The Annals of Mathematical Statistics
1968, Vol. 38, No. 1, 23–28

ON A NECESSARY AND SUFFICIENT CONDITION FOR ADMISSIBILITY OF ESTIMATORS WHEN STRICTLY CONVEX LOSS IS USED[1]

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1. Introduction. In this paper we consider a necessary and sufficient condition for the admissibility of estimators when strictly convex loss is used. The result is stated as Theorem 1. The sufficiency of the condition is obvious and has served as the basis of admissibility proofs in [1], [6], [11], [3], and [2]. The necessity of such a method of proof is relatively deep. The author claims no practical use of Theorem 1. He has been moved primarily by curiosity about the necessity part of the theorem together with a desire to strengthen the tools of decision theory. The results of this paper depend on Farrell [5] to which one can refer for definitions of some common terms like "Bayes" if these are not clear.

In the sequel $\mathbb{R}_n$ will denote Euclidean n-space $n \geqq 1$, and $\mathbb{R} = \mathbb{R}_1$ the set of real numbers. If $n \geqq 1$ we let $\mathcal{B}_n$ be the σ-algebra of Borel subsets of $\mathbb{R}_n$.

THEOREM 1. *Let $X = \mathbb{R}_k$. Let the decision space $\mathfrak{D}$ be an open convex subset of $\mathbb{R}_m$. We suppose the parameter space Ω is a separable locally compact metric space and $\mathcal{C}$ is the σ-algebra of Borel subsets of Ω. In addition we assume*

(i) *μ is a σ-finite (regular) measure on $\mathcal{B}_k$.*

(ii) *$\{f(\cdot, \omega), \omega \in \Omega\}$ is a family of density functions in $L_1(X, \mathcal{B}_k, \mu)$.*

(iii) *$f: X \times \Omega \rightarrow [0, \infty)$ is (jointly) measurable and if $x \in X$ then $f(x, \cdot): \Omega \rightarrow [0, \infty)$ is a continuous function.*

(iv) *The measure of loss W satisfies, $W: \mathfrak{D} \times \Omega \rightarrow [0, \infty)$ is a continuous function. If $\omega \in \Omega$, $W(\cdot, \omega): \mathfrak{D} \rightarrow [0, \infty)$ is strictly convex. If $E \subset \Omega$ is compact then* $\lim_{|t| \to \infty} \inf_{\omega \in E} W(t, \omega) = \infty$.

(v) *If $\omega \in \Omega$, $x \in X$, then $f(x, \omega) > 0$.*

Then (vi) *and* (vii) *stated below are equivalent.*

(vi) *The estimator δ is admissible;*

(vii) *The procedure δ is non-randomized and has risk function $r(\delta, \cdot)$. There exists an increasing sequence of compact subsets $\{F_n, n \geqq 1\}$ of Ω, $F_n \uparrow \Omega$, a sequence of finite measures $\{\eta_n, n \geqq 1\}$, and a sequence of Bayes procedures $\{\delta_n, n \geqq 1\}$, such that*

(vii a) *there exists a compact subset $E_0 \subset \Omega$ such that $\inf_{n \geqq 1} \eta_n(E_0) \geqq 1$;*

(vii b) *if $E \subset \Omega$ is a compact subset then $\sup_{n \geqq 1} \eta_n(E) < \infty$;*

(vii c) *if $n \geqq 1$ then δ_n is Bayes relative to η_n with risk function*

$$r(\delta_n, \cdot) \cdot \mathrm{Lim}_{n \to \infty} \int (r(\delta_n, \omega) - r(\delta, \omega)) \eta_n(d\omega) = 0;$$

(vii d) $\lim_{n \to \infty} r(\delta_n, \omega) = r(\delta, \omega)$, $\omega \in \Omega$.

Received 6 December 1967.

[1]This research was supported in part by the Office of Naval Research under Contract number Nonr 401(50). Reproduction in whole or in part is permitted for any purpose of the United States Government.

[Note: throughout we consider only decision procedures having everywhere finite risk.]

A restricted version of Theorem 1 was proven in Farrell [4] in the case Ω was a sudset of $\mathbb{R}$. The proof there required a very detailed analysis of the nature of generalized Bayes procedures. The purpose of the present paper is to generalize the earlier result and to show that it really is a consequence of the geometry of the problem alone.

The examples of admissibility proofs mentioned above are all examples of admissibility of generalized Bayes procedures. An example due to L. D. Browns Example 6.2 of Farrell [5], shows that in problems of testing hypotheses there exist examples of admissible tests which cannot be generalized Bayes procedures. Although we believe the same to be true of some estimation problems we do not have a conclusive example of an admissible estimator which is not a generalized Bayes procedure. Further Farrell [4] and Sacks [7] have shown for estimation problems that in certain cases generalized Bayes procedures do form a complete class. Therefore, our statement of the theorem speaks only of admissible estimators without trying to show that such estimators must be generalized Bayes procedures.

The proof makes use of a strengthened form of Stein's [9] necessary and sufficient condition for admissibility. The required result is given in Theorem 3.7 of Farrell [5].

The proof given below requires that a special case discussed in Section 2 be treated. The case discussed there does arise in the decision theory of testing hypotheses but we do not know whether it arises in estimation theory. Section 3 contains a proof of Theorem 1.

2. Admissibility outside of compact sets. In this section $\mathcal{R}$ will be a convex set of functions on Ω to $[0, \infty)$. In this section no further restriction on the individual elements of $\mathcal{R}$ is needed. It will however be helpful to think of $\mathcal{R}$ as the set of risk functions for some statistical problem.

DEFINITION 2.1. $\mathcal{R}$ is sequentially weakly subcompact if and only if to every countably infinite subset $\{f_n, n \geqq 1\}$ of $\mathcal{R}$ there is an $f \in \mathcal{R}$ and a countably infinite subsequence $\{f_{a_n}, n \geqq 1\}$ such that if $\omega \in \Omega$ then $f(\omega) \leqq \liminf_{n\to\infty} f_{a_n}(\omega)$.

DEFINITION 2.2. Let $\{E_n, n \geqq 1\}$ be a sequence of subsets of Ω. $f \in \mathcal{R}$ is admissible outside $\{E_n, n \geqq 1\}$ if and only if given $g \in \mathcal{R}$, $g \neq f$, then if $n \geqq 1$ there exists $\omega \notin E_n$ such that $g(\omega) > f(\omega)$.

DEFINITION 2.3. An admissible point of $\mathcal{R}$ is a function $f \in \mathcal{R}$ such that if $g \in \mathcal{R}$, $g \neq f$, then there exists an $\omega \in \Omega$ such that $g(\omega) > f(\omega)$.

The concept of Definition 2.2 may prove hard for the reader and we offer an example. Suppose the family of density functions is exponential, the problem is a testing problem, $\Omega = X = \mathbb{R}_k$, H_0 is an (unbounded) proper subspace of $\mathbb{R}_k$ and $H_1 = \Omega - H_0$. Let $\{E_n, n \geqq 1\}$ be any sequence of *bounded* subsets of $\mathbb{R}_k$ and let φ be a non-randomized test having a convex acceptance region. Then it follows from Stein [10] that φ is admissible outside $\{E_n, n \geqq 1\}$. For, given n, let ω_0 be a point of H_0 not in E_n. Then ω_0 is a boundary point of H_0 and continuity of the power function implies every test φ' as good as φ outside E_n must have the same power function as φ on $H_0 \cap (\Omega - E_n)$, and in particular at ω_0. It now follows at once from Stein's proof, op. cit., that if φ' and φ are distinct yet have equal power at ω_0 then φ' must have power less than φ somewhere far out, that is, outside E_n.

The basic result of this section, needed for the proof of Theorem 1, is really about functions f admissible outside $\{E_n, n \geqq 1\}$. However a slightly stronger result is needed for the proof of Theorem 1, and this stronger version is given below.

THEOREM 2. *Assume Ω is a σ-compact locally compact Hausdorff space. Let $\mathcal{R}$ be a convex sequentially weakly subcompact set of real valued functions on Ω. Let $\{E_n, n \geqq 1\}$ be an increasing sequence of compact subsets of Ω, $E_n \uparrow \Omega$, such that if $n \geqq 1$ then the interior E_n^0 of E_n is nonvoid and $E_n^0 \supset E_{n-1}$, $E_0 = \{\omega_0\}$. Let f be an admissible point of $\mathcal{R}$. Suppose that it is false that there exists a compact subset $E \subset \Omega$ and a function $g \in \mathcal{R}$ such that $g \neq f$, $\sup_{\omega \in E} (g(\omega) - f(\omega)) < \infty$, and if $\omega \notin E$ then $g(\omega) \leqq f(\omega)$. Then there exists a continuous function $h:\Omega \to \mathbb{R}$ such that $h(\omega_0) > 0$, if $\omega \notin E_1$ then $h(\omega) < 0$, and f is an admissible point of the convex hull of $\mathcal{R}$ and $\{f + h\}$.*

PROOF. If $n \geqq 1$ let $A_n = \{g \mid g \in \mathcal{R}, g \neq f, g(\omega_0) + (1/n) \leqq f(\omega_0)$, and $\sup_{\omega \in E_n} (g(\omega) - f(\omega)) \leqq n\}$. Define a sequence of constants $\{\pi_n, n \geqq 1\}$ by $\pi_n = \inf_{g \in A_n} \sup_{\omega \notin E_n} (g(\omega) - f(\omega))$, $\pi_n = \infty$ if A_n is empty.

We show that $\pi_n > 0$, $n \geqq 1$. Since $\mathcal{R}$ is sequentially weakly subcompact, if $\pi_n \leqq 0$, we may choose in A_n a sequence $\{g_m, m \geqq 1\}$ and a function $g \in \mathcal{R}$ such that if $\omega \in \Omega$ then $g(\omega) \leqq \liminf_{m \to \infty} g_m(\omega)$, and $\sup_{\omega \notin E_n} (g_m(\omega) - f(\omega)) \leqq 1/m$. It follows that $g \in A_n$, that $g(\omega_0) < f(\omega_0)$ so that $g \neq f$, and that if $\omega \notin E_n$ then $g(\omega) \leqq f(\omega)$. This contradicts the hypothesis of the theorem.

A compact Hausdorff space is a normal topological space. See Kelley [7]. Let $\{\delta_n, n \geqq 1\}$ be a nonincreasing real number sequence such that $\delta_n \downarrow 0$. Let $h:\Omega \to \mathbb{R}$ be a continuous function which satisfies $h(\omega_0) > 0$, and if $\omega \in E_1$ then $h(\omega) \geqq -1$. Further, if $n \geqq 1$ and $\omega \notin E_n$ then $0 > h(\omega) > \max(-1, -\delta_n \pi_n/2)$. To obtain such a function define constants τ_n by $\tau_n = \max(-1, -\delta_1 \pi_1/2, \cdots, -\delta_n \pi_n/2)$, $n \geqq 1$. Since $E_{n+2} - E_{n+1}^0$ is compact, since E_n is compact, and since $E_n \subset E_{n+2}$, by the normality of E_{n+2} as a topological space we may find a continuous function h_n such that if $\omega \in E_{n+2}$ then $0 \leqq h_n(\omega) \leqq 1$, if $\omega \in E_n$ then $h_n(\omega) = 1$, and if $\omega \notin E_{n+1}^0$ then $h_n(\omega) = 0$. Extend h_n to all of Ω by giving it the value zero outside E_{n+2}. Define

$$h(\omega) = h_1(\omega) - \tau_2 h_1(\omega) - \sum_{n=1}^{\infty} (\tau_{n+1} - \tau_n) h_n(\omega).$$

The infinite series is uniformly convergent so the continuity of h follows. Since $\tau_n \uparrow 0$, $h(\omega_0) = h_1(\omega_0) > 0$ follows. If $\omega \in E_{n+1} - E_n$ then $h_1(\omega) = 0, \cdots, h_{n-1}(\omega) = 0$, and $h_j(\omega) = 1$ if $j \geqq n + 1$. Thus $h(\omega) \geqq -(\tau_{n+1} - \tau_n) - (\tau_{n+2} - \tau_{n+1}) - \cdots = \tau_n$. Therefore the function h has the desired properties.

We form the convex hull of $\mathcal{R}$ and $\{h + f\}$. Suppose for some $\beta \in \mathbb{R}$, $0 \leqq \beta \leqq 1$, and $g \in \mathbb{R}$ such that $g \neq f$, that $\beta(h + f) + (1 - \beta)g \leqq f$. We show that this assumption leads to a contradiction.

Since $h(\omega_0) > 0$ it must be that $\beta \neq 0$ and $\beta \neq 1$. Therefore we obtain $\beta h(\omega_0) \leqq (1 - \beta)(f(\omega_0) - g(\omega_0))$ which implies that $g(\omega_0) - f(\omega_0) < 0$. It also follows that if $\omega \varepsilon \Omega$ then $\beta h(\omega) + (1 - \beta)(g(\omega) - f(\omega)) \leqq 0$ which implies that $(1 - \beta) \sup_{\omega \varepsilon \Omega} (g(\omega) - f(\omega)) \leqq \beta$. We use here the assumption that $\inf_{\omega \varepsilon \Omega} h(\omega) \geqq -1$. If n_0 is the least integer such that $\sup_{\omega \varepsilon \Omega} (g(\omega) - f(\omega)) \leqq n_0$ and $f(\omega_0) - g(\omega_0) \geqq 1/n_0$, then $g \varepsilon A_n$, $n \geqq n_0$. Therefore if $n \geqq n_0$ there exists $\omega_n \not\varepsilon E_n$ such that $g(\omega_n) - f(\omega_n) > \pi_n/2$.

Substitution of these inequalities now gives, if $n \geqq n_0$ then $\beta h(\omega_n) + (1 - \beta)(g(\omega_n) - f(\omega_n)) \leqq 0$. Therefore $\beta(-\delta_n \pi_n/2) + (1 - \beta)(\pi_n/2) \leqq 0$. If in this inequality we divide out $\pi_n/2$, then take a limit on n, we obtain $1 - \beta \leqq 0$. Since $0 < \beta < 1$ this is a contradiction.

Therefore it must be that f is an admissible point of the convex hull of $\mathcal{R}$ and $\{f + h\}$, as was to be proven.

3. Proof of Theorem 1. In the sequel we let $\mathcal{R}$ be the set of risk functions obtainable by considering all randomized decision functions, and assume as stated earlier that if $g \varepsilon \mathcal{R}$ then $g(\omega) < \infty$ for all $\omega \varepsilon \Omega$. Our hypotheses require that if $\omega \varepsilon \Omega$ then $g(\omega) \geqq 0$. As is well known, the hypotheses of Theorem 1 imply the compactness of the set of decision functions and the sequentially weakly subcompactness of $\mathcal{R}$. See Wald [12] and the appendix to Farrell [4]. As a first step in the proof we need a lemma.

LEMMA 3. *The points of $\mathcal{R}$ are lower semicontinuous functions. If $f \varepsilon \mathcal{R}$ and $E \subset \Omega$ is compact there exists $g \varepsilon \mathcal{R}$ such that g is continuous and if $\omega \varepsilon E$ then $g(\omega) \leqq f(\omega)$.*

PROOF. Using hypothesis (iii), it follows that $\lim_{\omega \to \omega_0} f(\cdot, \omega) = f(\cdot, \omega_0)$ in $L_1(X, \mathcal{B}_k, \mu)$. Using this, hypothesis (iv) and considering truncations of the loss function the lower-semicontinuity assertion follows.

Let $E \subset \Omega$, and suppose E is compact. Let $t_0 \varepsilon \mathfrak{D}$ and let C be a compact subset of $\mathfrak{D}$ such that $\sup_{\omega \varepsilon E} W(t_0, \omega) < \inf_{t \not\varepsilon C} \inf_{\omega \varepsilon E} W(t, \omega)$. This is possible by virtue of hypothesis (iv). Given a nonrandomized procedure δ define δ' by, if $x \varepsilon X$ and if $\delta(x) \varepsilon C$ then $\delta'(x) = \delta(x)$, and if $\delta(x) \not\varepsilon C$ then $\delta'(x) = t_0$. As is well known it follows from the definition of δ' that if $\omega \varepsilon E$ then $r(\delta', \omega) \leqq r(\delta, \omega)$. Further, since W is a continuous function, if $F \subset \Omega$ is any compact subset of Ω then $\sup_{x \varepsilon X} \sup_{\omega \varepsilon F} W(\delta'(x), \omega) < \infty$, so that using the bounded convergence theorem it follows that if $\omega_0 \varepsilon \Omega$ then $\lim_{\omega \to \omega_0} r(\delta', \omega) = r(\delta', \omega_0)$. The proof of Lemma 3 has been completed.

In the proof of Theorem 1 we need to consider an admissible point $f \varepsilon \mathcal{R}$ and relative to f we consider two cases as follows.

CASE I. There exists a compact subset $E \subset \Omega$ and $g \varepsilon \mathcal{R}$ such that $g \neq f$ and if $\omega \not\varepsilon E$ then $g(\omega) \leqq f(\omega) \cdot \sup_{\omega \varepsilon E} (g(\omega) - f(\omega)) < \infty$.

CASE II. It is false that case I obtains.

We begin with Case I. Let $E \subset \Omega$ be a compact subset, and let $g \varepsilon \mathcal{R}$, $g \neq f$, such that if $\omega \not\varepsilon E$ then $g(\omega) \leqq f(\omega)$. We may suppose g is the risk function of a nonrandomized procedure δ' and f is the risk function of a nonrandomized procedure δ, and we will use the alternative notations $g(\omega) = r(\delta', \omega)$ and

$f(\omega) = r(\delta, \omega)$. Let $\delta'' = (\delta + \delta')/2$. Since the measure of loss is strictly convex and since we assume $f(x, \omega) > 0$ everywhere, it follows that if $\omega \not\varepsilon E$ then $r(\delta'', \omega) < r(\delta, \omega)$. Farrell [4] has shown that if $F \subset \Omega - E$ is compact then $\inf_{\omega \varepsilon F} (r(\delta, \omega) - r(\delta'', \omega)) > 0$.

We apply Theorem 3.7 of Farrell [5]. In the application we let E' be a compact set such that E is contained in the interior of E', and define the function $a(\cdot)$ by, if $\omega \varepsilon E'$ then $a(\omega) = 1$, if $\omega \not\varepsilon E'$ then $a(\omega) = 0$. Then the sequences $\{\eta_n, n \geqq 1\}$, $\{g_n, n \geqq 1\}$ and $\{F_n, n \geqq 1\}$ referred to in Farrell, op. cit., satisfy (vii a), (vii c) and (vii d) of Theorem 1. It remains to verify (vii b). Since g_n is the risk function of some decision procedure, say δ_n, we write $g_n = r(\delta_n, \cdot)$, $n \geqq 1$.

Since δ_n is Bayes for η_n, $n \geqq 1$, we find

$$(3.1) \qquad \int (r(\delta_n, \omega) - r(\delta, \omega))\eta_n(d\omega) \leqq \int (g(\omega) - f(\omega))\eta_n(d\omega).$$

Using (vii c) and E as above, since the left side of (3.1) goes to zero as n goes to infinity, it follows that

$$(3.2) \qquad \limsup_{n\to\infty} \int_{\Omega - E} (f(\omega) - g(\omega))\eta_n(d\omega) \leqq \limsup_{n\to\infty} \int_E (g(\omega) - f(\omega))\eta_n(d\omega).$$

The right side of (3.2) is $< \infty$ since f is nonnegative and since g satisfies the hypothesis of Case I.

It may happen that for some $\omega \not\varepsilon E$ that $g(\omega) = f(\omega)$. In this case we may repeat the above argument with g replaced by g' defined by $g'(\omega) = r(\delta'', \omega)$, $\omega \varepsilon \Omega$. Thus, without loss of generality we may suppose that if $\omega \not\varepsilon E$ then $g(\omega) < f(\omega)$. Then since F is a compact set and $F \subset \Omega - E$, we have $\inf_{\omega \varepsilon F} (r(\delta', \omega) - r(\delta, \omega)) > 0$. It follows that if $n \geqq 1$ then $\eta_n(E') = 1$ and $\sup_{n \geqq 1} \eta_n(F) < \infty$.

In Case II we apply Theorem 2. We are able to find a continuous function h such that $h(\omega_0) > 0$ and if $\omega \not\varepsilon E$, then $h(\omega) < 0$. Let $\mathcal{R}'$ be the convex hull of $\{h + f\}$ and $\mathcal{R}$. $\mathcal{R}'$ is again a convex and sequentially weakly subcompact set, $f \varepsilon \mathcal{R}'$ is an admissible point, the functions in $\mathcal{R}'$ are lower semicontinuous, and if $E \subset \Omega$ is compact, if $g \varepsilon \mathcal{R}'$, then there is a continuous $g' \varepsilon \mathcal{R}'$ such that if $\omega \varepsilon E$ then $g(\omega) \geqq g'(\omega)$. Therefore Theorem 3.7 of Farrell [5] applies to $\mathcal{R}'$. Let $\{\eta_n, n \geqq 1\}$ and $\{F_n, n \geqq 1\}$ be as in Theorem 3.7, op. cit., and let $g_n' = \beta_n(h + f) + (1 - \beta_n)g_n$, $\beta_n \varepsilon R$, $0 \leqq \beta_n \leqq 1$, $g_n \varepsilon \mathcal{R}$, $n \geqq 1$, be the corresponding Bayes points. Since $\mathcal{R}$ is sequentially weakly subcompact, from any subsequence $\{\beta_{2n}, n \geqq 1\}$ such that $\beta = \lim_{n\to\infty} \beta_{2n}$ exists we may choose a subsequence $\{\beta_{1n}, n \geqq 1\}$, a sequence $\{g_{1n}, n \geqq 1\}$ of $\{g_n, n \geqq 1\}$ and a point $g \varepsilon \mathcal{R}$ such that $g \leqq \liminf_{n\to\infty} g_{1n}$. Part of the conclusion of Theorem 3.7, op. cit., asserts that $f = \lim_{n\to\infty} g_n'$. Therefore it follows that $f = \lim_{n\to\infty} g_n' \geqq \beta(h + f) + (1 - \beta)g$. Since $g \geqq 0$, and since $h(\omega_0) > 0$, it follows that $\beta = 0$ and $g \neq f$. We use here the hypothesis that f is an admissible point of $\mathcal{R}$. Since every convergent subsequence of $\{\beta_n, n \geqq 1\}$ converges to zero it follows that $\lim_{n\to\infty} \beta_n = 0$. Therefore $f = \lim_{n\to\infty} g_n' = \lim_{n\to\infty} g_n$. Since if $n \geqq 1$,

g_n' is Bayes for η_n, it follows that

$$(3.3)\quad 0 = \lim_{n\to\infty} \int (g_n'(\omega) - f(\omega))\eta_n(d\omega) \leqq \liminf_{n\to\infty} \int (h(\omega) + f(\omega) - f(\omega))\eta_n(d\omega) = \liminf_{n\to\infty} \int h(\omega)\eta_n(d\omega).$$

Since h is a continuous function, since $h(\omega) < 0$ if $\omega \not\in E$, since E is compact, and since $\eta_n(E) \leqq 1, n \geqq 1$, from (3.3) it follows that

$$\limsup_{n\to\infty} \int |h(\omega)|\eta_n(d\omega) < \infty.$$

Therefore $\sup_{n\geqq 1} \eta_n(F) < \infty$ for all compact subsets of Ω, as before, and

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int (\beta_n(h(\omega) + f(\omega)) + (1 - \beta_n)g_n(\omega) - f(\omega))\eta_n(d\omega) \\ (3.4)\qquad &= \lim_{n\to\infty} (\beta_n \int h(\omega)\eta_n(d\omega) + (1 - \beta_n) \int (g_n(\omega) - f(\omega))\eta_n(d\omega)) \\ &= \lim_{n\to\infty} \int (g_n(\omega) - f(\omega))\eta_n(d\omega). \end{aligned}$$

Lastly, since $\lim_{n\to\infty} \beta_n = 0$ there exists an integer n_0 such that if $n \geqq n_0$ then $\beta_n \neq 1$. Therefore if $n \geqq n_0$, g_n is Bayes relative to η_n within the class $\mathcal{R}'$ and hence within $\mathcal{R}$.

The proof of Theorem 1 has been completed.

REFERENCES

[1] Blyth, Colin R. (1951). On minimax statistical decision procedures and their admissibility. *Ann. Math. Statist.* **22** 22–42.

[2] Brown, Lawrence David. (1966). On the admissibility of invariant estimators of one or more location parameters. *Ann. Math. Statist.* **37** 1087–1136.

[3] Farrell, R. H. (1964). Estimators of a location parameter in the absolutely continuous case. *Ann. Math. Statist.* **35** 949–998.

[4] Farrell, R. H. (1966). Weak limits of sequences of Bayes procedures in estimation theory. To appear in the *Fifth Berkeley Symp. Math. Statist. Prob.*

[5] Farrell, R. H. (1968). Towards a theory of generalized Bayes tests. *Ann. Math. Statist.* **38** 1–22.

[6] Karlin, Samuel. (1958). Admissibility for estimation with quadratic loss. *Ann. Math. Statist.* **29** 406–436.

[7] Kelley, John L. (1955). *General topology*. Van Nostrand, New York.

[8] Sacks, J. (1963). Generalized Bayes solutions in estimation problems. *Ann. Math. Statist.* **34** 751–768.

[9] Stein, C. (1955). A necessary and sufficient condition for admissibility. *Ann. Math. Statist.* **26** 518–522.

[10] Stein, C. (1956). The admissibility of Hotelling's T^2-test. *Ann. Math. Statist.* **27** 616–623.

[11] Stein, C. (1959). The admissibility of Pitman's estimator of a single location parameter. *Ann. Math. Statist.* **30** 970–979.

[12] Wald, A. (1950). *Statistical Decision Functions*. Wiley, New York.