambda\mathbf{\Lambda})^{-1}) C_\kappa(\mathbf{I})/C_\eta(\mathbf{I}).$$

3.4. *Distribution of* f_p. We use Lemma 2 by taking $g(F) = f_p$. Changing $f_i = x_i f_p$, $i = 1, 2, \cdots, p - 1$, in (1), we get the joint density function of $f_p, x_1, \cdots, x_{p-1}$ as

$$(15) \quad C\, |\mathbf{\Lambda}|^{-\frac{1}{2}n_1} f_p^{\frac{1}{2}pn_1-1}\, |\mathbf{I} + f_p\mathbf{\Lambda}^{-1}|^{-\frac{1}{2}(n_1+n_2)}\, |\mathbf{X}|^{\frac{1}{2}(n_1-p-1)}\, |\mathbf{I}_{p-1} - \mathbf{X}|\, \alpha_{p-1}(\mathbf{X})$$
$$\cdot {}_1F_0^{(p)}(\tfrac{1}{2}n_1 + \tfrac{1}{2}n_2\,;\, f_p(\mathbf{\Lambda} + f_p\mathbf{I})^{-1}, \mathbf{I}_{p-1} - \mathbf{X})$$

for $0 < f_p < \infty$, $0 < x_1 < \cdots < x_{p-1} < 1$ and $\mathbf{X} = \operatorname{diag}(x_1, x_2, \cdots, x_{p-1})$. Using Lemma 3, we can integrate $\mathbf{X}$ and get the density function of f_p as

$$(16) \quad c_2\, |\mathbf{\Lambda}|^{-\frac{1}{2}n_1} f_p^{\frac{1}{2}pn_1-1}\, |\mathbf{I} + f_p\mathbf{\Lambda}^{-1}|^{-\frac{1}{2}(n_1+n_2)}$$
$$\cdot {}_3F_2(\tfrac{1}{2}n_1 + \tfrac{1}{2}n_2, \tfrac{1}{2}p + 1, \tfrac{1}{2}p - \tfrac{1}{2};\, \tfrac{1}{2}p, \tfrac{1}{2}(n_1 + p + 1);\, f_p(\mathbf{\Lambda} + f_p\mathbf{I})^{-1})$$

where

$$c_2 = \Gamma(\tfrac{1}{2})\Gamma_p(\tfrac{1}{2}n_1 + \tfrac{1}{2}n_2)\Gamma_{p-1}(\tfrac{1}{2}p + 1)$$
$$\cdot \{\Gamma(\tfrac{1}{2}p)\Gamma(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2)\Gamma_{p-1}(\tfrac{1}{2}n_1 + \tfrac{1}{2}p + \tfrac{1}{2})\}^{-1}.$$

If $p = 1$, we get the usual distribution, because the hypergeometric function ${}_3F_2 = 1$ and $\Gamma_0(x) = 1$. We may note the difference in (10) and (16). Note that when $\mathbf{\Lambda} = \mathbf{I}_p$, ${}_3F_2$ can be written as ${}_2F_1(\tfrac{1}{2}n_1 + \tfrac{1}{2}n_2, \tfrac{1}{2}p + 1;\, \tfrac{1}{2}(n_1 + p + 1);\, f_p(1 + f_p)^{-1}\mathbf{I}_{p-1})$.

4. Certain null distributions of statistics for testing H_0'. In this section, let $\lambda\mathbf{\Lambda} = \mathbf{I}$ in (15) and integrating f_p, we get the joint density function of $(x_1, x_2, \cdots, x_{p-1})$ as

$$c\, |\mathbf{X}|^{\frac{1}{2}(n_1-p-1)}\, |\mathbf{I}_{p-1} - \mathbf{X}|\, \alpha_{p-1}(\mathbf{X})$$
$$(17) \qquad \cdot \textstyle\sum_{k=0}^{\infty} \Gamma(\tfrac{1}{2}pn_1 + k)\Gamma(\tfrac{1}{2}pn_2)\{k!\, \Gamma(\tfrac{1}{2}pn_1 + \tfrac{1}{2}pn_2 + k)\}^{-1}$$
$$\cdot \textstyle\sum_{\kappa} (\tfrac{1}{2}n_1 + \tfrac{1}{2}n_2)_\kappa C_\kappa(\mathbf{I}_{p-1} - \mathbf{X}) \quad \text{for} \quad 0 < x_1 < x_2 < \cdots < x_{p-1} < 1.$$

First of all we may note that if $\mathbf{Z} = \operatorname{diag}(z_1, \cdots, z_m)$, $\mathbf{Z}_1 = \operatorname{diag}(z_2, \cdots, z_m)$, $n > 0$ and $k = (k_1, \cdots, k_m)$, $k_1 \geqq k_2 \geqq \cdots \geqq k_m \geqq 0$ and $k_1 + k_2 + \cdots + k_m = k$, then

$$\sum_\kappa (n)_\kappa C_\kappa(\mathbf{Z}) = \sum_{j=0}^{k} k!\binom{n+k-j-1}{k-j} z_1^{k-j} \sum_J (n)_J C_J(\mathbf{Z}_1)/j!. \tag{18}$$

(18) is easy to prove and hence its proof is not given.

Let us use the transformation $y_i = (x_i - x_1)/(1 - x_1)$ for $i = 2, 3, \cdots, p - 1$ in (17). With the help of (20), the joint density function of x_1 and $(y_2, \cdots, y_{p-1})$ can be written as

$$\begin{aligned} & c x_1^{\frac{1}{2}(n_1-p-1)}(1 - x_1)^{\frac{1}{2}p(p+1)-2} |\mathbf{I}_{p-2} - (1 - x_1)(\mathbf{I}_{p-2} - \mathbf{Y})|^{\frac{1}{2}(n_1-p-1)} \\ & \quad \cdot|\mathbf{Y}|\,|\mathbf{I}_{p-2} - \mathbf{Y}|\alpha_{p-2}(\mathbf{Y}) \textstyle\sum_{k=0}^{\infty} \sum_{j=0}^{k} \Gamma(\frac{1}{2}pn_1 + k)\Gamma(\frac{1}{2}pn_2) \\ & \quad \cdot\{j!(k - j)!\Gamma(\frac{1}{2}pn_1 + \frac{1}{2}pn_2 + k)\}^{-1}(\frac{1}{2}n_1 + \frac{1}{2}n_2)_{k-j}(1 - x_1)^k \\ & \quad \cdot\textstyle\sum_J (\frac{1}{2}n_1 + \frac{1}{2}n_2)_J C_J(\mathbf{I}_{p-2} - \mathbf{Y}) \quad \text{for} \quad 0 < x_1 < 1 \\ & \quad \text{and} \quad 0 < y_2 < \cdots < y_{p-1} < 1 \end{aligned} \tag{19}$$

with $\mathbf{Y} = \operatorname{diag}(y_2, y_3, \cdots, y_{p-1})$. Integrating x_1, we get the joint density function of $y_i = (f_i - f_1)/(f_p - f_1)$ for $i = 2, 3, \cdots, p - 1$ as

$$\begin{aligned} & c|\mathbf{Y}||\mathbf{I}_{p-2} - \mathbf{Y}|\alpha_{p-2}(\mathbf{Y}) \sum_{n=0}^{\infty} \sum_{\eta} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{J} \Gamma(\tfrac{1}{2}(n_1 - p + 1)) \\ & \quad \cdot\Gamma(\tfrac{1}{2}p^2 + \tfrac{1}{2}p + n + k - 1)\Gamma(\tfrac{1}{2}pn_1 + k)\Gamma(\tfrac{1}{2}pn_2) \\ & \quad \cdot[j!\Gamma(\tfrac{1}{2}n_1 + \tfrac{1}{2}p^2 - \tfrac{1}{2} + n + k)\Gamma(\tfrac{1}{2}pn_1 + \tfrac{1}{2}pn_2 + k)(k - j)!]^{-1} \\ & \quad \cdot(\tfrac{1}{2}n_1 + \tfrac{1}{2}n_2)_J\, C_\eta(\mathbf{I}_{p-2} - \mathbf{Y})C_J(\mathbf{I}_{p-2} - \mathbf{Y}) \\ & \qquad\qquad \text{for} \quad 0 < y_2 < y_3 < \cdots < y_{p-1} < 1. \end{aligned} \tag{20}$$

If we integrate $\mathbf{Y}$ from (19), we get the density function of x_1, but it is much complicated and hence we do not like to give it here. Similarly, we can obtain the density function of x_{p-1} by making the transformation $x_i/x_{p-1} = z_i$ for $i = 1, 2, \cdots, p - 2$ in (17) and then integrating z_i for $i = 1, 2, \cdots, p - 2$.

REFERENCES

[1] ANDERSON, T. W. (1958). *An Introduction to Multivariate Statistical Analysis*. Wiley, New York.

[2] CONSTANTINE, A. G. (1963). Some non-central distribution problems in multivariate analysis. *Ann. Math. Statist.* **34** 1270–1285.

[3] CONSTANTINE, A. G. (1966). The distribution of Hotelling's generalized T_0^2. *Ann. Math. Statist.* **37** 215–225.

[4] HOTELLING, H. (1951). A generalized T-test and measure of multivariate dispersion. *Proc. Second Berkeley Symp. Math. Statist. Prob.* 23–42. Univ. of California Press.

[5] JAMES, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples. *Ann. Math. Statist.* **35** 475–501.

[6] KHATRI, C. G. (1966). On certain distribution problems based on positive definite quadratic functions in normal vectors. *Ann. Math. Statist.* **37** 468–479.

[7] ROY, S. N. (1958). *Some aspects on multivariate analysis*. Wiley, New York.