NOTES

CAUCHY-DISTRIBUTED FUNCTIONS OF CAUCHY VARIATES

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1. Summary. A certain class of functions of Cauchy variates whose members are also Cauchy variates is described. Some independence properties of Cauchy variates are also discussed.

2. Introduction. The Cauchy distribution used to be regarded as a pathological distribution, having a superficial resemblance (symmetrical, bell-shaped, with infinite range) to the normal distribution, yet not subject to the central limit theorem. With increasing interest in the stable distributions and their domains of attraction, the Cauchy distribution is found to occupy a less isolated position; indeed the normal distribution is extremal and rather special among stable distributions, while the Cauchy distribution shows many of their typical properties.

This note gives some simple properties of the Cauchy distribution that show that it is reproductive under functions other than sums. We shall use the abbreviation C for "Cauchy-distributed", and the symbol $\sim$ for "is distributed like". The density function of the Cauchy distribution is

$$c/\pi\{(x-a)^2+c^2\}, \qquad -\infty < x < \infty, \quad c > 0.$$

If $a = 0$, $c = 1$, the distribution is standard Cauchy.

Certain reproductive properties of the Cauchy distribution are well known. For instance, if X and Y are independent standard Cauchy and c_1 and c_2 have the same sign,

$$\begin{aligned}(c_1X + a_1) + (c_2Y + a_2) &\sim (c_1 + c_2)X + (a_1 + a_2)\\ &= (c_1X + a_1) + (c_2X + a_2)\end{aligned}$$

In this sense "independence of summands" is not important for Cauchy variates. It is the object of this note to give a striking general result of this type.

3. Meromorphic functions. A meromorphic function is a function which is analytic everywhere except at countably many poles (see Titchmarsh, (1939), Section 3.2). We consider a meromorphic function $G(z)$ whose poles are all real and simple, and which can be expanded in the form

$$(1) \qquad G(z) = kz - b/z + \sum (b_n/(a_n - z) - b_n/a_n),$$

a finite or infinite sum, where the constants are all real, $a_n \neq 0$, and those of

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k, b, b_n which are not 0 all have the same sign, which we shall take to be positive. The validity of the expansion (1) implies that $|a_n| \to \infty$ as $n \to \infty$, and that $\sum b_n/a_n^2$ is convergent.

It is easy to see that if U is C, each term in the sum,

$$G(U) = kU - b/U + \sum (b_n/(a_n - U) - b_n/a_n) \tag{2}$$

is also C; for if X is C so is its reciprocal. The striking fact is that the sum is also C. We prove this by evaluating the characteristic function $\phi(t), = E(e^{itG(U)})$.

Without loss of generality we take U to be standard Cauchy. We then have

$$\phi(t) = \int_{-\infty}^{\infty} [e^{itG(z)}/\pi(1 + z^2)] \, dz.$$

This is evaluated by contour integration. For $t > 0$, $k, b, b_n \geqq 0$, the contour of integration is in the positive direction along the real axis indented to pass above each of the poles of G, followed by a semi-circle of large radius R in the counter-clockwise sense.

Denoting the real and imaginary parts of the complex number z by $R(z), I(z)$, we note that if $I(z) \geqq 0$, then $I\{G(z)\} \geqq 0$, and therefore

$$|e^{itG(z)}| = e^{-tI\{G(z)\}} \leqq 1 \qquad (t > 0).$$

Hence the poles on the real axis contribute nothing to the integral, neither does the integral round the semicircle. The residue at $z = i$, resulting from the pole of the density function, gives the characteristic function

$$\phi(t) = e^{itG(i)} = e^{itG_1 - tG_2},$$

where $G(i) = G_1 + iG_2$. Hence, for all real t,

$$\phi(t) = e^{itG_1 - |G_2 t|}$$

From this we see that $G(U) \sim G_1 + G_2 U$ or

$$(G(U) - G_1)/G_2 \sim U. \tag{3}$$

The distribution of $G(U)$ is the same as that of a sum of independent terms which are the summands in (2).

The result of this section may be expressed in the following form:

If $k, b, b_n \geqq 0$, the a_n are real, $\sum b_n/a_n^2$ is convergent, $|a_n| \to \infty$ as $n \to \infty$, and U is a standard Cauchy variate, then

$$\frac{kU - b/U + \sum [b_n/(a_n - U) - a_n b_n/(a_n^2 + 1)]}{k + b + \sum [b_n/(a_n^2 + 1)]}$$

is a standard Cauchy variate.

4. Examples. This general result includes many interesting special cases.

(i) *Addition of angles.* The standard Cauchy variate U may be regarded as the tangent of an angle Q, uniformly distributed in $(0, \pi)$. Accordingly from now on we represent all angles by their residues modulo π.

If P is any angle, and Q is independently uniformly distributed in $(0, \pi)$, then $P + Q$ is also uniformly distributed independently of P. Accordingly, if $X = \tan P$ has any distribution whatever, the variate

$$\tan (P + Q) = (X + U)/(1 - XU) \sim U.$$

Here the meromorphic function depends (functionally not statistically) on an arbitrarily distributed random variable. Examples in which arbitrary random variables are involved can be multiplied indefinitely if we replace the constants in (2) by variates distributed independently of U, and express the results in the form (3).

(ii) *Angle-multiplication.* If Q is a uniformly distributed angle, so is any integral multiple of Q. Hence, if $U = \tan Q$ and n is any integer, $\tan nQ$ has the same distribution as U. Particular examples are

$$U_2 = 2U/(1 - U^2), \qquad U_3 = (3U - U^3)/(1 - 3U^2), \quad \text{etc.}$$

(iii) *Odd functions.* For these functions, $G_1 = 0$, so $G(U) \sim UG_2$. An interesting example is $\tan aU \sim U \tanh a$; similarly,

$$\tan (a \tan (bU)) \sim U \tanh (a \tanh b).$$

It is noted that $\tan aU$ is distributed with reduced scale, whatever the value of a. This fact follows from the fact that the standard Cauchy distribution 'wrapped round' the unit circle has a non-uniform distribution with greatest density at zero.

5. Independence properties. Many independence properties of Cauchy variates follow from the fact that the standard Cauchy variate is distributed as $\tan Q$, Q being uniformly distributed in a range π.

If Q_1 and Q_2 are two such angles, independently distributed, $Q_1 + Q_2$ has the same distribution and is independent of Q_1 and Q_2 individually. More generally, if m_1, m_2, n_1, n_2 are integers, positive, negative or zero, such that $m_1n_2 \neq m_2n_1$ (in order to exclude obvious exceptions), then $m_1Q_1 + m_2Q_2$ and $n_1Q_1 + n_2Q_2$ are independent. On the other hand, any three such linear forms are functionally related.

Accordingly the corresponding Cauchy variates are pairwise independent, but any three of them are functionally related. The simplest example of this property follows from putting

$$U_1 = \tan Q_1, \qquad U_2 = \tan Q_2, \qquad U_3 = -\tan (Q_1 + Q_2).$$

Then, although U_1, U_2 and U_3 are pairwise independent, they are subject to the relation $U_1 + U_2 + U_3 = U_1U_2U_3$. This result is a strong but natural example to show that pairwise independence of three or more variates does not imply independence. It is relatively easy to define functional relations among discrete variates that are pairwise independent but not mutually independent, but this is the first example we have found for continuous variates.

REFERENCE

TITCHMARSH, E. C. (1939). *The Theory of Functions* (2nd Ed.). Oxford Univ. Press.