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TOWARDS A THEORY OF GENERALIZED BAYES TESTS[1]

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1. Introduction. The main result of this paper is Theorem 7.1. Stated there is a necessary and sufficient condition for the admissibility of tests in the exponential case when the hypothesis set H_0 is compact and "topologically" separated from the alternative H_1. In the theorem we ask that the entire parameter space Ω be a closed convex cone in Euclidean k-space $\mathbb{R}_k$. The proof that tests satisfying the stated condition are admissible is relatively easy and is almost a direct consequence of an admissibility theorem for generalized Bayes tests proven in Section 5. The proof that the stated condition is necessary is much harder and requires the results of Section 2, Section 3, Parts of Section 4, and a lengthy argument in Section 7.

This paper originated out of efforts on the author's part to see what could be done with a theory of generalized Bayes tests. This can be considered to be a continuation of work begun in Farrell [5] in which some complete class theorems in estimation problems were obtained.

It was discovered that in the case of Birnbaum's [2] necessary and sufficient condition all admissible tests are generalized Bayes tests (see Theorem 4.1) and conversely under much less restricted conditions generalized Bayes tests are admissible (see Theorem 5.1 and 5.2). L. D. Brown has given the author several examples presented in Section 6. In Example 6.1 the hypothesis set H_0 is a compact convex set and $H_1 = \Omega - H_0$. The probabilities form an exponential family (see below) and *every* test function is admissible. This example completely destroys the hope of completely describing admissible tests in the exponential case by generalized Bayes procedures.

Example 6.2 of L. D. Brown led the author to Theorem 7.1. In this example H_0 contains two points x_1, x_2, (hence is compact) and H_1 may be considered to be any closed subset of Ω disjoint from H_0 so long as H_1 contains $(x_1 + x_2)/2$ and a sequence of points (x_n, y_n) with $\lim_{n\to\infty} y_n = \infty$. An admissible test is described that is not a generalized Bayes test but within a certain subfamily of tests is in fact a Bayes test. The necessary and sufficient condition stated in Theorem 7.1 has to do with choice of the right subfamily of tests.

Throughout subsequent sections the parameter space will be denoted by Ω, the hypothesis set by H_0 and the alternative set by H_1. We assume $H_0 \cap H_1 =$ null set, but allow $H_0 \cup H_1$ to be a proper subset of Ω.

$\{f_\omega(\cdot), \omega \varepsilon \Omega\}$ will be a family of generalized density functions on a set X,

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measurable in a σ-algebra of subsets $\mathcal{B}$, and integrated with respect to a σ-finite measure μ.

If the given family of density functions has the special form $f_\omega(x) = k(\omega) \exp(\omega \cdot x)$, $\omega, x \in \mathbb{R}_k$, Ω an open subset of $\mathbb{R}_k$, then we will speak of an exponential family of density functions.

In the sequel, if η is a finite measure on the Borel subsets of Ω such that the total support of η is on $H_0 \cup H_1$ and if φ_0 is a test function satisfying

$$\int\int_{H_0} \varphi_0(x) f_\omega(x)\mu(dx)\eta(d\omega) + \int\int_{H_1} (1 - \varphi_c(x)) f_\omega(x)\mu(dx)\eta(d\omega)$$
$$= \inf_\varphi \int\int_{H_0} \varphi(x) f_\omega(x)\mu(dx)\eta(d\omega) + \int\int_{H_1} (1 - \varphi(x)) f_\omega(x)\mu(dx)\eta(d\omega),$$

then we shall say that φ_0 is a Bayes test relative to (or for) η even though η may not be a probability measure. If η is a σ-finite measure and

$$\varphi_0(x) \int_{H_0} f_\omega(x)\eta(d\omega) + (1 - \varphi_0(x)) \int_{H_1} f_\omega(x)\eta(d\omega)$$
$$= \inf_\varphi \varphi(x) \int_{H_0} f_\omega(x)\eta(d\omega) + (1 - \varphi(x)) \int_{H_1} f_\omega(x)\eta(d\omega)$$

then we shall say that φ_0 is a generalized Bayes test for η.

A slightly different usage will also be made as follows. If $\mathcal{R}$ is a convex set of nonnegative real valued functions on a locally compact Hausdorff space X, and if η is a finite Borel measure on the Borel subsets of X then we will say that $f \in \mathcal{R}$ is Bayes for η if

$$\int f(x)\eta(dx) = \inf_{g \in \mathcal{R}} \int g(x)\eta(dx).$$

2. Representation of positive linear functionals. In the sequel we need to know that certain positive linear functionals can be represented as integrals. Although results of this type are standard, see for example Bourbaki [3], Chapter III, Section 3, and Neveu [8], for the sake of completeness we have written a short section about these results.

Throughout Ω is a set with locally compact Hausdorff topology assigned. $C(\Omega, \mathbb{R})$ will be the linear space of all real valued continuous functions on Ω to $\mathbb{R}$. We take on $C(\Omega, \mathbb{R})$ the topology of uniform convergence on compact sets defined as follows: If A is a directed index set, $\{f_a, a \in A\}$ an indexed set of functions in $C(\Omega, \mathbb{R})$, then $\lim_{a \in A} f_a = f$ if and only if to every compact subset $E \subset \Omega$ and every $\epsilon > 0$ there exists $a_0 \in A$ such that if $a > a_0$ then $\sup_{\omega \in E} |f_a(\omega) - f(\omega)| < \epsilon$.

The basic result needed is as follows.

THEOREM 2.1. *Let Ω have a locally compact Hausdorff topology and $C(\Omega, \mathbb{R})$ have the topology of uniform convergence on compact sets. If $I: C(\Omega, \mathbb{R}) \to \mathbb{R}$ is a positive continuous linear functional then there exists a nonnegative countably additive Borel measure μ such that μ has compact support and if $f \in C(\Omega, \mathbb{R})$ then $I(f) = \int f(\omega)\mu(d\omega)$.*

PROOF. By the theory of the Daniell integral, see Loomis [7], a measure μ and a σ-algebra of sets $\mathcal{B}$ exist such that if $f \in C(\Omega, \mathbb{R})$ and $\sup_{\omega \in \Omega} |f(\omega)| < \infty$ then f is $\mathcal{B}$ measurable and $I(f) = \int f(\omega)\mu(d\omega)$. Since $C(\Omega, \mathbb{R})$ contains the constant functions, $\mu(\Omega) < \infty$ follows.

We show μ to have compact support. We suppose that Ω is not compact. Suppose there exists $\epsilon \geqq 0$ such that if $E \subset \Omega$, E is compact, then $\mu(E) + \epsilon \leqq \mu(\Omega)$. We show $\epsilon = 0$. For to each compact subset E choose a function $f_E \varepsilon C(\Omega, \mathbb{R})$ such that if $\omega \varepsilon E$ then $f_E(\omega) = 0$, and such that if $\omega \varepsilon \Omega$ then $0 \leqq f_E(\omega) \leqq 1$. We may further suppose for some compact set E', if $\omega \not\varepsilon E'$ then $f_E(\omega) = 1$. Then $\int f_E(\omega)\mu(d\omega) \geqq \mu(\Omega - E') \geqq \epsilon$. But the set of compact subsets of Ω is directed under inclusion and $\lim_E f_E = 0$. Therefore since I is continuous, $0 = I(\lim_E f_E) = \lim_E I(f_E) \geqq \epsilon \geqq 0$. Therefore $\epsilon = 0$. If μ does not have compact support we may therefore find an increasing sequence $\{E_n, n \geqq 1\}$ of compact subsets such that if $n > 1$ then E_n is interior to E_{n+1} and $\lim_{n\to\infty} \mu(E_n) = \mu(\Omega)$. Then it is clear we may find a nonnegative function $f \varepsilon C(\Omega, \mathbb{R})$ such that $I(f) \geqq \int f(\omega)\mu(d\omega) = \infty$. This contradiction shows μ must have compact support.

Since μ has compact support it now readily follows that $I(f) = \int f(\omega)\mu(d\omega)$ for all $f \varepsilon C(\Omega, \mathbb{R})$.

3. A necessary and sufficient condition for admissibility. Stein [9] has given a necessary and sufficient condition for admissibility. Stein's condition has been generalized somewhat by LeCam and has been given a very elegant proof by LeCam. In as much as the better version is needed here, and has not been published, we sketch the details here.

We suppose Ω is given a locally compact Hausdorff topology, $C(\Omega, \mathbb{R})$ is the topological linear space of continuous functions on Ω to $\mathbb{R}$ with the topology of uniform convergence on compact sets. We will consider $\mathcal{R} \subset C(\Omega, \mathbb{R})$, a convex set satisfying the following definition of weak subcompactness.

DEFINITION 3.1. $\mathcal{R}$ is said to be weakly subcompact if given a directed index set A and a sequence $\{f_a, a \varepsilon A\} \subset \mathcal{R}$ there exists in $\mathcal{R}$ a function f such that if $\omega \varepsilon \Omega$ then $f(\omega) \leqq \lim_a \sup f_a(\omega)$.

In addition we need the following definitions:

DEFINITION 3.2. $f \varepsilon \mathcal{R}$ is an admissilbe point of $\mathcal{R}$ if $g \varepsilon \mathcal{R}$ and $f \neq g$ implies there exists $\omega \varepsilon \Omega$ such that $f(\omega) < g(\omega)$.

DEFINITION 3.3. Let $\mathcal{B}$ be the least σ-algebra of subsets of Ω in which the functions of $C(\Omega, \mathbb{R})$ are measurable. To each nonnegative $\mathcal{B}$-measurable function a let V_a be the set of all finite nonnegative measures μ on $\mathcal{B}$ such that $\int a(\omega)\mu(d\omega) = 1$. If $\mathcal{R}$ is a convex subset of $C(\Omega, \mathbb{R})$ and $f \varepsilon \mathcal{R}$, then *$f$ is Wald in the direction a* if and only if

$$\inf_{\mu \varepsilon V_a} \{\int f(\omega)\mu(d\omega) - \inf_{g \varepsilon \mathcal{R}} \int g(\omega)\mu(d\omega)\} = 0. \tag{3.1}$$

DEFINITION 3.4. Let Ω, $C(\Omega, \mathbb{R})$, $\mathcal{B}$ and $\mathcal{R}$ be as above. If $f \varepsilon \mathcal{R}$ and if $a \geqq 0$ is a $\mathcal{B}$-measurable function then *f is low in the direction a* if and only if for all $\epsilon > 0$, $f - \epsilon a \not\varepsilon \mathcal{R}$.

THEOREM 3.5. (Stein-LeCam). *Suppose Ω is a locally compact Hausdorff space, $C(\Omega, \mathbb{R})$ has the topology of uniform convergence of compact sets and $\mathcal{R} \subset C(\Omega, \mathbb{R})$ is a weakly subcompact convex subset of $C(\Omega, \mathbb{R})$. Then the following conditions are equivalent*:

(i) *$f \varepsilon \mathcal{R}$ and f is an admissible point of $\mathcal{R}$,*

(ii) $f \varepsilon \mathfrak{R}$ *and* f *is low in every direction* $a \geqq 0$ *such that* a *is bounded and continuous.*

(iii) $f \varepsilon \mathfrak{R}$ *and* f *is Wald in the direction* a *for every* $a \geqq 0$ *such that* a *is bounded and continuous.*

PROOF. The equivalence of (i) and (ii) is obvious. We show first that (ii) follows from (iii). Let $a \geqq 0$ be a $\mathfrak{B}$-measurable function. Let $f \varepsilon \mathfrak{R}$ be Wald in direction a, let $\epsilon \geqq 0$, and suppose $f - \epsilon a \varepsilon \mathfrak{R}$. By definition, if $\delta > 0$ we may choose $\mu \varepsilon V_a$ such that

$$\int f(\omega)\mu(d\omega) - \inf_{g \varepsilon \mathfrak{R}} \int g(\omega)\mu(d\omega) < \delta. \tag{3.2}$$

Therefore

$$\int f(\omega)\mu(d\omega) < \delta + \int (f(\omega) - \epsilon a(\omega))\mu(d\omega), \tag{3.3}$$

or,

$$0 \leqq \epsilon \int a(\omega)\mu(d\omega) = \epsilon < \delta. \tag{3.4}$$

The inequalities (3.4) hold for all $\delta > 0$. Therefore $\epsilon = 0$, as was to be shown.

We now show (iii) follows from (ii). Let $a \geqq 0$ be a $\mathfrak{B}$-measurable function, let $f \varepsilon \mathfrak{R}$, and let f be low in the direction a. From $\mathfrak{R}$ construct a closed convex set $\mathfrak{R}^* = \{h \mid h \varepsilon C(\Omega, \mathbb{R})$, for some $g \varepsilon \mathfrak{R}, h \geqq g\}$. The assumed weak subcompactness of $\mathfrak{R}$ easily implies $\mathfrak{R}^*$ to be closed in the topology of uniform convergence on compact sets. It is clear that since $f - \epsilon a \not\varepsilon \mathfrak{R}$ then $f - \epsilon a \not\varepsilon \mathfrak{R}^*$. Therefore the compact convex set $\{f - \epsilon a\}$ may be separated from the closed convex set $\mathfrak{R}^*$ by a continuous linear functional I such that $I(f - \epsilon a) < \inf_{g \varepsilon \mathfrak{R}^{**}} I(g)$. See Dunford and Schwartz [4]. If $h \geqq 0$, $h \varepsilon C(\Omega, \mathbb{R})$, and if $g \varepsilon \mathfrak{R}^*$, then $\alpha h + g \varepsilon \mathfrak{R}^*$, $\alpha \geqq 0$. Therefore $\alpha^{-1}(I(f - \epsilon a) - I(g)) < I(h)$, and letting $\alpha \to \infty$, we find $I(h) \geqq 0$. By Theorem 2.1 there exists a finite Borel measure μ having compact support which gives a representation of I. Then

$$\int (f(\omega) - \epsilon a(\omega))\mu(d\omega) = I(f - \epsilon a) < I(f) = \int f(\omega)\mu(d\omega). \tag{3.5}$$

Therefore $\int a(\omega)\mu(d\omega) > 0$ and without loss of generality we may assume μ has been normalized so that $\mu \varepsilon V_a$. Then

$$\int f(\omega)\mu(d\omega) - \inf_{g \varepsilon \mathfrak{R}^*} \int g(\omega)\mu(d\omega) < \epsilon. \tag{3.6}$$

Since

$$\inf_{g \varepsilon \mathfrak{R}} \int g(\omega)\mu(d\omega) = \inf_{g \varepsilon \mathfrak{R}^*} \int g(\omega)\mu(d\omega), \tag{3.7}$$

we see that f is Wald in the direction a. The proof is complete.

The applications made in this paper require a stronger theorem which we now state. We shall require Ω to be a σ-compact locally compact Hausdorff space. If $\{E_n, n \geqq 1\}$ is a countable cover of Ω by compact subsets then the topology of uniform convergence on compact sets is equivalent to the topology of uniform convergence on the sets E_n, $n \geqq 1$. This latter topology is a metric topology so that in the discussion of convergence only countable sequences need be considered. We make a definition.

DEFINITION 3.6. $\mathcal{R}$ is sequentially weakly subcompact if given a countable subset $\{f_n, n \geqq 1\}$ of $\mathcal{R}$ there is an $f \in \mathcal{R}$ and a subsequence $\{f_{a_n}, n \geqq 1\}$ such that $f \leqq \liminf_{n\to\infty} f_{a_n}$.

THEOREM 3.7. *Assume Ω is a σ-compact locally compact metric space. Let $\mathcal{R}$ be weakly subcompact (Definition 3.1) and sequentially weakly subcompact (Definition 3.6) convex set of nonnegative lower semicontinuous real valued functions on Ω. Assume that if $f \in \mathcal{R}$, $E \subset \Omega$, and E is compact then there exists $g \in \mathcal{R}$ such that g is continuous and $g(\omega) \leqq f(\omega)$, $\omega \in E$. Let $a \geqq 0$, $a \neq 0$, be a bounded Baire measurable function on Ω. If f is an admissible point of $\mathcal{R}$ then there exists a sequence of functions $\{g_n, n \geqq 1\}$ in $\mathcal{R}$, an increasing sequence of compact sets $\{F_n, n \geqq 1\}$ and a sequence of finite Baire measures $\{\eta_n, n \geqq 1\}$ on $\mathcal{B}$ such that $\Omega = \bigcup_{n=1}^{\infty} F_n$ and*

(i) *if $n \geqq 1$ then η_n is supported on F_n and $\int a(\omega)\eta_n(d\omega) = 1$;*

(ii) *if $n \geqq 1$ then g_n is Bayes relative to η_n (see the last paragraph of the introductory section) and*

$$\lim_{n\to\infty} \int (f(\omega) - g_n(\omega))\eta_n(d\omega) = 0;$$

(iii) *if $\omega \in \Omega$ then $\lim_{n\to\infty} g_n(\omega) = f(\omega)$.*

PROOF. Taking the discrete topology on Ω and using Theorem 3.5, since f is an admissible point of $\mathcal{R}$, there exists $\mu_n \in V_a$, μ_n supported on a finite set E_n, such that

$$(3.8) \qquad \int f(\omega)\mu_n(d\omega) < 1/n + \inf_{h \in \mathcal{R}} \int h(\omega)\mu_n(d\omega).$$

Define an affine map T_n by, if g is a bounded function,

$$(3.9) \qquad (T_n g)(\omega) = g(\omega) + (\mu_n(\Omega))^{-1} \int g(\omega)\mu_n(d\omega) - f(\omega) - (\mu_n(\Omega))^{-1} \int f(\omega)\mu_n(d\omega), \qquad \omega \in \Omega.$$

We will apply T_n to the set $\mathcal{R}$ to obtain the convex set $T_n\mathcal{R}$.

Let $E \subset \Omega$. Let $(T_n\mathcal{R} \mid E)$ be the functions of $T_n\mathcal{R}$ restricted to E. We show $(T_n\mathcal{R} \mid E)$ has a minimax point. The zero function is a point of $(T_n\mathcal{R} \mid E)$ so that $-\epsilon_n = \inf_{h\in\mathcal{R}} \sup_{\omega\in E} (T_n h)(\omega) \leqq 0$. We show $\epsilon_n < \infty$. For if $\epsilon_n = \infty$ then to each integer $m \geqq 1$ we may find $g_m' \in \mathcal{R}$ such that

$$(3.10) \qquad \text{if } \omega \in E \text{ then } g_m'(\omega) + (\mu_n(\Omega))^{-1} \int g_m'(\omega)\mu_n(d\omega) \leqq f(\omega) + (\mu_n(\Omega))^{-1} \int f(\omega)\mu_n(d\omega) - m.$$

Using (3.8) and $g_m' \geqq 0$, this implies $\limsup_{m\to\infty} g_m'(\omega) = -\infty$, $\omega \in E$, which is a contradiction. Therefore $\epsilon_n < \infty$.

If $m \geqq 1$ let $g_m'' \in \mathcal{R}$ such that $-\epsilon_n \leqq \sup_{\omega\in E} (T_n g_m'')(\omega) < -\epsilon_n + 1/m$. Since $\mathcal{R}$ is sequentially weakly subcompact there exists $g_n \in \mathcal{R}$ and a subsequence $\{g_{a_m}'', m \geqq 1\}$ such that $g_n \leqq \liminf_{m\to\infty} g_{a_m}''$. By Fatou's lemma, $\int g_n(\omega)\mu_n(d\omega) \leqq \liminf_{m\to\infty} \int g_{a_m}''(\omega)\mu_n(d\omega)$. Therefore $\sup_{\omega\in E} (T_n g_n)(\omega) \leqq -\epsilon_n$ and $T_n g_n$ is minimax.

We shall let

$$(3.11) \qquad (T_n\mathcal{R} \mid E)_C = \{h \mid h \in C(\Omega, R),$$

and there exists $h' \varepsilon T_n\mathcal{R}$ such that if $\omega \varepsilon E$ then $h(\omega) \geqq h'(\omega)\}$. If e is the function $e(\omega) = 1$, $\omega \varepsilon \Omega$, then $(-\epsilon_n)e \varepsilon (T_n\mathcal{R} \mid E)_C$ but if $\alpha < -\epsilon_n$ then $\alpha e \not\varepsilon (T_n\mathcal{R} \mid E)_C$. Therefore $(-\epsilon_n)e$ is minimax in $(T_n\mathcal{R} \mid E)_C$.

If $E = F_n$ is compact then by the minimax theorem of Wald [11] there exists a hyperplane of support to $(T_n\mathcal{R} \mid E)_C$ at $(-\epsilon_n)e$ given by a probability measure ν_n supported on F_n, so that if $\sup_{\omega \varepsilon F_n} h(\omega) \leqq -\epsilon_n$ then $\int h(\omega)\nu_n(d\omega) \leqq -\epsilon_n$ and if $h \varepsilon (T_n\mathcal{R} \mid F_n)_C$ then $\int h(\omega)\nu_n(d\omega) \geqq -\epsilon_n$.

Since μ_n is supported on E_n, let $\{F_n, n \geqq 1\}$ be an increasing sequence of compact subsets of Ω, $F_n \uparrow \Omega$, such that if $n \geqq 1$ then $E_n \subset F_n$. Let T_ng_n be minimax in $T_n\mathcal{R} \mid F_n$ and let g_n' be a continuous function in $\mathcal{R}$ such that if $\omega \varepsilon F_n$ then $g_n'(\omega) \leqq g_n(\omega)$. Then if $\omega \varepsilon F_n$, $g_n'(\omega) + (\mu_n(\Omega))^{-1} \int g_n'(\omega)\mu_n(d\omega) \leqq g_n(\omega) + (\mu_n(\Omega))^{-1} \int g_n(\omega)\mu_n(d\omega)$ from which it follows that $(T_ng_n')(\omega) \leqq (T_ng_n)(\omega)$, $\omega \varepsilon F_n$. Let $\{f_n, n \geqq 1\}$ be a sequence of nonnegative continuous functions on Ω such that $f_n \uparrow f$. The functions $g_n'(\cdot) + (\mu_n(\Omega))^{-1} \int g_n'(\omega)\mu_n(d\omega) - f_m(\cdot) - (\mu_n(\Omega))^{-1} \int f_m(\omega)\mu_n(d\omega)$ are in $(T_n\mathcal{R} \mid F_n)_C$. Therefore by construction of ν_n, if $n \geqq 1$, $m \geqq 1$,

$$-\epsilon_n \leqq (\mu_n(\Omega))^{-1} \int (g_n'(\omega) - f_m(\omega))(\mu_n(\Omega)\nu_n(d\omega) + \mu_n(d\omega)). \tag{3.12}$$

Using the monotone convergence theorem and (3.12) gives, if $n \geqq 1$,

$$-\epsilon_n \leqq (\mu_n(\Omega))^{-1} \int (g_n'(\omega) - f(\omega))(\mu_n(\Omega)\nu_n(d\omega) + \mu_n(d\omega)), \tag{3.13}$$

and since g_n' is minimax (3.13) implies

$$\begin{aligned} -\epsilon_n &= (\mu_n(\Omega))^{-1} \int (g_n'(\omega) - f(\omega))(\mu_n(\Omega)\nu_n(d\omega) + \mu_n(d\omega)) \\ &= \int (T_ng_n')(\omega)\nu_n(d\omega). \end{aligned} \tag{3.14}$$

And since on the support of ν_n we have $T_ng_n' \leqq T_ng_n \leqq -\epsilon_n e$, (3.14) holds also with g_n' replaced by g_n.

From (3.14) we may now obtain

$$\begin{aligned} 0 \geqq -\epsilon_n &= \int (T_ng_n)(\omega)\nu_n(d\omega) \\ &\geqq (\mu_n(\Omega))^{-1} \int (T_ng_n)(\omega)\mu_n(d\omega) \\ &= 2(\mu_n(\Omega))^{-1} \int (g_n(\omega) - f(\omega))\mu_n(d\omega) \\ &\geqq 2(\mu_n(\Omega))^{-1} \inf_{h\varepsilon\mathcal{R}} \int (h(\omega) - f(\omega))\mu_n(d\omega) \\ &\geqq -2n^{-1}(\mu_n(\Omega))^{-1}. \end{aligned} \tag{3.15}$$

Since a is a bounded function and $\mu_n \varepsilon V_a$, $n \geqq 1$, we obtain

$$\liminf_{n\to\infty} \mu_n(\Omega) > 0. \tag{3.16}$$

Therefore

$$\lim_{n\to\infty} (n\mu_n(\Omega))^{-1} = 0. \tag{3.17}$$

From (3.15) it follows that

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \mu_n(\Omega)\epsilon_n = 0. \tag{3.18}$$

Define

$$\text{(3.19)} \qquad \text{if } n \geqq 1,\ c_n^{-1} = \mu_n(\Omega) \int a(\omega)\nu_n(d\omega) + \int a(\omega)\mu_n(d\omega).$$

Then $c_n^{-1} \geqq 1$ and we let

$$\text{(3.20)} \qquad \text{if } n \geqq 1,\ \eta_n(\cdot) = c_n(\mu_n(\Omega)\nu_n(\cdot) + \mu_n(\cdot)).$$

Then $\eta_n \varepsilon V_a$ and η_n is supported on F_n, $n \geqq 1$. From (3.14) and (3.15) we find

$$\text{(3.21)} \qquad \lim_{n\to\infty} \int (g_n(\omega) - f(\omega))\eta_n(d\omega) = 0.$$

By construction g_n is Bayes relative to η_n. To complete the proof, observe from (3.15) that it follows that if $\omega \varepsilon \Omega$.

$$\begin{aligned} &\limsup_{n\to\infty} (g_n(\omega) - f(\omega)) \\ \text{(3.22)} \qquad &= \limsup_{n\to\infty} (g_n(\omega) - f(\omega) + (\mu_n(\Omega))^{-1} \int (g_n(\omega) - f(\omega))\mu_n(d\omega)) \\ &\leqq 0. \end{aligned}$$

We use here the assumption $T_n g_n$ is minimax for $T_n \mathcal{R} \mid F_n$ and that $F_n \uparrow \Omega$. Since f is an admissible point of $\mathcal{R}$, (3.22) and the sequential weak subcompactness of $\mathcal{R}$ imply

$$\text{(3.23)} \qquad \liminf_{n\to\infty} g_n(\omega) \geqq f(\omega).$$

The proof has been completed.

4. The complete class theorem of Birnbaum. In case the sample space X is Euclidean k-space $\mathbb{R}_k$, $\Omega = X = \mathbb{R}_k$, μ is a σ-finite measure on the σ-algebra of Borel subsets of $\mathbb{R}_k$, and $\{f_\omega, \omega \varepsilon \Omega\}$ is an exponential family of densities in $L_1(X, \mathcal{B}, \mu)$, Birnbaum [2] gave a complete class theorem. If $H_0 = \{0\}$, $H_1 = \Omega - \{0\}$, and if μ is absolutely continuous with respect to a nonatomic product measure, then a test φ is admissible if and only if there exists a convex set C with indicator function $\chi(C,\cdot)$ such that $\mu(\{x \mid \varphi(x) \neq \chi(C, x)\}) = 0$. Parts of Birnbaum's result have been extended by Stein [10] and Stein's result will enter our discussion later.

It is the main purpose of Section 4 to prove that in Birnbaum's problem every admissible test is a generalized Bayes test and conversely. We state this formally.

THEOREM 4.1. *Let $\Omega = X = \mathbb{R}_k$; let $\mathcal{B}$ be the σ-algebra of Borel subsets of X; and let μ be a σ-finite measure on $\mathcal{B}$ which is absolutely continuous relative to a nonatomic product measure. Let $H_0 = \{0\}$ and $H_1 = \Omega - \{0\}$. Let $\{f_\omega, w \varepsilon \Omega\}$ be an exponential family of density functions in $L_1(X, \mathcal{B}, \mu)$. A test φ is admissible if and only if φ is a generalized Bayes test.*

In order to prove Theorem 4.1 several lemmas are needed. We proceed at once to the statements and proofs of the lemmas.

LEMMA 4.2. *Let $\mu_k = \mu \times \cdots \times \mu$ be a nonatomic product measure on $\mathcal{B}$. Let A be a convex subset of $\mathbb{R}_k$ and suppose $\mu_k(A) > 0$. Suppose f is an analytic function of k real variables defined on A and $\mu_k(\{x \mid f(x) = 0\}) > 0$. Then $f(x) = 0$ at every interior point of A.*

PROOF. To each $r \varepsilon \mathbb{R}$ let $g_r : \mathbb{R}_{k-1} \to \mathbb{R}$ be defined by $g_r(y) = f((y, r))$. Let $A_r = \{y \mid y \varepsilon \mathbb{R}_{k-1},\ (y, r) \varepsilon A\}$. The sets A_r are then convex sets. Let $B = \{r \mid r \varepsilon \mathbb{R};\ \mu_{k-1}(\{y \mid y \varepsilon A_r,\ g_r(y) = 0\}) > 0\}$. By Fubini's theorem, $\mu(B) > 0$.

We make an inductive argument on k. If $k = 1$, $\mu(A) > 0$ together with μ being nonatomic implies there is an accumulation point interior to A in the neighborhood of which f vanishes infinitely often. Hence $f(x) = 0$ at all x interior to A.

By induction, if $r \varepsilon B$ then $g_r(y') = 0$ for all y' interior to A_r. Since A has interior points we may choose real numbers $a < b$ and $\epsilon > 0$, and an interior point $y_0 \varepsilon A$, such that $\mu(B \cap [a, b]) > 0$, and a sphere $S_\epsilon(y_0)$ of radius ϵ about y_0 so that $S_\epsilon(y_0) \times [a, b] \subset A$. Write $f((y, r)) = \sum_{n=0}^{\infty} f_n(y) r^n$. Then if $r \varepsilon B \cap [a, b]$ and $y \varepsilon S_\epsilon(y_0)$ we have $0 = \sum_{n=0}^{\infty} f_n(y) r^n$. This implies $f_n(y) = 0$, $n \geqq 0$, $y \varepsilon S_\epsilon(y_0)$. Therefore $f_n(y) = 0$, $n \geqq 0$, $y \varepsilon A_r$, $r \varepsilon B \cap [a, b]$. This clearly implies $f(x) = 0$ for all x interior to A. For f is an analytic function.

LEMMA 4.3. *Let $\mu_k = \mu \times \cdots \times \mu$ be a nonatomic product measure on $\mathcal{B}$. Let $A \subset \mathbb{R}_k$, A a closed convex set with topological boundary B. Then $\mu_k(B) = 0$.*

PROOF. By induction on the dimension k. If $k = 1$ then A is a (finite or infinite) line segment, and B consists of at most two points. Therefore $\mu(B) = 0$.

To each $r \varepsilon \mathbb{R}$ let $A_r = \{y \mid y \varepsilon \mathbb{R}_{k-1},\ (y, r) \varepsilon A\}$ and $B_r = \{y \mid y \varepsilon \mathbb{R}_{k-1},\ (y, r) \varepsilon B\}$. Then A_r is a closed convex set. Either B_r is a convex set parallel to one of the coordinate axes so that $\mu_k(B_r) = 0$ or B_r is the topological boundary of A_r in $\mathbb{R}_{k-1}$. Therefore by the inductive hypothesis $\mu_{k-1}(B_r) = 0$. By Fubini's theorem, $\mu_k(B) = \int u_{k-1}(B_r)\mu(dr) = 0$.

LEMMA 4.4. *Let μ be a σ-finite measure on $\mathcal{B}$ which is absolutely continuous with respect to a nonatomic product measure. Suppose Ω is a convex set with nonvoid interior. Let $\eta_0 \neq \eta_1$ be σ-finite measures on the Borel subsets of Ω. Then $0 = \mu(\{x \mid \int \exp(\omega \cdot x)\eta_0(d\omega) < \infty$ and $\int \exp(\omega \cdot x)\eta_0(d\omega) = \int \exp(\omega \cdot x)\eta_1(d\omega)\})$.*

PROOF. We write $\omega \cdot x$ for the dot product of vectors ω and x. Let $A = \{x \mid \int \exp(\omega \cdot x)\eta_0(d\omega) < \infty$ and $\int \exp(\omega \cdot x)\eta_1(d\omega) < \infty\}$. Then A is convex, if $\mu(A) > 0$ then A has nonvoid interior. Further, if $\mu(A \cap \{x \mid \int \exp(\omega \cdot x)\eta_0(d\omega) = \int \exp(\omega \cdot x)\eta_1(d\omega)\}) > 0$ then by Lemma 4.2, $\int \exp(\omega \cdot x)\eta_0(d\omega) = \int \exp(\omega \cdot x)\eta_1(d\omega)$ if x is interior to A. Therefore η_0 and η_1 have Laplace transforms equal on a set with nonvoid interior and $\eta_0 = \eta_1$ follows. $\eta_0 \neq \eta_1$ by hypothesis. Therefore the conclusion of the lemma follows.

We now prove Theorem 4.1. If φ is generalized Bayes relative to measures η_0 supported on $\{0\}$ and η_1 supported on $\Omega - \{0\}$, we may suppose $\eta_0(\{0\}) = 0$ or $\eta_0(\{0\}) = 1$. In the latter case we find that, writing $f\omega(x) = k(\omega) \exp(\omega \cdot x)$,

$$\text{(4.1)} \qquad \begin{aligned} &\text{if } k(0) < \int k(\omega) \exp(\omega \cdot x)\eta_1(d\omega) \quad \text{then} \quad \varphi(x) = 1; \\ &\text{if } k(0) > \int k(\omega) \exp(\omega \cdot x)\eta_1(d\omega) \quad \text{then} \quad \varphi(x) = 0. \end{aligned}$$

Therefore $\{x \mid \varphi(x) = 0\} = A$ is a convex set. The test φ will randomize only on the boundary of A, hence randomization takes place with μ-measure zero (see Lemma 4.3). From Birnbaum [2] the test is admissible. If $\eta_0(\{0\}) = 0$ then $\varphi(x) \neq 1$ with μ-measure zero and we take $A = \varnothing$ having μ measure zero. Again this test is admissible. Therefore generalized Bayes tests are admissible. (This result will also follow from Theorem 5.1).

Conversely, an admissible test φ has the form $\mu(\{x \mid \varphi(x) \neq \chi(A, x)\}) = 0$ where A is a suitable convex set and $\chi(A, \cdot)$ is the indicator function of A. We will now show the test function $\chi(A, \cdot)$ is generalized Bayes. The conclusion is clear if A has void interior. If A has nonvoid interior we argue as follows.

A closed convex set A is the intersection of all closed half-spaces containing A. The general closed half-space $\{y \mid \xi \cdot y \leqq c\}$ supporting A may be represented by a triple (ξ, x, c), ξ a unit vector of $\mathbb{R}_k$, $x \in \mathbb{R}_k$ a boundary point of A, $c \in \mathbb{R}$ and $\xi \cdot x = c$. We choose a countable dense subset $\{(\xi_n, x_n, c_n), n \geqq 1\}$ of these points and let $\pi_n = \{y \mid \xi_n \cdot y \leqq c_n, \xi_n \cdot x_n = c_n\}$, $n \geqq 1$. Then $A = \bigcap_n \pi_n$.

Let λ be Lebesque measure on R, and let $g: \mathbb{R} \to [0, \infty)$ be a measurable function satisfying $\int g(\alpha)\lambda(d\alpha) = 1$ and $\int g(\alpha) \exp(\alpha\beta)\lambda(d\alpha) = \infty$ if $\beta > 0$. Let η_n be defined on $\mathcal{B}$ by, if $E \in \mathcal{B}$ then $\eta_n(E) = \int_{[\alpha \mid \alpha\xi_n \in E]} (k(\alpha\xi_n))^{-1} \exp(-\alpha\xi_n \cdot x_n)\lambda(d\alpha)$, $n \geqq 1$. Then if $n \geqq 1$ and $x \in A$, $\xi_n \cdot (x - x_n) \leqq 0$ and

$$\int k(\xi) \exp(\xi \cdot x)\eta_n(d\xi) = \int g(\alpha) \exp(\alpha\xi_n \cdot (x - x_n))\lambda(d\alpha) \leqq 1;$$

if $\xi_n \cdot (x - x_n) > 0$ then $\int k(\xi_n) \exp(\xi_n \cdot x)\eta_n(d\xi) = \infty$. Let η be defined by $\eta(E) = \sum_{n=1}^{\infty} 2^{-n}\eta_n(E)$, $E \in \mathcal{B}$. Then, if $x \in A$, $\int k(\xi) \exp(\xi \cdot x)\eta(d\xi) \leqq 1$; and if $x \notin A$, then since A is a closed set, $\int k(\xi) \exp(\xi \cdot x)\eta(d\xi) = \infty$. The given test $\chi(A, \cdot)$ is thus generalized Bayes for the pair η_0, η, where $\eta_0(\{0\}) = 1$ and $\eta_0(\Omega - \{0\}) = 0$.

In order to obtain the complete class statement above it has been necessary to allow integrals which are divergent. We now show that use of divergent integrals is necessary.

THEOREM 4.5. *Let A be a convex subset of $\mathbb{R}_k$ and suppose the boundary of A contains a line segment $\{x \mid x = \alpha\xi + \tau, 0 \leqq \alpha \leqq 1\}$. Let η be a σ-finite measure on $\mathcal{B}$ such that $A = \{x \mid \int k(\omega) \exp(\omega \cdot x)\eta(dx) \leqq 1\}$. Suppose*

$$\int \exp(\omega \cdot x)\eta(dx) = 1$$

if $x = \alpha\xi + \tau$, $0 \leqq \alpha \leqq 1$. Then the support of η lies in the orthogonal complement of ξ.

PROOF. $\int \exp(\omega \cdot (\alpha\xi + \tau))\eta(d\omega)$ is analytic in α, and by hypothesis is a constant function of α, $0 \leqq \alpha \leqq 1$. Thus if the integral converges for $0 \leqq \alpha \leqq 1$ we take $\alpha = 0$ and find $\int \exp(\omega \cdot \xi)\eta(d\omega) = 1$. By Jensen's inequality, if $\frac{1}{2} \leqq \alpha \leqq 1$,

$$(4.2) \qquad 1 = \int (\exp(\omega \cdot (\xi/2)))^{2\alpha} \exp(\omega \cdot \tau)\eta(d\omega)$$
$$\geqq \left(\int \exp(\omega \cdot ((\xi/2) + \tau))\eta(d\omega)\right)^{2\alpha} = 1.$$

If $\alpha > \frac{1}{2}$ the inequality will be strict unless $\exp(\omega \cdot \tau)\eta(d\omega)$ is concentrated on a line $\omega \cdot (\xi/2) = c$. Substitution into (4.2) gives $1 = (\exp c)^{2\alpha}$ or $c = 0$. Therefore the support of η is perpendicular to ξ.

By way of an example let $k = 2$ and suppose the boundary of A contains two nonparallel flats with normals ξ_1 and ξ_2. If $A = \{x \mid \int \exp(\omega \cdot x)\eta(d\omega) \leqq 1\}$ and if $\int \exp(\omega\cdot x)\eta(d\omega) < \infty$ for *all* $x \varepsilon \mathbb{R}_2$, then on the boundary of A we have $1 = \int \exp(\omega\cdot x)\eta(d\omega)$. By Theorem 4.5 the support of η must be contained on the 1-spaces generated by ξ_1 and ξ_2. Hence η is completely concentrated at $(0, 0)$, which is impossible. Therefore if $k = 2$ the generalized Bayes test with acceptance region $\{x \mid |x_1| \leqq 1, |x_2| \leqq 1, x = (x_1, x_2)\}$ can be obtained only by measures with an integral somewhere divergent.

5. A sufficient condition for admissibility of tests. It is the purpose of this section to show that a very large class of generalized Bayes tests are admissible. The main theorem is as follows.

THEOREM 5.1. *Let* $(X, \mathcal{B}, \mu)$ *be a* σ*-finite measure space* $(X \varepsilon \mathcal{B})$, *and let* $\{f_\omega, \omega \varepsilon \Omega\}$ *be a family of density functions in* $L_1(X, \mathcal{B}, \mu)$. *Let* $H_0 \subset \Omega, H_0 \neq \emptyset$, $H_1 = \Omega - H_0, H_1 \neq \emptyset$. *Let* $\mathcal{C}$ *be a* σ*-algebra of subsets of* Ω *such that* $H_0 \varepsilon \mathcal{C}$. *In addition we make the following hypotheses.*

(1) $f_{(\cdot)}(\cdot)$ *is jointly measurable as a function on* $\Omega \times X$, $\mathcal{C} \times \mathcal{B}$ *to* $\mathbb{R}$.

(2) η_0 *is a probability measure on* $\mathcal{C}$ *such that* $\eta_0(H_1) = 0$.

(3) η_1 *is a* σ*-finite measure on* $\mathcal{C}$ *such that* $\eta_1(H_0) = 0$.

(4) $\mu(\{x \mid \int f_\omega(x)\eta_0(dx) = \int f_\omega(x)\eta_1(dx)\}) = 0$.

(5) χ *is a generalized Bayes test for the pair* η_0, η_1.

Then χ *is an admissible test. In addition, if* β *is the power function of* χ, *then*

$$\int (1 - \beta(\omega))\eta_1(d\omega) \leqq \int (1 - \beta(\omega))\eta_0(d\omega), \quad \textit{and,} \tag{5.1}$$

$$\begin{aligned} &\int \beta(\omega)\eta_0(d\omega) + \int (1 - \beta(\omega))\eta_1(d\omega) \\ &\quad = \inf_{\chi'} \{\textstyle\int\int \chi'(x)f_\omega(x)\mu(dx)\eta_0(d\omega) \\ &\qquad + \textstyle\int\int (1 - \chi'(x))f_\omega(x)\mu(dx)\eta_1(d_\omega)\}. \end{aligned} \tag{5.2}$$

PROOF. Let $\{C_n, n \geqq 1\}$ be a nondecreasing sequence of sets from $\mathcal{C}$ such that $\Omega = \bigcup_{n=1}^\infty C_n$ and if $n \geqq 1$, $\eta_1(C_n) < \infty$. In $n \geqq 1$, define $\eta_{1_n}(E) = \eta_1(E \cap C_n)$, $E \varepsilon \mathcal{C}$.

Relative to the pair η_0, η_{1n}, let χ_n be Bayes, chosen so that if $\int f_\omega(x)\eta_0(d\omega) > \int f_\omega(x)\eta_{1n}(d\omega)$, then $\chi_n(x) = 0$; if $\int f_\omega(x)\eta_0(d\omega) \leqq \int f_\omega(x)\eta_{1n}(d\omega)$ then $\chi_n(x) = 1$; $n \geqq 1$. Since the integrals on the right increase with n, it follows that if $x \varepsilon X$, $n \geqq 1$, then $\chi_n(x) \leqq \chi_{n+1}(x)$. By the monotone convergence theorem, $\lim_{n\to\infty} \int f_\omega(x)\eta_{1n}(d\omega) = \int f_\omega(x)\eta_1(d\omega)$ (which may $= \infty$), so that if $x \not\varepsilon \{x \mid \int f_\omega(x)\eta_0(d\omega) = \int f_\omega(x)\eta_1(d\omega)\}$ then $\lim_{n\to\infty} \chi_n(x) = \chi(x)$. By hypothesis (4), $\lim_{n\to\infty} \chi_n(x) = \chi(x)$ a.e. $[\mu]$.

Let β_n be the power function of χ_n, $n \geqq 1$. Since η_0 and η_{1n} are finite measures and χ_n is Bayes relative to $\eta_0, \eta_{1n}, n \geqq 1$, we find

$$\begin{aligned} &\int\beta_n(\omega)\eta_0(d\omega) + \int (1 - \beta_n(\omega))\eta_{1n}(d\omega) \\ &\qquad \leqq \int \beta(\omega)\eta_0(d\omega) + \int (1 - \beta(\omega))\eta_{1n}(d\omega) < \infty. \end{aligned} \tag{5.3}$$

Since all the integrals involved are absolutely convergent, we find that

$$\begin{aligned} &\text{if } n \geqq 1 \text{ then } 0 \leqq \textstyle\int\int (\chi(x) - \chi_n(x))f_\omega(x)\eta_{1n}(d\omega)\mu(dx) \\ &\qquad \leqq \textstyle\int\int (\chi(x) - \chi_n(x))f_\omega(x)\eta_0(d\omega)\mu(dx). \end{aligned} \tag{5.4}$$

By the monotone convergence theorem, the second integral in (5.4) converges to zero as $n \to \infty$, while, since $\chi(x) - \chi_n(x) \geqq 0$, $x \varepsilon X$, $n \geqq 1$, it then follows from (5.4) that

$$\lim_{n\to\infty} \int (\beta(\omega) - \beta_n(\omega))\eta_0(d\omega) = 0; \tag{5.5}$$
$$\lim_{n\to\infty} \int (\beta(\omega) - \beta_n(\omega))\eta_{1n}(d\omega) = 0.$$

We now show (5.1) must hold. Consider the test, if $x \varepsilon X$ then $\chi'(x) = 1$. It has a power function $\beta'(\omega) = 1$, $\omega \varepsilon \Omega$. We find therefore, using the fact that if $n \geqq 1$ then χ_n is Bayes relative to η_0, η_{1n}, that

$$\int (\beta_n(\omega) - \beta(\omega))\eta_0(d\omega) + \int (\beta(\omega) - \beta_n(\omega))\eta_{1n}(d\omega) \tag{5.6}$$
$$\leqq \int (\beta'(\omega) - \beta(\omega))\eta_0(d\omega) + \int (\beta(\omega) - \beta'(\omega))\eta_{1n}(d\omega).$$

As $n \to \infty$, the left side of (5.6) tends to zero. Substituting for β' its values, and passing to the limit, we find

$$\limsup_{n\to\infty} \int (1 - \beta(\omega))\eta_{1n}(d\omega) = \int (1 - \beta(\omega))\eta_1(d\omega) \tag{5.7}$$
$$\leqq \int (1 - \beta(\omega))\eta_0(d\omega).$$

From this it follows that the left side of (5.2) is finite. We may now prove that (5.2) holds. Let $\epsilon \geqq 0$ and suppose χ^* is a test with power function β^*. Then if

$$\int \beta^*(\omega)\eta_0(d\omega) + \int (1 - \beta^*(\omega))\eta_1(d\omega) + \epsilon$$
$$\leqq \int \beta(\omega)\eta_0(d\omega) + \int (1 - \beta(\omega))\eta_1(d\omega),$$

we find from (5.3) that

$$\epsilon + \int \beta_n(\omega)\eta_0(d\omega) + \int (1 - \beta_n(\omega))\eta_{1n}(d\omega)$$
$$\leqq \epsilon + \int \beta^*(\omega)\eta_0(d\omega) + \int (1 - \beta^*(\omega))\eta_{1n}(d\omega) \tag{5.8}$$
$$\leqq \int \beta(\omega)\eta_0(d\omega) + \int (1 - \beta(\omega))\eta_{1n}(d\omega).$$

From (5.5) we find that $\epsilon \leqq 0$. Therefore $\epsilon = 0$ and (5.2) is proven.

Hypotheses (4) and (5) imply that χ is the essentially unique test function giving the minimum established in (5.2). Therefore χ is admissible.

THEOREM 5.2. *Let X, $\mathcal{B}$, μ, Ω, $\mathcal{C}$, $\{f_\omega, \omega \varepsilon \Omega\}$, H_0 and H_1 be as for Theorem* 5.1. *In addition assume*

(1) *η_0 is a σ-finite measure on $\mathcal{C}$ with $\eta_0(H_1) = 0$.*

(2) *η_1 is a σ-finite measure of $\mathcal{C}$ with $\eta_1(H_0) = 0$.*

(3) *χ is a test function having power function β which satisfies $\int (1 - \beta(\omega))\eta_1(d\omega) < \infty$.*

(4) *If η_0' and η_1' are σ-finite measures on $\mathcal{C}$ satisfying $\eta_0'(H_1) = \eta_1'(H_0) = 0$ then*

$$\mu(\{x \mid \int f_\omega(x)\eta_0'(d\omega) < \infty, \int f_\omega(x)\eta_0'(d\omega) = \int f_\omega(x)\eta_1'(d\omega)\}) = 0.$$

(5) $\mu(\{x \mid \int f_\omega(x)\eta_0(d\omega) = \infty$ *and* $\int f_\omega(x)\eta_1(d\omega) = \infty\}) = 0$.

(6) *χ is a generalized Bayes test for η_0, η_1.*

Then, χ is an admissible test.

PROOF. Let $\{D_n, n \geqq 1\}$ be an increasing sequence of subsets in $\mathfrak{C}$ such that $H_0 = \bigcup_{n=1}^{\infty} D_n$, and such that if $n \geqq 1$ then $\eta_0(D_n) < \infty$. If $n \geqq 1$ define η_{0n} by $\eta_{0n}(E) = \eta_0(E \cap D_n)$. Let χ_n be a generalized Bayes test for η_{0n}, η_1. Hypotheses (3), (4) and (5) imply that χ_n is well defined, is essentially unique, and that $\chi_n \downarrow \chi$ a.e. $[\mu]$ as $n \to \infty$. Therefore $(1 - \chi_n) \uparrow (1 - \chi)$ and by the monotone convergence theorem, $\lim_{n\to\infty} \int\int (1 - \chi_n(x))f_\omega(x)\eta_1(d\omega)\mu(dx) = \int\int (1 - \chi(x))f_\omega(x)\eta_1(d\omega)\mu(dx) < \infty$.

The hypotheses of Theorem 5.1 are satisfied. Thus, if χ_n has power function β_n, $n \geqq 1$, if χ^* is as good as χ, and if χ^* has power function β^*, then using Theorem 5.1 we find

$$(5.9)\quad \begin{aligned} &\int [\chi_n(x) \int f_\omega(x)\eta_{0n}(d\omega) + (1 - \chi_n(x)) \int f_\omega(x)\eta_1(d\omega)]\mu(dx) \\ &\leqq \int [\chi(x) \int f_\omega(x)\eta_{0n}(d\omega) + (1 - \chi(x)\ (1-x(x)) \int f_\omega(x)\eta_1(d\omega)]\mu(dx) \end{aligned}$$

so that

$$(5.10)\quad \begin{aligned} 0 &\leqq \int\int (\chi_n(x) - \chi(x))f_\omega(x)\eta_{0n}(d\omega)\mu(dx) \\ &\leqq \int\int (\chi_n(x) - \chi(x))f_\omega(dx)\eta_1(d\omega)\mu(dx). \end{aligned}$$

As the integral on the right side of (5.10) tends to zero, we find that

$$(5.11)\quad \begin{aligned} &\lim_{n\to\infty} \int (\beta_n(\omega) - \beta(\omega))\eta_{0n}(d\omega) = 0 \quad \text{and} \\ &\lim_{n\to\infty} \int (\beta_n(\omega) - \beta(\omega))\eta_1(d\omega) = 0. \end{aligned}$$

Further, using Theorem 5.1 and the assumption that χ^* is as good as χ, we obtain

$$(5.12)\quad \begin{aligned} &\int \beta_n(\omega)\eta_{0n}(d\omega) + \int (1 - \beta_n(\omega))\eta_1(d\omega) \\ &\leqq \int \beta^*(\omega)\eta_{0n}(d\omega) + \int (1 - \beta^*(\omega))\eta_1(d\omega) \\ &\leqq \int \beta(\omega)\eta_{0n}(d\omega) + \int (1 - \beta(\omega))\eta_1(d\omega) < \infty. \end{aligned}$$

Therefore from (5.11) and (5.12) we conclude that

$$(5.13)\quad 0 = \lim_{n\to\infty} (\int (\beta(\omega) - \beta^*(\omega))\eta_{0n}(d\omega) + \int (\beta^*(\omega) - \beta(\omega))\eta_1(d\omega)).$$

We write $(1 - \chi) - (1 - \chi^*) = \chi^* - \chi = \chi_+ - \chi_-$, where $\chi_+ \geqq 0$, $\chi_- \geqq 0$, and $\chi_+(x)\chi_-(x) = 0$ for all x. Then $(1 - \chi) \geqq \chi_+$ and $(1 - \chi^*) \geqq \chi_-$. From (5.12) conclude

$$(5.14)\quad \int\int \chi_+(x)f_\omega(x)\eta_1(d\omega)\mu(dx) < \infty; \quad \int\int \chi_-(x)f_\omega(x)\eta_1(d\omega)\mu(dx) < \infty.$$

Therefore

$$(5.15)\quad \begin{aligned} &\int (\beta(\omega) - \beta^*(\omega))\eta_{0n}(d\omega) + \int (\beta^*(\omega) - \beta(\omega))\eta_1(d\omega) \\ &= \int \chi_-(x)(\int f_\omega(x)\eta_{0n}(d\omega) - \int f_\omega(x)\eta_1(d\omega))\mu(dx) \\ &\quad - \int \chi_+(x)(\int f_\omega(x)\eta_{0n}(d\omega) - \int f_\omega(x)\eta_1(d\omega))\mu(dx). \end{aligned}$$

Since $\chi_-(x) > 0$ implies $\chi(x) = 1$, which implies

$$\int f_\omega(x)\eta_{0n}(d\omega) \leqq \int f_\omega(x)\eta_0(f\omega) \leqq \int f_\omega(x)\eta_1(d\omega),$$

the third integral of (5.15) is always nonpositive. As $n \to \infty$ the integrand increases monotonely. By the monotone convergence theorem we obtain

$$(5.16) \quad \lim_{n\to\infty} \int \chi_-(x)(\int f_\omega(x)\eta_{0n}(d\omega) - \int f_\omega(x)\eta_1(d\omega))\mu(dx) = \int \chi_-(x)(\int f_\omega(x)\eta_0(d\omega) - \int f_\omega(x)\eta_1(d\omega))\mu(dx) \leqq 0.$$

Similarly, $\chi_+(x) > 0$ implies $\chi^*(x) \neq 0$ and $\chi(x) = 0$. This means that $\int f_\omega(x)\eta_0(d\omega) \geqq \int f_\omega(x)\eta_1(d\omega)$. Therefore by the monotone convergence theorem,

$$(5.17) \quad \lim_{n\to\infty} - \int \chi_+(x)(\int f_\omega(x)\eta_{0n}(d\omega) - \int f_\omega(x)\eta_1(d\omega))\mu(dx) = - \int \chi_+(x)(\int f_\omega(x)\eta_0(d\omega) - \int f_\omega(x)\eta_1(d\omega))\mu(dx) \leqq 0.$$

By (5.13), (5.15), (5.16), and (5.14) we concluded

$$(5.18) \quad \begin{aligned} 0 &= \int \chi_-(x)|\int f_\omega(x)\eta_0(d\omega) - \int f_\omega(x)\eta_1(d\omega)|\mu(dx); \\ 0 &= \int \chi_+(x)|\int f_\omega(x)\eta_0(d\omega) - \int f_\omega(x)\eta_1(d\omega)|\mu(dx). \end{aligned}$$

Hypothesis (4) now implies $\chi_- = 0$ a.e. $[\mu]$ and $\chi_+ = 0$ a.e. $[\mu]$. Therefore χ is admissible. □

6. Examples of L. D. Brown. The results of Sections 4 and 5 strongly suggest a false result. The author is indebted to L. D. Brown for two examples. The exposition of these examples is the contents of Section 6. Throughout we deal with $k = 2$ and exponential families $\{f_\omega(\cdot), \omega \varepsilon \Omega\}$ of density functions.

Example 6.1 is an example of a hypothesis set H_0 which is convex but which by virtue of its structure requires that *all* tests are admissible. Thus in particular the test function $\varphi(x) = \frac{1}{2}$ for all $x \varepsilon X$ is an admissible test which is not generalized Bayes. As we shall see in Section 7, Example 6.1 is related to the fact that H_0 and H_1 are not topologically separated and yet a discontinuous measure of loss is used.

Example 6.2 is an example of a situation in which H_0 and H_1 are topologically separated and in which an admissible test φ is for some x generalized Bayes while for other x, φ is not generalized Bayes. It will be the main work of Section 7 to abstract this form and prove a complete class theorem.

EXAMPLE 6.1. Let $\{\alpha_n, n \geqq 0\}$ be a strictly increasing sequence of positive real numbers such that $\alpha_0 = 0$ and if $n \geqq 1$ then $\alpha_{n+1} - \alpha_n < \pi$ and $\lim_{n\to\infty} \alpha_n = 2\pi$. Let H_0 be the convex hull of the points $\{(\cos \alpha_n, \sin \alpha_n), n \geqq 0\}$. Then the boundary of H_0 contains a countable number of line segments which converge towards the point (1, 0). If $n \geqq 1$ we let L_n be the line segment between $(\cos \alpha_n, \sin \alpha_n)$ and $(\cos \alpha_{n+1}, \sin \alpha_{n+1})$.

Let the parameter space Ω be any convex set containing H_0. If φ and φ' are test functions with power functions β, β' respectively and if φ' is as good as φ, then if $\omega \varepsilon H_0$, $\beta'(\omega) \leqq \beta(\omega)$, and if $\omega \varepsilon H_1$, $\beta'(\omega) \geqq \beta(\omega)$. In the exponential case power functions are continuous, and we find $\beta'(\omega) = \beta(\omega)$ if ω is on one of the line segments L_n.

If we write $L_n = \{x \mid x = \alpha\xi_n + \tau_n, 0 \leqq \alpha \leqq 1\}$ then $\beta(\alpha\xi_n + \tau_n)$ is an analytic function of the real variable α, $n \geqq 1$. Similarly $\beta'(\alpha\xi_n + \tau_n)$ is an analytic function of α, $n \geqq 1$. Since $\beta(\alpha\xi_n + \tau_n) = \beta'(\alpha\xi_n + \tau_n)$, $0 \leqq \alpha \leqq 1$, $n \geqq 1$, it follows from the analyticity of β, β' that $\beta(\alpha\xi_n + \tau_n) = \beta'(\alpha\xi_n + \tau_n)$, $-\infty < \alpha < \infty$, $n \geqq 1$. If $n \geqq 1$ let L_n' be the line $\{x \mid x = \alpha\xi_n + \tau_n, -\infty < \alpha < \infty\}$. If L is any line through $(1, 0)$ then all but at most two of the lines L_n', $n \geqq 1$, intersect L in a sequence of points $\{\omega_n, n \geqq 1\}$ such that $\lim_{n\to\infty} \omega_n = (1, 0)$ and if $n \geqq 1$, $\beta(\omega_n) = \beta'(\omega_n)$. The analyticity of β, β' then requires $\beta(\omega) = \beta'(\omega)$ for all $\omega \varepsilon L$. As this holds for all lines through $(1, 0)$ and since Ω is convex, $\beta(\omega) = \beta'(\omega)$, $\omega \varepsilon \Omega$. Since an exponential family is boundedly complete, $\mu(\{x \mid \varphi(x) \neq \varphi'(x)\}) = 0$.

This proves that every test is admissible.

Example 6.2. We will first describe the example, then a few remarks about its relevance are made. Then the actual verification of details which involves a considerable amount of work.

We assume $k = 2$, and $\Omega = X = \mathbb{R}_2$. μ will be Lebesque measure. We use the family of normal density functions $(2\pi)^{-1} \exp\left(-\frac{1}{2}\right)((x - \theta)^2 + (y - \eta)^2)$. We set $H_0 = \{(-1, 0), (1, 0)\}$ and $H_1 = \{(\theta, \eta) \mid \eta \geqq 1\} \cup \{(0, 0)\}$. For acceptance region A of a test take all points (x, y) satisfying $y \leqq \beta$ and $|x| \geqq \alpha$, $\alpha > 0, \beta > 0$.

In the sequel we prove several lemmas. From the lemmas it will follow that A is the acceptance region of an admissible test. We will then show A cannot be the acceptance region of a generalized Bayes test. The analysis will show that there exists a convex set C (here C is the half space $\{(x, y) \mid y \leqq \beta\}$) such that among all tests ψ satisfying, if $\psi(x) \neq 1$ then $x \varepsilon C$, A is a Bayes acceptance region. The complete class theorem of Section 7 will abstract this idea.

Lemma 6.3. (Stein) *Let $\mathcal{B}$ be the σ-algebra of Borel subsets of $\mathbb{R}_k$ and μ be a σ-finite measure on $\mathcal{B}$. Let $\{k(\omega) \exp(\omega \cdot x), \omega \varepsilon \Omega\}$ be an exponential family of density functions in $L_1(X, \mathcal{B}, \mu)$. Let H_0 and H_1 be disjoint nonempty subsets of Ω. Let A be a closed convex subset of $\mathbb{R}_k$ such that if $\xi \varepsilon \mathbb{R}$, $c \varepsilon \mathbb{R}$, and $A \cap \{x \mid \xi \cdot x > c\} = \varnothing$ then there exists ω_1 such that $\int \exp(\omega_1 \cdot x)\mu(dx) < \infty$ and a sequence $\lambda_n \uparrow \infty$ with $\omega_1 + \lambda_n\xi \varepsilon H_1$. Let φ be a test such that if $\varphi(x) \neq 1$ then $x \varepsilon A$. If the test ψ is as good as φ then $\mu(\{x \mid \psi(x) \neq 1 \text{ and } x \not\varepsilon A\}) = 0$.*

Proof. The proof given here is an almost direct copy of Stein [10]. If $\omega \varepsilon \Omega$ let P_ω be the probability measure on Ω determined by the density $k(\omega) \exp(\omega \cdot x)$. Let φ be as in the hypotheses of Lemma 6.3 and ψ as good as φ satisfying $\mu(\{x \mid \psi(x) \neq 1, x \not\varepsilon A\}) > 0$. Then there exists $\xi \varepsilon \mathbb{R}_k$ and $c \varepsilon \mathbb{R}$ such that $A \cap \{x \mid \xi \cdot x > c\} = \varnothing$ and $\mu(\{x \mid \psi(x) \neq \varphi(x) \text{ and } \xi \cdot x > c\}) > 0$. Then

$$\begin{aligned} &\int (\varphi(x) - \psi(x)) P_{\omega_1 + \lambda_n \xi}(dx) \\ (6.1) \qquad &= (k(\omega_1 + \lambda_n\xi)/k(\omega_1)) \exp(c\lambda_n) \\ &\quad \cdot \{\textstyle\int_{\{x \mid \xi \cdot x > c\}} (\varphi(x) - \psi(x)) \exp(\lambda_n(\xi \cdot x - c)) P_{\omega_1}(dx) \\ &\quad + \textstyle\int_{\{x \mid \xi \cdot x \leqq c\}} (\varphi(x) - \psi(x)) \exp(\lambda_n(\xi \cdot x - c)) P_{\omega_1}(dx)\}. \end{aligned}$$

As $n \to \infty$ the first integral tends to ∞ while the second integral is bounded.

Therefore at some parameter points $\omega_1 + \lambda_n \xi$ the test φ has better power than ψ. The negative of this implication is the conclusion of Lemma 6.3.

LEMMA 6.4. *Suppose the hypotheses of Lemma 6.3 about X, $\mathcal{B}$, μ, $\{k(\omega) \exp(\omega \cdot x), \omega \varepsilon \Omega\}$ and A hold. Suppose H_0 and H_1 are disjoint nonempty measurable subsets of Ω and η_0, η_1 are totally finite measures on $\mathcal{B}$ such that $\eta_0(H_1) = \eta_1(H_0) = 0$. Let φ be a test such that, if $\varphi(x) \neq 1$ then $x \varepsilon A$, and, if $x \varepsilon A$ then*

$$\begin{aligned} \varphi(x) \int k(\omega) \exp(\omega\cdot x)\eta_0(d\omega) + (1 - \varphi(x)) \int k(\omega) \exp(\omega\cdot x)\eta_1(d\omega) \\ = \inf_{0 \leqq a \leqq 1} (a \int k(\omega) \exp(\omega\cdot x)\eta_0(d\omega) \\ + (1 - a) \int k(\omega) \exp(\omega\cdot x)\eta_1(d\omega)). \end{aligned} \tag{6.2}$$

Then the test φ is admissible.

PROOF. If ψ is a test as good as φ then by Lemma 6.3, $\mu(\{x \mid x \not\varepsilon A$ and $\psi(x) \neq 1\}) = 0$. However within the class of all tests satisfying $\mu(\{x \mid x \not\varepsilon A$ and $\psi(x) \neq 1\}) = 0$ the test φ is the essentially unique Bayes procedure. We use here Lemma 4.4. Therefore φ is admissible.

We return to Example 6.2 and show the acceptance region described there is admissible. To apply Lemmas 6.3 and 6.4 let $A = \{(x, y) \mid y \leqq \beta\}$. Then relative to the class of tests which accept H_0 only in A the test φ is Bayes. For put mass $p/2$ at $(-1, 0)$, $p/2$ at $(1, 0)$ and $(1 - p)$ at $(0, 0)$. The expression to be minimized is then

$$(2\pi)^{-1} \exp(-\tfrac{1}{2}(x^2 + y^2))[\psi(x)p(e^{-x} + e^x) + (1 - \psi(x))(1 - p)].$$

Then $\psi(x) = 0$ if $|x| \geqq \alpha$ is the form of the acceptance region. By proper choice of p between 0 and 1 all $\alpha > 0$ may be obtained. Therefore φ is an admissible test. We use here Lemma 6.4.

We now show that φ cannot be generalized Bayes. For let $0 \leqq p \leqq 1$, put mass p at $(-1, 0)$, mass $1 - p$ at $(1, 0)$, and call this measure η_0.

Suppose η_1 is a σ-finite measure on $\mathcal{B}$ for which $\eta_1(H_0) = 0$. Let φ be generalized Bayes for the pair η_0, η_1. Then φ minimizes

$$\begin{aligned} \psi(x) \int \exp(-\tfrac{1}{2}(x - \theta)^2 - \tfrac{1}{2}(y - \eta)^2)\eta_0(d\theta, d\eta) \\ + (1 - \psi(x)) \int \exp(-\tfrac{1}{2}(x - \theta)^2 - \tfrac{1}{2}(y - \eta)^2)\eta_1(d\theta, d\eta). \end{aligned}$$

By hypothesis $\varphi(x) = 0$ if $y < \beta$ and $|x| > \alpha$. From convexity it follows that if $y < \beta$ then

$$\int \exp(-\tfrac{1}{2}(x^2 + y^2)) \exp(x\theta + y\eta) \exp(-\tfrac{1}{2}(\theta^2 + \eta^2))\eta_1(d\theta, d\eta) < \infty.$$

Therefore along the line segments $x = \pm\alpha$, $y < \beta$, we must have

$$\begin{aligned} \int \exp(x\theta + y\eta) \exp(-\tfrac{1}{2}(\theta^2 + \eta^2))\eta_0(d\theta, \theta\eta) \\ = \int \exp(x\theta + y\eta) \exp(-\tfrac{1}{2}(\theta^2 + \eta^2))\eta_1(d\theta, d\eta). \end{aligned} \tag{6.3}$$

Taking partial derivatives with respect to y under the integrals we find that if

$x = \pm\alpha$ and $y < \beta$ then

$$\int \eta \exp(x\theta + y\eta) \exp(-\tfrac{1}{2}(\theta^2 + \eta^2))\eta_0(d\theta, d\eta) = \int \eta \exp(x\theta + y\eta) \exp(-\tfrac{1}{2}(\theta^2 + \eta^2))\eta_1(d\theta, d\eta). \tag{6.4}$$

Therefore the two measures (of equal mass) represented in (6.3) have by (6.4) the same mean value in the y direction. In order that the acceptance region be contained in the half space $y \leqq \beta$ it is necessary that η_1 place positive mass *above* the line $y = 0$. Therefore in order to obtain the same mean value in the y-direction, it is necessary that η_1 place positive mass *below* the line $y = 0$. From this it follows that

$$\infty = \lim_{y \to -\infty} \int \exp(x\theta + y\eta) \exp(-\tfrac{1}{2}(\theta^2 + \eta^2))\eta_1(d\theta, d\eta).$$

Since $\int \exp(x\theta + y\eta) \exp(-\frac{1}{2}(\theta^2 + \eta^2))\eta_0(d\theta, d\eta)$ is independent of y, it follows that the generalized Bayes test for η_0, η_1 must always reject the hypothesis if $y \leqq y_0 < 0$ for some y_0. Therefore η_0, η_1 cannot determine the acceptance region of φ.

The analysis given has shown that φ is an admissible test which is not a generalized Bayes test. We now give a necessary and sufficient condition for admissibility which includes Example 6.2.

7. A necessary and sufficient condition for admissibility.

THEOREM 7.1. *Let $X \subset R_k$ and $\mathcal{B}$ be as previously. We suppose the parameter set Ω is a convex cone containing 0 such that Ω is a closed subset of R_k. Let μ be a σ-finite measure on $\mathcal{B}$ such that μ is absolutely continuous with respect to a non-atomic product measure. Let $\{f_\omega, \omega \in \Omega\}$ be an exponential family of density functions in $L_1(X, \mathcal{B}, \mu)$. Let H_0 and H_1 be disjoint subsets of Ω such that H_0 is compact and H_1 is closed. We suppose that if $\xi \in \Omega$ there exists $\omega_1 \in \Omega$ and a real number sequence $\lambda_n \uparrow \infty$ such that if $n \geqq 1$ then $\omega_1 + \lambda_n\xi \in H_1$. Then the following are equivalent.*

(i) *The test φ is admissible.*

(ii) *There exists a probability measure η_0 on $\mathcal{B}$ such that $\eta_0(H_1) = 0$, and a σ-finite measure η_1 on $\mathcal{B}$ such that $\eta_1(H_0) = 0$.*

There exists a convex set C such that

$$\mu(\{x \mid x \notin C \text{ and } \varphi(x) \neq 1\}) = 0. \tag{7.1}$$

Within the class of tests satisfying (7.1) φ is generalized Bayes for η_0, η_1. The convex set C is the intersection of Ω and half spaces whose normals are in Ω.

The remainder of Section 7 consists of a proof of Theorem 7.1. We begin with a proof that a test satisfying (ii) is admissible. The proof is an obvious modification of the proof of Theorem 5.1.

We begin by considering a truncation. Let $\{C_n, n \geqq 1\}$ be a sequence of measurable parameter sets such that $C_n \uparrow \Omega$ and if $n \geqq 1$, $\eta_1(C_n) < \infty$. If $n \geqq 1$ define η_{1n} by, if $E \in \mathcal{B}$ then $\eta_{1n}(E) = \eta_1(E \cap C_n)$. If $n \geqq 1$ let φ_n^* be Bayes related to η_0, η_{1n} and define φ_n by, if $n \geqq 1$ and $x \in A$ then $\varphi_n(x) = \varphi_n^*(x)$, if $x \not\varepsilon A$ then $\varphi_n(x) = 1$. Then $\varphi_n(x) = \varphi(x)$ if $x \not\varepsilon A$, while if $x \varepsilon A$, then $\varphi_n(x) = \varphi_n^*(x) \leqq \varphi(x)$. Further, it is clear by the nature of the truncation that $\varphi_n \uparrow \varphi$ a.e. $[\mu]$, since in the exponential case, generalized Bayes solutions are uniquely determined. Therefore

$$\mu(\{x \mid \lim_{n\to\infty} \varphi_n(x) \neq \varphi(x)\}) = 0. \tag{7.2}$$

Let φ have power function β and φ_n have power function β_n, $n \geqq 1$. Then we have $\beta \geqq \beta_n$. Further, the construction of φ_n requires

$$\int \beta_n(\omega)\eta_0(d\omega) + \int (1 - \beta_n(\omega))\eta_{1n}(d\omega) \leqq \int \beta(\omega)\eta_0(d\omega) + \int (1 - \beta(\omega))\eta_{1n}(d\omega), \tag{7.3}$$

so that we obtain

$$0 \leqq \int (\beta(\omega) - \beta_n(\omega))\eta_{1n}(d\omega) \leqq \int (\beta(\omega) - \beta_n(\omega))\eta_0(d\omega).$$

Thus both sides converge to zero.

The test $\chi'(x) = 1$ for all x satisfies the condition that if $\chi'(x) \neq 1$ then $x \varepsilon A$. Therefore

$$\int (\beta_n(\omega) - \beta(\omega))\eta_0(d\omega) + \int (\beta(\omega) - \beta_n(\omega))\eta_{1n}(d\omega) \leqq \int (1 - \beta(\omega))\eta_0(d\omega) + \int (\beta(\omega) - 1)\eta_{1n}(d\omega).$$

From this we see that (7.3) implies

$$\limsup_{n\to\infty} \int (1 - \beta(\omega))\eta_{1n}(d\omega) = \int (1 - \beta(\theta))\eta_1(d\omega) \leqq \int (1 - \beta(\omega))\eta_0(d\omega).$$

It now follows that if ψ satisfies, $\psi(x) \neq 1$ implies $x \varepsilon A$, then the generalized Bayes risk of ψ is as great as that of φ. From the uniqueness of φ as a minimizing solution, it now follows φ is admissible. For by Lemma 6.3, if ψ is as good as φ then $\psi(x) \neq 1$ implies $x \varepsilon A$.

In order to prove the necessity of (ii) in Theorem 7.1 we need two lemmas on the convergence of sequences of convex sets. These lemmas are Lemma 7.2 and Lemma 7.3.

LEMMA 7.2. *Let μ be a σ-finite measure on $\mathfrak{B}$ such that μ gives zero mass to hyperplanes of $\mathbb{R}_k$. Let $\{C_n, n \geqq 1\}$ be a sequence of closed convex sets, let C be a bounded convex set, and assume if $n \geqq 1$ then $C_n \subset C$ and $\mu(C_n) \geqq \epsilon > 0$. Then there exists a subsequence $\{C_{a_n}, n \geqq 1\}$ such that $\bigcap_{n=1}^{\infty} C_{a_n}$ has nonvoid interior.*

PROOF. Below we shall show that if π is any hyperplane there exists a subsequence $\{C_{a_n}, n \geqq 1\}$ and a point x such that $x \not\varepsilon \pi$ and $x \varepsilon \bigcap_{n=1}^{\infty} C_{a_n}$. We show that Lemma 7.2 follows from this. Then we prove the assertion.

By the first paragraph there exists a subsequence $\{C_{0,n}, n \geqq 1\}$ such that there exists $x_0 \varepsilon \bigcap_{n=1}^{\infty} C_{0,n}$. Let π_0 be a hyperplane through x_0. By the first paragraph there exists a subsequence $\{C_{1,n}, n \geqq 1\}$ of $\{C_{0,n}, n \geqq 1\}$ and a point $x_1 \varepsilon \bigcap_{n=1}^{\infty} C_{1,n}$, $x_1 \not\varepsilon \pi_0$. Suppose sequences $\{C_{0,n}, n \geqq 1\}, \cdots, \{C_{m,n}, n \geqq 1\}$, points $x_0, \cdots, x_m$ and hyperplanes $\pi_0, \cdots, \pi_{m-1}$ have been obtained such that if $1 \leqq i \leqq m$ then $\{C_{i,n}, n \geqq 1\}$ is a subsequence of $\{C_{i-1,n}, n \geqq 1\}$, $x_0, \cdots, x_{i-1}$

are in π_{i-1} and $x_i \not\in \pi_{i-1}$. Let π_m be a hyperplane through $x_0, \cdots, x_m$. This will be possible provided $m \leqq k - 1$. By the first paragraph there is a subsequence $\{C_{m+1,n}, n \geqq 1\}$ of $\{C_{m,n}, n \geqq 1\}$ and a point $x_{m+1} \in \bigcap_{n=1}^{\infty} C_{m+1,n}$, such that $x_{m+1} \not\in \pi_m$. If $m < k - 1$, the dimension, then $x_0, \cdots, x_{m+1}$ are contained in a hyperplane π_{m+1} and we continue by induction. If $m = k - 1$ then the convex hull of $\{x_0, \cdots, x_{m+1}\} = \{x_0, \cdots, x_k\}$ is a simplex with nonvoid interior and $\{x_0, \cdots, x_{m+1}\} \subset \bigcap_{n=1}^{\infty} C_{m+1,n}$, proving the lemma.

To prove the assertion of the first paragraph, let π be a hyperplane, $\pi = \{x \mid \xi \cdot x = c\}$. To each real number δ let $\pi_\delta^+ = \{x \mid \xi \cdot x > c + \delta\}$, $\pi_\delta^- = \{x \mid \xi \cdot x < c + \delta\}$, and $\pi_\delta = \{x \mid \xi \cdot x = c + \delta\}$. By hypothesis $\mu(\pi_\delta) = 0$, $\delta \in \mathbb{R}$. Relative to one of π_0^+ and π_0^-, say π_0^+, it is possible to choose a subsequence $\{C_n', n \geqq 1\}$ of $\{C_n, n \geqq 1\}$ such that if $a \geqq 1$ then $\mu(\pi_0^+ \cap C_n') \geqq \epsilon/2$. Then it will be possible to choose $\delta > 0$ such that if $n \geqq 1$ then $\mu((\pi_\delta^+ \cup \pi_\delta) \cap C_n') \geqq \epsilon/4$. For $\mu(C) < \infty$ and $\lim_{\delta \to 0} \mu(C \cap \pi_\delta^- \cap \pi_0^+) = 0$.

By the recurrence theorem, Loève [6], we may choose a further subsequence $\{C_n'', n \geqq 1\}$ of $\{C_n', n \geqq 1\}$ such that if $n_0, \cdots, n_k$ are integers $\geqq 1$, then $\mu((\pi_\delta^+ \cup \pi_\delta) \cap C''_{n_0} \cap \cdots \cap C''_{n_k}) \geqq 2^{-1}(\epsilon/4)^{k+1}$. By Helley's theorem, see Berge [1], $(\pi_\delta^+ \cup \pi_\delta) \cap \bigcap_{n=1}^{\infty} C_n''$ is nonempty.

Therefore Lemma 7.2 is proven.

LEMMA 7.3. *Let μ be a σ-finite measure on $\mathcal{B}$ such that μ is absolutely continuous relative to a nonatomic product measure on $\mathcal{B}$. Let $\{C_n, n \geqq 1\}$ be a sequence of closed convex subsets of $\mathbb{R}_k$. If $E \subset \mathbb{R}_k$ let $\chi(E, \cdot)$ be the indicator function of E. Then there exists a convex set C and a subsequence $\{C_{a_n}, n \geqq 1\}$ such that*

$$(7.4) \qquad \lim_{n\to\infty} \chi(C_{a_n}, \cdot) = \chi(C, \cdot) \text{ a.e. } [\mu],$$

and such that if C has nonvoid interior then

$$(7.5) \qquad \lim_{n\to\infty} \chi(C_{a_n}, x) = \chi(C, x)$$

at every x interior to or exterior to C.

PROOF. If to every bounded convex set A, $\limsup_{n\to\infty} \mu(A \cap C_n) = 0$ then we may choose $C = \emptyset$ and find a subsequence $\{C_{a_n}, n \geqq 1\}$ such that $\lim_{n\to\infty} \chi(C_{a_n}, x) = 0$ for almost all x $[\mu]$.

If $\limsup_{n\to\infty} \mu(A \cap C_n) > 0$ for some bounded convex set A then by Lemma 7.2 we may suppose a subsequence $\{C_{1,n}, n \geqq 1\}$ chosen such that $\bigcap_{n=1}^{\infty} C_{1,n}$ has nonvoid interior.

Let $\{x_n, n \geqq 1\}$ be an enumeration of the points of R_n whose coordinates are all rational. By a diagonalization argument we may choose $\{C_{2,n}, n \geqq 1\}$ a subsequence of $\{C_{1,n}, n \geqq 1\}$ such that one of the following two conditions hold. If $n \geqq 1$ then

(i) $x_n \in C_{2,m}$ for all but a finite number of integers $m \geqq 1$;

(ii) $x_n \in C_{2,m}$ for at most a finite number of integers $m \geqq 1$. We define C to be the convex hull of those x_n, $n \geqq 1$, which satisfy (i) relative to $\{C_{2,n}, n \geqq 1\}$.

C has nonvoid interior. For $\bigcap_{n=1}^{\infty} C_{2,n}$ has an interior point x. Therefore we may find integers $n_0, \cdots, n_k$ such that the convex hull of $x_{n_0}, \cdots, x_{n_k}$ contains

x as an interior point and $x_{n_0}, \cdots, x_{n_k} \varepsilon \bigcap_{n=1}^{\infty} C_{2,n}$. Therefore x is interior to C.

Let x be an interior point of C. Then we may find integers $n_0, \cdots, n_k$ such that x is interior to the convex hull of $x_{n_0}, \cdots, x_{n_k}$ and $x_{n_0}, \cdots, x_{n_k}$ satisfy (i). Then there exists an m_0 such that if $m \geqq m_0$ then $x_{n_0}, \cdots, x_{n_k} \varepsilon C_{2,m}$. Therefore if $m \geqq m_0$, x is interior to $C_{2,m}$ and $\lim_{m\to\infty} \chi(C_{2,m}, x) = \chi(C, x)$.

Let x be exterior to C. We show there exists m_0 and $\delta > 0$ such that if $m \geqq m_0$ then the distance of x to $C_{2,m}$ is at least δ. For suppose to the contrary that on the subsequence C_{2,a_m} the point x is of distance $\leqq 1/m$ to C_{2,a_m}, $m \geqq 1$. Let y be interior to C, and let $m_0, x_{n_0}, \cdots, x_{n_k}$ be as in the preceding paragraph such that y is interior to the convex hull of $x_{n_0}, \cdots, x_{k_k} \varepsilon C_{2,m}$. Then if L is the line segment joining y and x, given $\epsilon > 0$ we may find a point x_ϵ interior to C_{2,a_m}, $m \geqq 1/\epsilon$, such that $\|x_\epsilon - x\| < 2\epsilon$. It follows that $x_\epsilon \varepsilon C$ for all $\epsilon > 0$ and thus that x is a boundary point of C. This contradiction proves the assertion of this paragraph.

The proof of Lemma 7.3 is therefore completed.

Proof of the Necessity of (ii) in Theorem 7.1. We will use Theorem 3.7. In order to apply Theorem 3.7 we verify the hypotheses. If we change the ordinary Euclidean topology on Ω to include among the open sets H_0 and H_1, then in the testing problem the risk function $r(\omega) = \beta(\omega)$, $\omega \varepsilon H_0$, $= 1 - \beta(\omega)$, $\omega \varepsilon H_1$, becomes continuous. The topology on Ω is a locally compact Hausdorff topology which is σ-compact.

Since $L_1(X, \mathcal{B}, \mu)$ is a separable Banach space, the set of test functions being a closed convex subset of the unit ball of $L_\infty(X, \mathcal{B}, \mu)$ is weakly compact. This at once implies that the set of risk functions are weakly subcompact in the sense of Definition 3.1.

By Theorem 3.7 we may find sequences of measures $\{\eta_{0,n}, n > 1\}$, $\{\eta_{1,n}, n \geqq 1\}$ such that if $n \geqq 1$ then $\eta_{0n}(H_0) = 1$, if $n \geqq 1$ and φ_n is Bayes for $\eta_{0,n}$, $\eta_{1,n}$ with power function β_n, then, if φ has power function β,

$$\lim_{n\to\infty} \beta_n(\omega) = \beta(\omega), \qquad \omega \varepsilon \Omega, \tag{7.6}$$

and

$$\lim_{n\to\infty} \left(\int (\beta_n(\omega) - \beta(\omega))\eta_{0n}(d\omega) + \int (\beta(\omega) - \beta_n(\omega))\eta_{1n}(d\omega)\right) = 0. \tag{7.7}$$

We may without loss of generality assume $\varphi' = \text{weak} \lim_{n\to\infty} \varphi_n$ exists (as linear functionals on L_1) and obtain from (7.6) that the power function β' of φ' is the same as β. Since an exponential family of density functions is boundedly complete, $\varphi = \varphi'$ a.e. $[\mu]$.

Using the assumption that φ_n is Bayes and comparing with the test which always accepts H_1, we find

$$\begin{aligned} \int (\beta_n(\omega) - \beta(\omega))\eta_{0n}(d\omega) + \int (\beta(\omega) - \beta_n(\omega))\eta_{1n}(d\omega) \\ \leqq \int (1 - \beta(\omega))\eta_{0n}(d\omega) + \int (\beta(\omega) - 1)\eta_{1n}(d\omega). \end{aligned} \tag{7.8}$$

Since H_0 is compact we may choose from $\{\eta_{0n}, n \geqq 1\}$ a weakly convergent subsequence. To simplify notation we suppose weak $\lim_{n\to\infty} \eta_{0n} = \eta_0$. Then as H_0

is compact, $\eta_0(H_0) = 1$. Similarly we may suppose weak $\lim_{n\to\infty} \eta_{1n} = \eta_1$ (otherwise take a subsequence), the weak limit being in the sense that if $g:\Omega \to \mathbb{R}$ is continuous and has compact support then $\lim_{n\to\infty} \int g(\omega)\eta_{1n}(d\omega)$ exists. From (7.7) and (7.8) follows

$$(7.9)\qquad \limsup_{n\to\infty} \int (1 - \beta(\omega))\eta_{1n}(d\omega) \leqq \int (1 - \beta(\omega))\eta_0(d\omega).$$

The hypothesis that H_0 and H_1 are topologically separated has been made to ensure that $\eta_0 \neq \eta_1$. We need this observation in the sequel.

Define sets $\{C_{m,n}, m \geqq 1, n \geqq 1\}$ by

$$(7.10)\qquad \text{if } m \geqq 1, n \geqq 1 \quad \text{then} \quad C_{m,n} = \{x \mid \int f_\omega(x)\eta_{1n}(d\omega) \leqq m\}.$$

Since $f_\omega(x) = k(\omega) \exp(\omega\cdot x)$ and since $\eta_{1n}(\cdot)$ is a finite measure with compact support, $C_{m,n}$ is a closed convex set, $m \geqq 1$, $n \geqq 1$. Further, $C_{m+1,n} \supset C_{m,n}$, $m \geqq 1, n \geqq 1$.

We apply Lemma 7.3 and choose a subsequence $\{a_n, n \geqq 1\}$ the integers such that

$$(7.11)\qquad \text{if } m \geqq 1 \lim_{n\to\infty} \chi(C_{m,a_n}, x) = \chi(C_m, x)$$

exists a.e. $[\mu]$, C_m a closed convex set. It follows from (7.11) and the inequalities $C_{m+1,n} \supset C_{m,n}$ that if $m \geqq 1$ and if C_m has nonvoid interior then $C_{m+1} \supset C_m$. Using (7.9) we obtain the existence of a constant $K > 0$ such that

$$(7.12)\qquad \sup_{n\leqq 1} \iint (1 - \varphi(x))f_\omega(x)\mu(dx)\eta_{1n}(d\omega) \leqq K.$$

Then

$$(7.13)\qquad \begin{aligned} K &\geqq \iint_{(\Omega - C_{m,a_n})} (1 - \varphi(x))f_\omega(x)\mu(dx)\eta_{1n}(d\omega) \\ &\geqq m \int_{(\Omega - C_{m,a_n})} (1 - \varphi(x))\mu(dx). \end{aligned}$$

It follows from (7.11) that we may pass to the limit on n and obtain

$$(7.14)\qquad K \geqq m \int_{(\Omega - C_m)} (1 - \varphi(x))\mu(dx).$$

By the monotone convergence theorem, if $C = \bigcup_{m=1}^{\infty} C_m$

$$(7.15)\qquad 0 = \int_{(\Omega - C)} (1 - \varphi(x))\mu(dx).$$

We take C to be the convex set whose existence is asserted in Theorem 7.1.

If $\mu(C) = 0$ then $0 = \int (1 - \varphi(x))\mu(dx)$ and using the test φ, H_0 is always accepted. This test is a Bayes test and Theorem 7.1 is satisfied. In the remainder of the proof we suppose $\mu(C) > 0$. Then it follows that if $m \geqq m_0$ then $\mu(C_m) > 0$. By Lemma 7.3 it follows that $\lim_{n\to\infty} \chi(C_{m,a_n}, x) = \chi(C_m, x)$ at all x interior to and exterior to C_m, $m \geqq m_0$.

We now show that if x is interior to C then

$$(7.16)\qquad \begin{aligned} &\varphi(x) \int f_\omega(x)\eta_0(d_\omega) + (1 - \varphi(x)) \int f_\omega(x)\eta_1(d\omega) \\ &\qquad = \inf_{0\leqq a\leqq 1} \left(a \int f_\omega(x)\eta_0(d\omega) + (1 - a) \int f_\omega(x)\eta_1(d\omega)\right). \end{aligned}$$

Let x be interior to C. Then $x \varepsilon C_m$, $m \geqq m_1$. If x is a boundary point of C_m, $m \geqq m_1$ then to each $m \geqq 1$ we may find $\xi m \in \mathbb{R}_k$ and $c_m \varepsilon \mathbb{R}$ such that $\xi m \cdot x = c_m$ and $\xi m \cdot y \leqq c_m$, $y \varepsilon C_m$, $m \geqq m_1$. We may further suppose $\|\xi_m\| = 1$, $m \geqq m_1$. A simple compactness argument then shows that since $C_m \uparrow C$, x is a boundary point of C. Contradiction.

Since x is interior to C_{m_1} we may choose n_1 such that if $n \geqq n_1$ and $m \geqq m_1$ then x is interior to C_{m,a_n}. We may further choose $x_{n_0}, \cdots, x_{n_k}$ (of the countable dense set introduced in the proof of Lemma 7.3) such that x is interior to the convex hull $x_{n_0}, \cdots, x_{n_k} \subset C_{m,a_n}$, $m \geqq m_1$, $n \geqq n_1$. Let $x = \sum \alpha_i x_{n_i}$, where $(\alpha_0, \cdots, \alpha_k)$ is a probability vector. Then

$$\lim_{\|\omega\| \to \infty} \left(\sum \alpha_i f_\omega(x_i)\right)/f_\omega(x) = \infty. \tag{7.17}$$

Since we assume H_0 to be a closed subset of $\mathbb{R}_k$, it follows from (7.17) and from $\eta_1 = \text{weak} \lim \eta_{1n}$ that

$$\int f_\omega(x_i)\eta_{1n}(d\omega) \leqq m, n \geqq n_1 \quad \text{and} \tag{7.18}$$

$$\int f_\omega(x)\eta_1(d\omega) = \lim_{n\to\infty} \int f_\omega(x)\eta_{1n}(d\omega).$$

We have assumed that C has nonvoid interior, and from (7.18) we obtain $\int f_\omega(x)\eta_1(d\omega) < \infty$ on the interior of C. By Lemma 4.4 either $\eta_0 = \eta_1$ or $\mu(\{x \mid x \varepsilon C, \int f_\omega(x)\eta_0(d\omega) = \int f_\omega(x)\eta_1(d\omega)\}) = 0$. As noted earlier, our hypotheses exclude the case $\eta_0 = \eta_1$. From these remarks, from (7.18), from the functional inequalities that express the fact φ_n is Bayes, $n \geqq 1$, it follows that

$$\mu(\{x \mid x \varepsilon C, \lim_{n\to\infty} \varphi_{a_n}(x) \text{ does not exist}\}) = 0. \tag{7.19}$$

If x is exterior to C then $\lim_{n\to\infty} \int f_\omega(x)\eta_{1a_n}(d\omega) = \infty$ which implies

$$\mu(\{x \mid x \not\varepsilon C, \lim_{n\to\infty} \varphi_{a_n}(x) \neq 1\}) = 0. \tag{7.20}$$

In particular, $\lim_{n\to\infty} \varphi_{a_n}$ exists a.e. $[\mu]$ and since $\varphi = \text{weak} \lim_{n\to\infty} \varphi_{a_n}$ we obtain

$$\varphi = \lim_{n\to\infty} \varphi_{a_n} \text{ a.e. } [\mu]; \tag{7.21}$$

if x is interior to C then (7.16) holds.

To complete the proof of necessity of (ii) we establish the shape of the boundary of C by considering the boundaries of C_{m,a_n}, $n \geqq 1$. Since $\int \exp(\omega\cdot x)k(\omega)\eta_{1n}(d\omega) < \infty$, $x \varepsilon \mathbb{R}_k$, $n \geqq 1$, we may take partial derivatives under the integral sign. We use here the fact that the measures η_{1n}, $n \geqq 1$, have compact support. See Theorem 3.7. Thus the boundary surface of C_{m,a_n} is given by

$$m = \int f_\omega(x)\eta_{1a_n}(d\omega) \tag{7.22}$$

which has normal the vector

$$\int \omega f_\omega(x)\eta_{1a_n}(d\omega). \tag{7.23}$$

Since Ω is a convex cone and $\eta_{1a_n}(\cdot)$ is supported on Ω the vector given in (7.23) is a vector in Ω.

C_m is a limit of the sets C_{m,a_n} in the sense that $\lim_{n\to\infty} \chi(C_{m,a_n}, x) = \chi(C_m, x)$ for all x interior to or exterior to C_m. Therefore if x is a boundary point of C_m we may find a point sequence $\{x_{a_n}, n \geqq 1\}$ such that $x = \lim_{n\to\infty} x_{a_n}$ and if $n \geqq 1$, x_{a_n} is a boundary point, of C_{m,a_n}. By considering planes of support $\xi_{a_n} \cdot y = c_{a_n}$ through x_{a_n}, $n \geqq 1$, normalized by $\|\xi_{a_n}\| = 1$, $n \geqq 1$, we see that C_m has a plane of support $\xi \cdot y = c$ through x such that $\|\xi\| = 1$ and $\xi \varepsilon \Omega$.

Since $C_m \uparrow C$, a similar argument shows every boundary point of C to have a plane of support $\xi \cdot y = c$ such that $\|\xi\| = 1$ and $\xi \varepsilon \Omega$. Since C has a unique normal at almost all boundary points of C, the last assertion of Theorem 7.1 is verified.

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