THE DISTRIBUTION OF A QUADRATIC FORM OF NORMAL RANDOM VARIABLES[1]

By Bruno Baldessari[2]

University of North Carolina

1. Introduction and notation. In this article we obtain a necessary and sufficient condition under which a quadratic form, in normal random variables, is distributed as a given linear combination of independent chi-square variates (Theorem 2). This result is a generalization of the known theorem that a quadratic form, in a set of normal random variables, is distributed as a chi-square variable (or a difference between two independent chi-squares) if and only if the product of the matrix of the quadratic form and the variance-covariance matrix is idempotent (see [3], p. 685) (or tripotent), (see [4], p. 683).

X will denote an n-dimensional random vector and we assume that X is distributed like $N(\mu, V)$ an n-variate normal distribution with mean vector μ and variance covariance matrix V. The matrix V is positive definite ($V > 0$). Also we denote by $\chi'^2(n, \lambda)$ a non-central chi-square random variable with n degrees of freedom and non-centrality parameter λ. We denote by S an $n \times n$ symmetric matrix, and by Λ an $n \times n$ diagonal matrix. If X and Y are random variables (or random vectors), we will write $\mathcal{L}(X) = \mathcal{L}(Y)$ to say that the distribution of X is the same as that of Y. Also we will write $\mathcal{L}(X) = N(\mu, V)$ for X is an $N(\mu, V)$ variable.

2. A simple lemma in matrix theory. First we recall the following definition: an $n \times n$ matrix A has spectral decomposition $A = \sum_{j=1}^{s} a_j E_j$, if a_j, $j = 1, \cdots, s$, are the distinct characteristic roots of A, and if the $n \times n$ matrices E_j are non-negative definite matrices satisfying the conditions $E_i E_j = 0$, $i \neq j$, $E_j^2 = E_j$, $j = 1, \cdots, s$ (see [2], p. 64).

Lemma 1. *If S and V are $n \times n$ real symmetric matrices and if $V > 0$, then the matrix SV has a spectral decomposition.*

Proof. Lemma 1 follows from the known fact that there exists an $n \times n$ matrix M such that $|M| \neq 0$, $M'V^{-1}M = I$, $M'SM = \Lambda$, (where the diagonal elements $\lambda_1, \cdots, \lambda_n$ of Λ are the n roots of the equation $|\lambda V^{-1} - S| = 0$). In fact, let a_j, $j = 1, \cdots, s$, be the distinct roots of the equation $|\lambda I - SV| = 0$, and let B_j be the $n \times n$ diagonal matrix which has elements 1 where Λ has elements a_j and 0 otherwise. Then $SV = M'^{-1}\Lambda M' = \sum_{j=1}^{s} a_j M'^{-1} B_j M' = \sum_{j=1}^{s} a_j E_j$, is the required spectral decomposition with $E_j = M'^{-1} B_j M'$, $j = 1, \cdots, s$.

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[2] Now at the University of Rome.

REMARK. Rank $(E_j) = r_j, j = 1, \cdots, s$, where $|\lambda I - SV| = \prod_{j=1}^{s} (\lambda - a_j)^{r_j}$, $a_j \neq a_k$, $k \neq j$, i.e. the rank of E_j is the order of multiplicity of the root a_j.

3. The moment generating function of $X'SX$. If X is a set of normal random variables, then the moment generating function of the random variable $X'SX$ is well known, but we need the following new form.

LEMMA 2. *If $\mathfrak{L}(X) = N(\mu, V)$, $V > 0$, and if S is a real symmetric matrix, then the moment generating function $m_{X'SX}(\theta; \mu, V)$ of the random variable $X'SX$ is*

$$(1) \quad m_{X'SX}(\theta; \mu, V) = |I - 2SV|^{-\frac{1}{2}} \exp \{\textstyle\sum_{j=1}^{s} \frac{1}{2}(\mu' E_j V^{-1}\mu)[2\theta a_j/(1 - 2\theta a_j)]\},$$

where a_j, $j = 1, \cdots, s$, are the distinct roots of the equation $|\lambda I - SV| = 0$, (a_j with multiplicity r_j), and where E_j is defined by the spectral decomposition $SV = \sum_{j=1}^{s} a_j E_j$.

PROOF. By definition, we have

$$m_{X'SX}(\theta; \mu, V) = (2\pi)^{-\frac{1}{2}}|V|^{-\frac{1}{2}} \textstyle\int \exp \{\theta X'SX - \frac{1}{2}(x - \mu)'V^{-1}(x - \mu)\} \, dx,$$

where "$\int$" is an n fold integral and where $dx = dx_1 \cdots dx_n$. We make the transformation $x = My + \mu$ and write $c' = \mu' M'^{-1} = (c_1, \cdots, c_n)$, say, so that

$$\begin{aligned} m_{X'SX}(\theta; \mu, V) &= \textstyle\prod_{i=1}^{n} (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \exp \{\theta\lambda_i(y_i + c_i)^2 - \frac{1}{2}y_i^2\} \, dy_i \\ &= [\textstyle\prod_{i=1}^{n} (1 - 2\theta\lambda_i)^{-\frac{1}{2}}] \exp \{\sum_{i=0}^{n} \frac{1}{2}c_i^2[2\theta\lambda_i/(1 - 2\theta\lambda_i)]\} \\ &= |I - 2\theta SV|^{-\frac{1}{2}} \exp \{\textstyle\sum_{j=1}^{s} \frac{1}{2}C_j[2\theta a_j/(1 - 2\theta a_j)]\}, \end{aligned}$$

where we suppose that we have r_j roots λ_i with value a_j, $j = 1, \cdots, s$, and we write $C_j = \sum^{(j)} c_i^2$, where $\sum^{(j)}$ is the sum on the r_j subscripts i such that $\lambda_i = a_j$.

In order to prove the lemma we must show that

$$(2) \quad \textstyle\sum_{j=1}^{s} \frac{1}{2}C_j[2\theta a_j/(1 - 2\theta a_j)] = \sum_{j=1}^{s} \frac{1}{2}\mu' E_j V^{-1}\mu[2\theta a_j/(1 - 2\theta a_j)],$$

where the matrices E_j are defined by the spectral decomposition $SV = \sum_{j=1}^{s} a_j E_j$, by means of the matrix M.

Decomposing the matrix M^{-1} into its row vectors $m_1', \cdots, m_n'$ it follows from from $c = M^{-1}\mu$ that $c_i = m_i'\mu$. We will denote by $B^{(i)}$ the diagonal matrix which has all elements equal to 0 except for the ith element of the diagonal which has the value 1.

By calculation it can be verified that $c_i^2 = \mu' m_i m_i' \mu = \mu M'^{-1} B^{(i)} M^{-1}\mu$. So we have $C_j = \sum^{(j)} c_j^2 = \sum^{(j)} \mu' m_i m_i' \mu = \mu' M'^{-1} \sum^{(j)} B^{(i)} M^{-1}\mu = \mu' M'^{-1} B_j M^{-1}\mu$, $j = 1, \cdots, s$. We have $E_j = M'^{-1} B_j M'$ so that $M'^{-1} B_j M^{-1} = M'^{-1} B_j M' M'^{-1} M^{-1} = E_j V^{-1}$, $j = 1, \cdots, s$. Therefore $C_j = \mu' E_j V^{-1}\mu$, $j = 1, \cdots, s$, which proves (2) and so also Lemma 2.[3]

4. The distribution of $X'SX$ and SV. In the following Theorem 1 we will consider the random variable Y which has distribution

[3] The proof of Lemma 2 is very simple if we use the properties $E_j V^{-1} = (E_j V^{-1})'$, $(E_j V^{-1}) \cdot V(E_k V^{-1}) = 0$, $j \neq k$, and $[(E_j V^{-1})V]^2 = (E_j V^{-1})V$, but the proof given is independent from the theorem of Graybill and Marsaglia (see [3], p. 685) which is a particular case of our Theorem 2.

$$\mathcal{L}(Y) = \mathcal{L}(\textstyle\sum_{j=1}^{s} a_j\chi'^2(r_j, \frac{1}{2}\mu' L_j\mu)),$$

where : $a_j \neq a_{j'}$, $j \neq j'$, the χ'^2's are mutually independent and where the $n \times n$ matrix L_j is symmetric, positive semidefinite and of rank r_j, $j = 1, \cdots, s$.

THEOREM 1. *If $\mathcal{L}(X) = N(\mu, V)$, $V > 0$, and if S is a real symmetric matrix, then $\mathcal{L}(X'SX) = \mathcal{L}(Y)$ if and only if the matrix SV has the spectral decomposition $SV = \sum_{j=1}^{s} a_jE_j$, with $\mu' E_jV^{-1}\mu = \mu' L_j\mu$, rank $(E_j) = r_j$, $j = 1, \cdots, s$, $\sum_{j=1}^{s} r_j = n$.*

PROOF OF SUFFICIENCY. Follows immediately from Lemma 1.

PROOF OF NECESSITY. If $\mathcal{L}(X'SX) = \mathcal{L}(Y)$, then:

$$(3) \quad m_{X'SX}(\theta; \mu, V) = \textstyle\prod_{j=1}^{s} (1 - 2\theta a_j)^{-\frac{1}{2}r_j} \exp\{\sum_{j=1}^{s} \frac{1}{2}\mu' L_j\mu[2\theta a_j/(1 - 2\theta a_j)]\}.$$

We observe now that the spectral decomposition of SV is independent of μ, (being dependent on S and V only, through M) and so the spectral decomposition of SV corresponding to the random variable $X'SX$ where $\mathcal{L}(X) = N(\mu, V)$, is the same as the spectral decomposition of SV corresponding to the random variable $Z'SZ$ where $\mathcal{L}(Z) = N(0, V)$. But $m_{Z'SZ}(\theta; 0, V) = \prod_{j=1}^{s} (1 - 2\theta a_j)^{-\frac{1}{2}r_j} = |I - 2\theta SV|^{-\frac{1}{2}}$, in which we take $2\theta = \lambda^{-1}$ so that $|\lambda I - SV| = \prod_{j=1}^{s} (\lambda - a_j)^{r_j}$, and therefore we have $SV = \sum_{j=1}^{s} a_jE_j$, rank $(E_j) = r_j$, $j = 1, \cdots, s$, $\sum_{j=1}^{s} r_j = n$. From the sufficiency we know that this implies that

$$m_{X'SX}(\theta; \mu, V) = \textstyle\prod_{j=1}^{s} (1 - 2\theta a_j)^{\frac{1}{2}r_j} \exp\{\sum_{j=1}^{s} \frac{1}{2}(\mu' E_jV_\mu^{-1})[2\theta a_j/(1 - 2\theta a_j)]\}$$

so that, by the hypothesis (3), we must have

$$\textstyle\sum_{j=1}^{s} \frac{1}{2}\mu'(L_j - E_jV^{-1})\mu[2\theta a_j/(1 - 2\theta a_j)] = 0,$$

for all θ sufficiently small. Here we can, obviously, suppose that $a_j \neq 0$, $j = 1, \cdots, s$. Expanding $(1 - 2\theta a_j)^{-1}$ in a geometric series it is now easily seen that $\mu' L_j\mu = \mu' E_jV^{-1}\mu$, $j = 1, \cdots, s$. This completes the proof of Theorem 1.

In the following Theorem 2 we will use the notation $Y = \sum_{j=1}^{s} a_j\chi_j'^2(\cdot, \cdot)$, $(a_j \neq a_{j'}, j \neq j')$ to mean that the random variable Y is a linear combination, with coefficients a_j, $(a_j \neq a_{j'}, j \neq j')$, of s mutually independent non-central chi-squares, in which we do not specify the individual d.f.'s and non-centrality parameters except for the fact that the d.f.'s are positive and sum to n, and the non-centrality parameters are non-negative.

THEOREM 2. *If $\mathcal{L}(X) = N(\mu, V)$, $V > 0$, and if S is a real symmetric matrix, then $\mathcal{L}(X'SX) = \mathcal{L}(Y)$ if and only if*

$$(4) \quad (a) \quad \textstyle\prod_{j=1}^{s} (SV - a_jI) = 0,$$

$$(b) \quad \textstyle\prod_{j=1, j\neq k}^{s} (SV - a_jI) \neq 0, \qquad k = 1, \cdots, s.$$

PROOF OF NECESSITY. First we show that if $\mathcal{L}(X'SX) = \mathcal{L}(Y)$ then the condition (4) (a) is satisfied. In fact, by Theorem 1 and Lemma 1, since $\mathcal{L}(X'SX) = \mathcal{L}(Y)$ then $SV = \sum_{j=1}^{s} a_jE_j$, where rank (E_j), $j = 1, \cdots, s$, is unspecified except for the fact that rank $(E_j) \geqq 1$, $j = 1, \cdots, s$, $\sum_{j=1}^{s}$ rank $(E_j) = n$. This implies that, for every polynomial $p(x)$, we have (see [1], p. 170)

$p(SV) = \sum_{j=1}^{s} p(a_j)E_j$, and so, for the polynomial $(x - a_1) \cdots (x - a_n)$ we have $(SV - a_1I) \cdots (SV - a_sI) = (a_1 - a_1) \cdots (a_1 - a_s)E_1 + \cdots + (a_s - a_1) \cdots (a_s - a_s)E_s = 0$ and then the condition (4) (a) is satisfied.

We prove, now, that if $\mathfrak{L}(X'SX) = \mathfrak{L}(Y)$ then the conditions (4) (b) are satisfied. For, suppose that within $1, \cdots, s$, there exists a number k such that $\prod_{j=1,j\neq k}^{s}(SV - a_jI) = 0$. If it is so, since $SV = \sum_{t=1}^{s} a_tE_t$, $I = \sum_{t=1}^{s} E_t$, we would then have

$$\prod_{j=1,j\neq k}^{s}\left[\sum_{t=0}^{s}(a_t - a_j)E_t\right] = 0 \tag{5}$$

so that, multiplying (5) by E_k and recalling that $E_kE_j = 0$, $t \neq k$, we would have $\prod_{j=1,j\neq k}^{s}(a_k - a_j)E_k = 0$. This implies that either one of the numbers a_j, $j \neq k$, is equal to a_k (contradicting to the hypotheses $a_j \neq a_k$, $j \neq k$) or $E_k = 0$ which contradicts the hypothesis that each of the chi-squares involved in Y have at least one d.f. since this would imply (Theorem 1) that rank $(E_j) = 0$, $j = 1, \cdots, s$.

Proof of Sufficiency. First we prove that the condition (4) (a) implies that the random variable $X'SX$ is distributed like a linear combination of q $(1 \leqq q \leqq s)$ independent non-central chi-square variates where the coefficients are q of the a_j numbers, that is, $\mathfrak{L}(X'SX) = \mathfrak{L}(\sum_{p=1}^{q} a_{j_p}\chi_{j_p}'^2(\cdot, \cdot))$, where $1 \leqq q \leqq s$, and the numbers $j_1, \cdots, j_q$ constitute any non-empty sub-set of the set $\{1, 2, \cdots, s\}$. We shall then say that $X'SX$ is distributed "at most" like the random variable $Y = \sum_{t=1}^{s} a_j\chi_j'^2(\cdot, \cdot)$, $a_j \neq a_{j'}$, $j \neq j'$.

Now suppose that the condition (4) (a) is satisfied and that $\mathfrak{L}(X'SX) = \mathfrak{L}(\sum_{t=1}^{k} b_t\chi_t'^2(\cdot, \cdot))$, with $b_l \neq b_t$, $t \neq l$, and $k > s$. Then, by Theorem 1 and Lemma 1, the matrix SV has the spectral decomposition $SV = \sum_{t=1}^{k} b_tF_t$ where $b_t \neq b_l$, $t \neq l$ and $\sum_{t=1}^{k} F_t = I$. By (4) (a) we have

$$\left(\sum_{t=1}^{k} b_tF_t - a_1\sum_{t=1}^{k} F_t\right) \cdots \left(\sum_{t=1}^{k} b_tF_t - a_s\sum_{t=1}^{k} F_t\right) = 0$$

and multiplying by F_l^s, $l = 1, \cdots, k$, we would have

$$\prod_{j=1}^{s}(b_l - a_j)F_l = 0, \qquad l = 1, \cdots, k,$$

but this is possible only if either $F_l = 0$, which is excluded, or b_l is equal to one of the numbers a_j $(a_j \neq a_{j'}, j \neq j')$, in which case the characteristic roots of the equation $|I - SV| = 0$ would take their values among the numbers $a_1, \cdots, a_s$. Therefore, by Theorem 1, $X'SX$ is distributed like a linear combination of non-central chi-squares and the coefficients of this linear combination are among $a_1, \cdots, a_s$, i.e., $X'SX$ is distributed at most like Y.

Now we can prove the sufficiency. In fact, (4) (a) implies that $X'SX$ is distributed at most like Y, and this fact leaves open two possibilities. Either $\mathfrak{L}(X'SX) = \mathfrak{L}(Y)$ (and in this case sufficiency follows), or $\mathfrak{L}(X'SX) = \mathfrak{L}(\sum_{p=1}^{q} a_{j_p}\chi_{j_p}'^2(\cdot, \cdot))$, $q \leqq s - 1$, $a_{j_p} \neq a_{j_{p'}}$, $p \neq p'$, and we have a contradiction. In fact, in the latter case, by Theorem and Lemma 1, we have the spectral decomposition $SV = \sum_{p=1}^{q} a_{j_p}E_{j_p}$ and this implies that $(SV - a_1I) \cdots$

$(SV - a_q I) = 0$, $q \leqq s - 1$, which contradicts at least one of the equations (4) (b).

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