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OPTIMAL STOPPING IN A MARKOV PROCESS[1]

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1. Introduction and summary. Let $X = (X_t, \mathbf{F}_t, P^x)_{t \geqq 0}$ be a Markov process where $(X_t; t \geqq 0)$ is the trajectory or sample path, $\mathbf{F}_t$ is the definitive σ-algebra of events generated by $(X_s; 0 \leqq s \leqq t)$, and P^x is the probability distribution on sample paths corresponding to an initial state x. The state space is taken as the semi-compact $(E, \mathbf{C})$ where E is a locally compact separable metric space with family of open sets $\mathbf{C}$. A non-negative extended real valued random variable T such that for each $t \geqq 0$, $\{T \leqq t\} \varepsilon \mathbf{F}_t$ is called a *Markov time* or *stopping time*. This paper studies the problem of choosing a stopping time T which, for a fixed $\lambda \geqq 0$, maximizes one of the following criteria:

(1) $\Theta_T(x) = E^x e^{-\lambda T} g(X_T)$;

(2) $\Lambda_T(x) = E^x[e^{-\lambda T} g(X_T) - \int_0^T e^{-\lambda s} c(X_s)\, ds]$, where $E^x T < \infty$; or

(3) $\Phi_T(x) = E^x[g(X_T) - \int_0^T c(X_s)\, ds]/E^x T$, where $0 < E^x T < \infty$;

where g and c are non-negative continuous functions defined on the state space of the process.

Dynkin [9] studied criterion (1) where $\lambda = 0$ under the general assumption that X is a standard process with a possibly random lifetime and under very weak continuity assumptions concerning the return function g. He showed that criterion (2) can often be transformed into criterion (1), and thus his approach is applicable in this case as well.

This paper studies optimal stopping in a Markov process having a Feller transition function, a special case in Dynkin's development. We further specialize to exponentially distributed lifetimes which causes the appearance of a discount factor $e^{-\lambda t}$, with the natural interpretation that a dollar transaction t time units hence has a present value of $e^{-\lambda t}$. Criterion (3) often has the meaning of a long-run time average return and a means of transforming this criterion into criterion (2) is given. Finally, some techniques for implementing Dynkin's approach in a variety of commonly occurring situations are given along with examples of their use.

2. Notation and basic assumptions. Throughout we assume that X is a Hunt process. In particular we assume that X is strong Markov with trajectories which are right continuous and have left limits and that X is quasi-left continuous, e.g., for any sequence of stopping times $(T(n); n = 1, 2, \cdots, \infty)$, if $T(n) \uparrow T(\infty) < \infty$ as $n \to \infty$ then $X_{T(n)} \to X_{T(\infty)}$ almost surely P^x for every x.

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Let **B** be the σ-algebra of topological Borel sets in $(E, \mathbf{C})$. Where not otherwise stated, a function will mean an extended real valued function defined on E and and universally measurable; that is, measurable with respect to the completion of every finite measure on **B**. With or without affixex, f, g, and h will denote functions. Let $C(E)$ be the class of bounded continuous functions.

For $x \varepsilon E$ and $\Gamma \varepsilon \mathbf{B}$ let $P_t(x, \Gamma) = P^x(X_t \varepsilon \Gamma)$, and let $P_t^\lambda(x, \Gamma) = e^{-\lambda t}P_t(x, \Gamma)$ for $\lambda \geqq 0$. For any f let $P_t^\lambda f(x) = \int_E f(y)P_t^\lambda(x, dy)$ and similarly define $P_t f(x)$, provided, of course, these integrals exist.

In this paper we consider only transition functions P_t having the property that for every $f \varepsilon C(E)$ and $t \geqq 0$, $P_t f \varepsilon C(E)$. This property defines a *Feller transition function* [10], and is used in this paper to ensure that the excessive majorant to a continuous function is lower semi-continuous.

Let E^x be the expectation operator corresponding to P^x. The phrase "almost surely," abbreviated a.s., will be understood to mean almost surely with respect to P^x for every x.

With or without affixes, S and T denote Markov times. For any non-negative f, the convention $E^x f(X_T) = \int_{T<\infty} f(X_T)\, dP^x$ is adopted. We call T^* *optimal at* x if $E^x e^{-\lambda T^*} g(X_{T^*}) = \sup_T E^x e^{-\lambda T} g(X_T)$. If T^* is optimal at x for every $x \varepsilon E$, we call T^* *optimal.*

For any (nearly) Borel set $A \subset E$, $T(A) = \inf \{t: t \geqq 0 \text{ and } X_t \varepsilon A\}$ is a Markov time, called the *entry time* of A. (It is understood that whenever the set in braces is empty, then $T(A) = \infty$.) Similarly, $T(A+) = \inf \{t: t > 0 \text{ and } X_t \varepsilon A\}$ is a Markov time, called the *hitting time* of A (with again $T(A+) = \infty$ when the set in braces is empty). The *exit time* of a Borel set A is defined as the entry time of $E \backslash A$.

3. Excessive functions and excessive majorants. A non-negative function h is said to be λ-*excessive* (with respect to P_t) if $P_t^\lambda h \leqq h$ for all $t \geqq 0$ and $\lim_{t \geqq 0} P_t h(x) = h(x)$ for all x. We omit the λ and say h is *excessive* when $\lambda = 0$. From Hunt [15] we have the important property:

(3.1) For any λ-excessive h and Markov times T and S, $T \geqq S$ and $h(x) < \infty$ imply $E^x e^{-\lambda T} h(X_T) \leqq E^x e^{-\lambda S} h(X_S)$. In particular, for $S = 0$, $h(x) \geqq E^x e^{-\lambda T} h(X_T)$ for all x.

Theorem 12.4 of Dynkin [10] yields the following condition, useful for verifying that a continuous function is excessive:

(3.2) If h is non-negative and continuous and $E^x e^{-\lambda T} h(X_T) \leqq h(x)$ whenever T is the exit time from U where U is an arbitrary open set with compact closure, then h is λ-excessive.

Let g be a non-negative function. A function f is called a λ-*excessive majorant of* g if (a) f is λ-excessive, (b) $f \geqq g$, and (c) if h is λ-excessive and $h \geqq g$ then $h \geqq f$. In [9], Dynkin has shown that for every non-negative g which is nearly Borel measurable and intrinsically continuous from below there exists a λ-

excessive majorant f. Property (c) above implies the uniqueness of f. If g is bounded, then f is bounded, since if $c = \sup_x g(x)$ then $c \geqq g$, c is λ-excessive and by (c) $c \geqq f$. Under our assumptions that g is continuous and that P_t is a Feller transition function, the λ-excessive majorant f may be found by a simple iteration which often supplies additional information. The construction was first used by McKean [17] for a Brownian motion process and the proof in the general case was given by Grigelionis and Shiryaev in [14]. The construction is given here in

THEOREM 1. (Grigelionis and Shiryaev). *Suppose g is non-negative and continuous and P_t is a Feller transition function. Let $h_0 = g$ and define $h_n = \sup_{t\geqq 0} P_t^\lambda h_{n-1}$ for $n = 1, 2, \cdots$. Then $h_n \geqq h_{n-1}$ and $\lim_n h_n$ is the λ-excessive majorant to g and is lower semi-continuous.*

PROOF. (See [14]).

4. Maximizing $\Theta_T(x) = E^x e^{-\lambda T} g(X_T)$. In this section g is a non-negative continuous function defined on E and f is the λ-excessive majorant to g, which exists and is lower semi-continuous by Theorem 1. We let $\Gamma_\epsilon = \{x: f(x) \leqq g(x) + \epsilon\}$ for $\epsilon \geqq 0$ and let $T(\epsilon)$ be the hitting time for Γ_ϵ. By the continuity of g and the semicontinuity of f each Γ_ϵ is closed and $\Gamma_\epsilon \downarrow \Gamma_0$ as $\epsilon \downarrow 0$.

LEMMA 1. *$T(\epsilon) \uparrow T(0)$ as $\epsilon \downarrow 0$ almost surely.*

PROOF. As $\epsilon \downarrow 0$, clearly $T(\epsilon)$ increases, and hence has a limit, denoted by T. Clearly $T \leqq T(0)$. If $T = \infty$, then $T(0) = \infty$, and $T = T(0)$. Thus we need only consider sample paths for which $T < \infty$, where, since the process is quasi-left continuous, $X_{T(\epsilon)} \rightarrow X_T$. From the right continuity of the process and that each Γ_ϵ is closed, $X_{T(\epsilon)} \varepsilon \Gamma_\epsilon$ for $\epsilon \geqq 0$ and thus $f(X_{T(\epsilon)}) \leqq g(X_{T(\epsilon)}) + \epsilon$. Letting $\epsilon \downarrow 0$ and using the lower semi-continuity of f and the continuity of g, $f(X_T) \leqq \liminf_{\epsilon \downarrow 0} f(X_{T(\epsilon)}) \leqq \liminf_{\epsilon \downarrow 0}[g(X_{T(\epsilon)}) + \epsilon] = g(X_T)$, or $X_T \varepsilon \Gamma_0$. Thus, $T(0) \leqq T$ and consequently $T = \lim_{\epsilon \downarrow 0} T(\epsilon) = T(0)$. □

One considers $E^x e^{-\lambda T} g(X_T)$ as the expected discounted "reward" associated with a Markov time T. By convention, $E^x e^{-\lambda T} g(X_T) = \int_{T<\infty} e^{-\lambda T} g(X_T)\, dP^x$ so that a reward of zero is associated with never stopping.

Lemmas 2 and 3 and Theorem 2 which follow are a slight modification of a theorem by Dynkin in [9].

LEMMA 2. *Let g be a non-negative continuous function and let f^* be any λ-excessive function such that $f^* \geqq g$. Then f^* is an upper bound on expected incomes,* i.e., *$f^*(x) \geqq \sup_T E^x e^{-\lambda T} g(X_T)$.*

PROOF. $f^* \geqq g$ implies $e^{-\lambda T} f^*(X_T) \geqq e^{-\lambda T} g(X_T)$ for any T and $E^x e^{-\lambda T} f^*(X_T) \geqq E^x e^{-\lambda T} g(X_T)$. But by property (3.1) $f^*(x) \geqq E^x e^{-\lambda T} f^*(X_T)$. □

Lemma 2 yields a simple condition for verifying that a given stopping time T is optimal. If $f^* \geqq g$, f^* is λ-excessive and T is such that $f^*(x) = E^x e^{-\lambda T} g(X_T)$ then clearly T is optimal.

LEMMA 3. (Dynkin) *Let $\epsilon > 0$ be given and suppose g is bounded on $E \backslash \Gamma_0$. If $f_\epsilon(x) = E^x e^{-\lambda T(\epsilon)} f(X_{T(\epsilon)})$ then $f_\epsilon = f$.*

PROOF. (See [9].)

THEOREM 2. (Dynkin) *Let g be a non-negative continuous function with λ-excessive majorant f. Then*

(i) $f(x) = \sup_T E^x e^{-\lambda T} g(X_T)$, *and*

(ii) *if g is bounded on $E \backslash \Gamma_0$ then for any $\epsilon > 0$,*

$$f(x) - \epsilon \leqq E^x e^{-\lambda T(\epsilon)} g(X_{T(\epsilon)}) \leqq f(x).$$

PROOF. (See [9]).

COROLLARY 1. *If $g \varepsilon C(E)$ and either $\lambda > 0$ or $T(0) < \infty$ almost surely, then $T(0)$ is optimal.*

PROOF. By Lemma 1, $T(\epsilon) \uparrow T(0)$ as $\epsilon \downarrow 0$ on $\{T(0) < \infty\}$ a.s. P^x, and since X_t is quasi-left continuous, $X_{T(\epsilon)} \to X_{T(0)}$ as $\epsilon \downarrow 0$. Thus

$$\begin{aligned} f(x) - \epsilon &\leqq E^x e^{-\lambda T(\epsilon)} g(X_{T(\epsilon)}) \\ &= \int_{T(0)<\infty} e^{-\lambda T(\epsilon)} g(X_{T(\epsilon)})\, dP^x + \int_{T(0)=\infty, T(\epsilon)<\infty} e^{-\lambda T(\epsilon)} g(X_{T(\epsilon)})\, dP^x. \end{aligned}$$

Since g is bounded and continuous, by the bounded convergence theorem the first term on the right converges to $E^x e^{-\lambda T(0)} g(X_{T(0)})$ as $\epsilon \downarrow 0$ while if either $\lambda > 0$ or $T(0) < \infty$ the second term converges to zero. Thus $f(x) \leqq E^x e^{-\lambda T(0)} g(X_{T(0)})$ and $T(0)$ is optimal. □

COROLLARY 2. *If X is a continuous process and g is continuous and bounded on the closure of $E \backslash \Gamma_0$ and if either $\lambda > 0$ or $T(0) < \infty$ almost surely, then T^*, the entry time for Γ_0 is optimal.*

PROOF. Both $g(X_{T(\epsilon)})$ and $g(X_{T(0)})$ are bounded for processes starting at $x \not\varepsilon \Gamma_0$. Thus, as in Corollary 1,

$$\begin{aligned} f(x) &= \lim_{\epsilon \downarrow 0} [f(x) - \epsilon] \leqq \lim_{\epsilon \downarrow 0} E^x e^{-\lambda T(\epsilon)} g(X_{T(\epsilon)}) \\ &= E^x e^{-\lambda T(0)} g(X_{T(0)}) = E^x e^{-\lambda T^*} g(X_{T^*}), \text{ for } x \not\varepsilon \Gamma_0 . \end{aligned}$$

For $x \varepsilon \Gamma_0$, $f(x) = E^x e^{-\lambda T^*} g(X_{T^*})$ so that T^* is optimal. □

COROLLARY 3. *Let $\mathbf{D}$ be the closed sets in E and for $A \varepsilon \mathbf{D}$ $T(A)$ be the hitting time for A. Then*

$$f(x) = \sup_{A \varepsilon \mathbf{D}} E^x e^{-\lambda T(A)} g(X_{T(A)}).$$

PROOF. We need only note in Theorem 2, that each $T(\epsilon)$ is the hitting time to the closed set Γ_ϵ. When g is unbounded we truncate, consider $g_n = g \wedge n$ and let $n \to \infty$. □

Corollary 3 implies that if a closed set A^* exists whose hitting time $T^* = T(A^*)$ is optimal in the class of all hitting times to closed sets, then T^* is optimal in the wider class of all stopping times. One might hope for a converse, but one can easily construct examples in which every hitting time has a finite expected reward but there exist stopping times with infinite expected reward. That this is the only type of exception is indicated by the following theorem.

THEOREM 3. *If there exists an optimal stopping time T^* then $T(0)$ is optimal at all initial points x for which $f(x) < \infty$.*

PROOF. By (3.1), $f(x) \geqq E^x e^{-\lambda T^*} f(X_{T^*})$ for all x. Since $f \geqq g$ and T^* is optimal,

$E^x e^{-\lambda T^*} f(X_{T^*}) \geqq E^x e^{-\lambda T^*} g(X_{T^*}) = f(x)$ for all x. Thus $f(x) = E^x e^{-\lambda T^*} f(X_{T^*})$. Next we claim $f(X_{T^*}) = g(X_{T^*})$ a.s. on $\{T^* < \infty\}$. We have $f(X_{T^*}) \geqq g(X_{T^*})$ and suppose the contrary, that $f(X_{T^*}) > g(X_{T^*})$ on a set in $\{T^* < \infty\}$ of positive P^x probability. Then $E^x e^{-\lambda T^*} f(X_{T^*}) > E^x e^{-\lambda T^*} g(X_{T^*}) = f(x)$, a contradiction. Hence $f(X_{T^*}) = g(X_{T^*})$ a.s. on $\{T^* < \infty\}$ or $X_{T^*} \varepsilon \Gamma_0$ a.s. on $\{T^* < \infty\}$ which implies $T(0) \leqq T^*$ a.s. by the definition of $T(0)$. But, again using (3.1), $T(0) \leqq T^*$ implies $E^x e^{-\lambda T(0)} g(X_{T(0)}) = E^x e^{-\lambda T(0)} f(X_{T(0)}) \geqq E^x e^{-\lambda T^*} f(X_{T^*}) = f(x)$ when $f(x) < \infty$. Hence $T(0)$ is optimal for initial points x with $f(x) < \infty$. □

The fact that T^* is optimal in the hypothesis of Theorem 3 plays an important role. It is *not* true that for every stopping time T' there exists a hitting time T for which $E^x e^{-\lambda T} g(X_T) \geqq E^x e^{-\lambda T'} g(X_{T'})$ for all x. For example, suppose that $X_t = X_0 + t$ for $X_0 > 1$, and $g(x) = 1 - 1/x$. Then no hitting time can replace $T' \equiv 1$ without loss at some initial points X_0.

Note that when $\lambda = 0$, a problem where g takes on negative values but is bounded below may be formulated as above by adding an appropriate constant.

Example 1. Let $(X_t ; t \geqq 0)$ be a Brownian motion process with drift $\mu \leqq 0$ and variance coefficient $\sigma^2 = 1$. Let $g(x) = x^+ = \max(x, 0)$, and consider the criterion function $\Theta_T(x) = E^x e^{-\lambda T} g(X_T)$ for $\lambda > 0$.

If $a = -(\mu - (\mu^2 + 2\lambda)^{\frac{1}{2}})^{-1}$ and $v(x) = a \exp(x/a - 1)$ then

$$\begin{aligned} f(x) &= x \quad \text{for} \quad x > a, \\ &= v(x) \quad \text{for} \quad x \leqq a, \end{aligned}$$

will be shown to be the λ-excessive majorant to g. Note that: (i) $v(x) \geqq f(x) \geqq g(x)$; (ii) $f(x) = g(x) = x$ for $x \geqq a$; and (iii) $P_t^\lambda v(x) = v(x)$ for all x. Let $h_0 = g$ and define $h_n = \sup_{t \geqq 0} P_t^\lambda h_{n-1}$ so that by Theorem 1, $h_n \uparrow h$ where h is the λ-excessive majorant to g. But since v is λ-excessive and exceeds g, $v \geqq h$ so that $h(a) = v(a) = g(a)$.

Note that each h_n is increasing, convex, hence continuous, with slope less than or equal to one, so that h also inherits these properties. Then $h(a) = a$, $h(x) \geqq x$, and $dh/dx \leqq 1$ imply that $h(x) = x$ for $x \geqq a$.

Let T be the hitting time of $\Gamma = [a, \infty)$. Then by (3.1)

$$\begin{aligned} h(x) \geqq E^x e^{-\lambda T} h(X_T) &= a E^x e^{-\lambda T} \quad \text{for} \quad x \leqq a. \\ &= x \quad \text{for} \quad x > a. \end{aligned}$$

By direct calculation using well known results on first passage times in a Brownian motion process (Cox and Miller [5], p. 211) $a\, E^x e^{-\lambda T} = v(x)$ for $x \leqq a$. Thus $h = f$ and f is the λ-excessive majorant to g. Since $f(x) = E^x e^{-\lambda T} g(X_T)$, T is optimal according to the remark following Lemma 2. The same result holds for $\lambda = 0$ provided $\mu < 0$.

This example is motivated by the work of McKean [17] who studied a similar model but where $Y_t = \log X_t$ is a Brownian motion process. A heuristic proof of the optimality of T is in [23].

EXAMPLE 2. Let $(X_t ; t \geqq 0)$ be a Uhlenbeck process whose transition density

$$p_t(x, y) = P_t(x, dy)/dy$$

satisfies

$$\partial^2 p/\partial x^2 - x\partial p/\partial x = \partial p/\partial t.$$

We note that X is a Gaussian Markov process with continuous sample paths and consider the problem of maximizing $E^x e^{-T} X_T$ at $x = 0$. We first show that for some $b \varepsilon (0, 1)$, the entry T_b of the X process into $[b, \infty)$ maximizes over all stopping times T, $E^x e^{-T} g(X_T)$ where $g(x) = \max \{x, 0\}$. Then since $g(x) \geqq x$, $E^x e^{-T(b)} g(X_{T(b)}) \geqq E^x e^{-T} X_T$ for all stopping times T. Breiman [4] gives an approximation which shows $P^x(T_b < \infty) = 1$, and since $b > 0$, $g(X_{T(b)}) = X_{T(b)}$ so that T_b maximizes the possibly negative return $E^x e^{-T} X_T$ as well.

Let f be the 1-excessive majorant to g, and let $v_a(x) = \exp [(x^2 - a^2)/2]$ and $h_a(x) = a^{-1}[v_a(x) - 1] + a$. The derivatives are given by $h_a'(x) = a^{-1} x v_a(x)$ and $h_x''(x) = (1 + x^2) a^{-1} v_a(x)$. For $a \geqq 1$, $\min_x h_a(x) = a^{-1} [\exp(-a^2/2) - 1] + a \geqq a - 1/a \geqq 0$, $h_a(a) = a = g(a)$, $h_a'(a) = 1$, and $h_a''(x) \geqq 0$ which implies $h_a(x) \geqq g(x)$ for all x, again remembering that $a \geqq 1$.

If $y_a(x) = h_a(x) - a + 1/a$ and $L = \partial^2/\partial x^2 - x\partial/\partial x$, the differential operator corresponding to the X process, then $Ly_a - y_a = 0$. From Theorem 13.16, p. 51 of [10], we have $y_a(x) = E^x e^{-T} y_a(X_T)$ for T an exit time from an arbitrary bounded open interval. Consequently $h_a(x) = E^x e^{-T} h_a(X_T) + (a - 1/a) \cdot E^x(1 - e^{-T}) \geqq E^x e^{-T} h_a(X_T)$. By (3.2) then, h_a is 1-excessive. Thus $h_a \geqq f \geqq g$, and since $h_a(a) = g(a)$, provided $a \geqq 1$, we have $[1, \infty) \subset \Gamma_0$ where $\Gamma_0 = \{x : f(x) = g(x)\}$. We've shown that g is bounded off Γ_0 and from Corollary 2 to Theorem 2 we have T^*, the entry time to Γ_0, is optimal for all x.

We shall consider entry times $T(b)$ of $[b, \infty)$ and show that $f_b(x) > 0$ where $f_b(x) = E^x e^{-T(b)} g(X_{T(b)})$ and thus $f \geqq f_b > 0$, or $(-\infty, 0] \cap \Gamma_0 = \phi$. Hence, for processes starting at $X_0 = 0$, a stopping time of the form $T(b)$ with $0 < b < 1$ is optimal. Since $P[T(b) < \infty] = 1$ we have

$$f_b(x) = bE^x e^{-T(b)}, \qquad x < b.$$

The solution is given in [6] as

$$f_b(x) = be^{x^2/4} D_{-1}(-x)/e^{b^2/4} D_{-1}(-b), \qquad x < b,$$

where $D_\nu(z)$ is the parabolic cylinder function ([11], p. 116),

$$D_\nu(z) = e^{-z^2/4}(\Gamma(-\nu))^{-1} \int_0^\infty t^{-\nu-1} e^{-zt-t^2/2} dt, \qquad \nu < 0.$$

Since $(d/db) f_b(x) = f_b(x)[b^{-1} - D_{-2}(-b)/D_{-1}(-b)]$, equating to zero yields $D_{-1}(-b) = bD_{-2}(-b)$ which may be reduced to $(1 - b^2)\Phi(b) - b\phi(b) = 0$, with Φ and ϕ the standard normal distribution and density functions, respectively. Since the left hand side in the above equation is positive for $b = 0$ and negative for $b = 1$ we know the optimal $b = b^* \varepsilon (0, 1)$. A numerical solution yields $b = 0.839+$.

Now let $(Y(s); s \geqq 0)$ be Brownian motion with $Y(0) = 0$ and consider finding a stopping time S^* which maximizes over all stopping times S the expected averaged return $E^y[Y(S)/(1 + S)]$ for $y = 0$. Following Doob [7] make the time scale transformation $s = e^{2t} - 1$ and let $X_t = e^{-t}Y(e^{2t} - 1)$. Then X_t has the statistics of the previously considered Uhlenbeck process, and $e^{-t}X_t$ transforms back into $Y(s)/(1 + s)$. Thus $S^* = \inf \{s: Y(s) \geqq b(1 + s)^{\frac{1}{2}}\}$ with $b = 0.839+$ is the optimal stopping time for the averaged Brownian motion. This problem was suggested for study in [8].[2]

5. Maximizing $\Lambda_T(x) = E^x[e^{-\lambda T}g^*(X_T) - \int_0^T e^{-\lambda s}c(X_s)\, ds]$. Let g^* and c be non-negative continuous functions defined on E. Suppose that stopping at time T one receives the discounted reward $e^{-\lambda T}g^*(X_T)$ and incurs the costs $\int_0^T e^{-\lambda s}c(X_s)\, ds$.

If $R^\lambda c(x) = \int_0^\infty e^{-\lambda t}P_t c(x)\, dt < \infty$ for all x, then

$$\begin{aligned} E^x \int_0^T e^{-\lambda t}c(X_t)\, dt &= E^x \int_0^\infty e^{-\lambda t}c(X_t)\, dt - E^x \int_T^\infty e^{-\lambda t}c(X_t)\, dt \\ &= R^\lambda c(x) - E^x e^{-\lambda T}E^{X_T} \int_0^\infty e^{-\lambda t}c(X_t)\, dt \\ &= R^\lambda c(x) - E^x e^{-\lambda T}R^\lambda c(X_T). \end{aligned}$$

Thus

$$\Lambda_T(x) = E^x[e^{-\lambda T}g^*(X_T) + e^{-\lambda T}R^\lambda c(X_T)] - R^\lambda c(x).$$

This representation, which when $R^\lambda c$ is finite and continuous, translates a problem with an observation cost into a problem with no observation cost, was suggested by Dynkin [9] for use in optimal stopping problems. Now let $g(x) = g^*(x) + R^\lambda c(x)$ and apply the techniques given in Section 4. Note that when $\lambda = 0$, if g^* and $R^\lambda c$ are bounded below rather than non-negative, the problem can be expressed in the earlier form by adding an appropriate constant in the definition of g.

Example 3. Let $(X_t ; t \geqq 0)$ be a Poisson process with mean parameter μ. Suppose $g^*(x) = x$ and $c(x) = c > 0$ for $x = 1, 2, \cdots$. Then $R^\lambda c(x) = c/\lambda > 0$ and $g(x) = g^*(x) + R^\lambda c(x) = x + c/\lambda$, for any fixed $\lambda > 0$.

Let $k = \log_e (1 + \lambda/\mu)$, assume $k^{-1} \geqq c/\lambda$ and let $a = k^{-1} - c/\lambda \geqq 0$. For convenience suppose a to be an integer. Let $v(x) = k^{-1} \exp(-k[a - x])$ and define

$$\begin{aligned} f(x) &= x + c/\lambda \quad \text{for} \quad x > a \\ &= v(x) \qquad \text{for} \quad x \leqq a. \end{aligned}$$

Again f is the λ-excessive majorant to g. First apply the inequality $e^{-\theta} \geqq 1 - \theta$ where $\theta = 1 - k(x + c/\lambda)$, to get $e^{-[1-k(x+c/\lambda)]} \geqq k(x + c/\lambda)$ and $k^{-1}e^{-k(a-x)} =$

[2] L. A. Shepp has independently obtained this same result plus many further results in the context of similar problems. His extensive work will soon appear in a forthcoming paper.

$v(x) \geqq x + c/\lambda = g(x)$. Thus (i) $v(x) \geqq g(x)$ for all x; (ii) by definition $f(x) = v(x) = g(x)$ for $x \geqq a$; and finally,

$$\text{(iii)} \qquad P_t^\lambda v(x) = k^{-1}e^{-k(a-x)} \sum_{j=0}^{\infty} e^{-\lambda t} e^{kj} (\mu t)^j e^{-\mu t}/j!$$
$$= k^{-1}e^{-k(a-x)} = v(x).$$

Let $h_0 = g$ and define $h_n = \sup_{t \geqq 0} P_t^\lambda h_{n-1}$ so that by Theorem 1, $h_n \uparrow h$ where h is the λ-excessive majorant to g. Since v is λ-excessive and $v \geqq g$ we have $v \geqq h \geqq g$ so that $h(a) = v(a) = g(a) = a + c/\lambda$. Each h_n is increasing in x and $h_n(x + 1) - h(x) \leqq 1$. Combining this with $h(a) = g(a)$ shows that $h(x) = g(x) = x + c/\lambda$ for $x \geqq a$. Let T be the hitting time of $\Gamma = \{a, a + 1, \cdots\}$. By (3.1)

$$h(x) \geqq E^x e^{-\lambda T} h(X_T)$$
$$= (a + c/\lambda) E^x e^{-\lambda T} \quad \text{for} \quad x \leqq a$$
$$= x + c/\lambda \quad \text{for} \quad x > a.$$

Since T has a gamma distribution, $E^x e^{-\lambda T} = (1 + \lambda/\mu)^{-(a-x)} = kv(x)$. Thus $h = f$, and f is the λ-excessive majorant to g. Since $f(x) = E^x e^{-\lambda T} g(X_T)$, T is optimal according to the remark following Lemma 2.

EXAMPLE 4. Let $(X_t\,; t \geqq 0)$ be a Brownian motion process. Let $c^*(x) = x^2$, γ be a positive constant and $\phi(s, t) = \int_s^t [c^*(X_u) - \gamma]\, du$. To minimize $E^x\phi(0, T)$ over stopping times T, let T_n be the moment of first exit from $(-n, n)$ and set $g_n(x) = E^x\phi(0, T_n)$. Then

$$-E^x\phi(0, T \wedge T_n) = -E^x\phi(0, T_n) + E^x\phi(T \wedge T_n, T_n)$$
$$= -g_n(x) + E^x g_n(X_{T\wedge T_n}).$$

First we shall maximize $E^x g_n(X_{T\wedge T_n})$ and then take limits as $n \uparrow \infty$. Green's function for the process on $(-n, n)$ is given by the density

$$g_r(x, y)\, dy = n^{-1}(n - x)(n + y)\, dy \quad \text{for} \quad -n < y \leqq x < n$$
$$= n^{-1}(n - y)(n + x)\, dy \quad \text{for} \quad -n < x \leqq y < n.$$

From this one may calculate $g_n(x) = -\gamma n^2 + n^4/6 + \gamma x^2 - x^4/6$. For $x \in (-n, n)$ and $n \geqq (6\gamma)^{\frac{1}{2}}$ we see that $g_n(x) \geqq 0$. Let $v(x) = \gamma x^2 - x^4/6$, so that $g_n(x) = -v(n) + v(x)$. By elementary calculus, $g_n((3\gamma)^{\frac{1}{2}}) = \max_x g_n(x) = -v(n) + 3\gamma^2/2$. Thus $-v(n) + 3\gamma^2/2 \geqq g_n(x) \geqq 0$ for all $x \in (-n, n)$ and since $-v(n) + 3\gamma^2/2$ is a constant, it must exceed the excessive majorant to g_n.

As usual let $h_0 = g_n$ and $h_{m+1} = \sup_{t\geqq 0} P_t h_m$ so that $h_m \uparrow h$, the excessive majorant to g_n. Let

$$f_n(x) = -v(n) + 3\gamma^2/2 \quad \text{for} \quad |x| < (3\gamma)^{\frac{1}{2}}$$
$$= g_n(x) \quad \text{for} \quad (3\gamma)^{\frac{1}{2}} \leqq |x| < n.$$

On $(-n, n)$, f_n is concave so that $E^x f_n(X_{t\wedge T_n}) \leqq f_n(E^x X_{t\wedge T_n}) = f_n(x)$. Hence

f_n is excessive, $f_n \geqq g_n$ and thus $f_n \geqq h$. Hence for $(3\gamma)^{\frac{1}{2}} \leqq |x| < n$, $f_n = g_n = h$. Let $\Gamma = \{x: |x| \geqq (3\gamma)^{\frac{1}{2}}\}$, and let T_Γ be the entry time for Γ. Then $h(x) \geqq E^x h(X_{T_\Gamma \wedge T_n}) = f_n(x)$ so that f_n is the excessive majorant to g_n. Applying Corollary 2 to Theorem 2 shows that for $n \geqq (6\gamma)^{\frac{1}{2}}$ we have

$$E^x\phi(0, T_\Gamma \wedge T_n) \leqq E^x\phi(0, T \wedge T_n)$$

for any stopping time T. Clearly $\lim_{n\to\infty} E^x\phi(0, T_\Gamma \wedge T_n) = E^x\phi(0, T_\Gamma)$. If $E^xT < \infty$ then

$$\begin{aligned}
\lim_{n\to\infty} & E^x\phi(0, T \wedge T_n) \\
&= \lim_{n\to\infty} E^x \int_0^{T\wedge T_n}[c^*(X_s) - \gamma]\, ds \\
&= \lim_{n\to\infty} E^x \int_0^{T\wedge T_n} [c^*(X_s) - \gamma]^+\, ds - \lim_{n\to\infty} E^x \int_0^{T\wedge T_n}[c^*(X_s) - \gamma]^-\, ds \\
&= E^x \int_0^T[c^*(X_s) - \gamma]^+\, ds - E^x \int_0^T[c^*(X_s) - \gamma]^-\, ds\,,
\end{aligned}$$

by the monotone convergence theorem. Since $0 \leqq [c^*(X_s) - \gamma]^- \leqq \gamma$, we have

$$E^x \int_0^T [c^*(X_s) - \gamma]^-\, ds \leqq \gamma E^xT < \infty, \qquad \text{and}$$

$$\lim_{n\to\infty} E^x\phi(0, T \wedge T_n) = E^x\phi(0, T).$$

Thus T_Γ minimizes $E^x\phi(0, T)$ over all stopping times T for which $E^xT < \infty$.

6. Maximizing $\Phi_T(x) = E^x[g(X_T) - \int_0^T c(X_s)\, ds]/E^xT$. In this section let g and c be continuous functions and γ a constant. Let

$$\Phi_T(x) = E^x[g(X_T) - \int_0^T c(X_s)\, ds]/E^xT, \qquad \text{and}$$

$$\Theta_T(\gamma, x) = E^x[g(X_T) - \int_0^T (\gamma + c(X_s))\, ds]$$

where we consider only T's and x's for which $0 < E^xT < \infty$. $\Phi_T(x)$ represents the long-run time average return or negative cost if a sequence of statistically independent stopping games are played, each starting at x (see [16]). Simple algebra yields

$$\Theta_T(\gamma, x) = (\Phi_T(x) - \gamma)E^xT, \qquad \text{and}$$

$$\Phi_T(x) = \gamma + \Theta_T(\gamma, x)/E^xT,$$

which leads to the

THEOREM 4. *Let $x \in E$ be fixed and let $\mathbf{T} = \{T: 0 < E^xT < \infty\}$. If for some γ, $T^* \in \mathbf{T}$ maximizes $\Theta_T(\gamma, x)$ over $\mathbf{T}$, and $\Theta_{T^*}(\gamma, x) = 0$ then T^* maximizes $\Phi_T(x)$ over $\mathbf{T}$ and conversely.*

PROOF. $\Theta_{T^*}(\gamma, x) = 0$ implies $\Phi_{T^*}(x) = \gamma$. But $\Theta_{T^*}(\gamma, x) = 0 \geqq \Theta_T(\gamma, x)$ implies $\Phi_T(x) = \gamma + \Theta_T(\gamma, x)/E^xT \leqq \gamma$. For the converse, set $\gamma = \Phi_{T^*}(x)$. Then $\Theta_{T^*}(\gamma, x) = 0$ while $\Theta_T(\gamma, x) = (\Phi_T(x) - \gamma)E^xT = (\Phi_T(x) - \Phi_{T^*}(x))E^xT \leqq 0$. □

EXAMPLE 5. Let $(X_t\,; t \geqq 0)$ be a Brownian motion process, $c(x) = x^2$, and $g(x) = K > 0$ for all x, and consider maximizing $\Phi_T(x) = (-K - E^x \int_0^T X_s^2\, ds)/E^xT$. By Theorem 4, if a T optimal among the class $0 < E^xT < \infty$ exists, then it may be found by finding a γ and a T^* such that $0 = \Theta_{T^*}(\gamma, x) = -K + E^x \int_0^{T^*} (\gamma - X_s^2)\, ds$, and $\Theta_{T^*}(\gamma, x) \geqq \Theta_T(\gamma, x)$ for all T. This is the problem considered in Section 5. From Example 4 for any $\gamma > 0$ the optimal T^* for $\Theta_T(\gamma, x)$ is given as the entry time for $\Gamma = \{x: |x| \geqq (3\gamma)^{\frac{1}{2}}\}$ and for initial state $x = 0 \not\in \Gamma$, $\Theta_{T^*}(\gamma, 0) = -K + 3\gamma^2/2$. Thus for $\Theta_{T^*}(\gamma, 0) = 0$ one needs $\gamma = (2K/3)^{\frac{1}{2}}$ and thus the optimal T^* for $\Phi_T(x)$ is given as the entry time to $\{x: |x| > \lambda\}$, where $\lambda = (6K)^{\frac{1}{4}}$.

This example is drawn from Bather [1] who treated the more difficult case where the X_t process is observable only with error.

EXAMPLE 6. Let $(X_t\,; t \geqq 0)$ be a diffusion process on $E = (0, \infty)$ with drift coefficient $\mu(x) = x^2/(1 + x) + (1 + x)$ and diffusion coefficient $\sigma^2(x) = x^2$. Let $c(x) = x/(1 + x)$ and consider finding a stopping time T which maximizes

$$\Theta_T(\gamma, x) = -K - E^x \int_0^T [\gamma + c(X_s)]\, ds$$

where $0 < -\gamma < 1$. Such a problem arises in quality control where X_t is a function of the posterior probability that a manufacturing process is out of control, given the history of previous production. (See [20], [22]. But note that [22] treats costs while we treat returns. Thus γ in [22] becomes $-\gamma$ here.)

Let T_n be the first exit time from $(0, n]$. As before we shall first maximize $\Theta_{T\wedge T_n}(\gamma, x)$ over T and then let $n \to \infty$. One can show (Dynkin [10] or Shiryaev [20]) that $g_n(x) = E^x \int_0^{T_n} (c(X_s) + \gamma)\, ds$ is a solution to

$$[\tfrac{1}{2}\, \sigma^2(x)\, d^2/dx^2 + \mu(x)\, d\cdot/dx]g_n(x) = -[x/(1 + x) + \gamma],$$

for $x \,\varepsilon\, (0, n)$ and $g_n(n) = 0$. Let $\phi(x) = z^2 \exp(-2/z)$, $\psi(z) = \int_0^z \phi(y)\, dy$, $A(z) = \psi(z)/\phi(z)$ and $C(z) = \int_0^z A(y)/(1 + y)^2\, dy$. Then

$$dg_n(x)/dx = (1 + \gamma)A(x)(1 + x)^{-2} - (1 + \gamma)(1 + x)^{-1} + (1 + x)^{-2}$$

for $x \,\varepsilon\, (0, n)$ and $g_n(x) = -\int_x^n g_n'(y)\, dy = u(x) - u(n)$ where $u(z) = (1 + \gamma)C(z) - (1 + \gamma)\log_e(1 + z) + z/(1 + z)$. These computations including discussions of boundary conditions may be found in [2], [20], [22]. As before

$$\Theta_{T\wedge T_n}(\gamma, x) = -K - [u(x) - u(n)] + E^x[u(X_{T\wedge T_n}) - u(n)],$$

so that the current problem is to maximize $E^x g_n(X_{T\wedge T_n})$.

For n sufficiently large $g_n(x) \geqq 0$ for all $x \,\varepsilon\, (0, n)$ and $g_n(x)$ has a maximum at the unique solution $x = \lambda^*$ to $x - A(x) = -\gamma/(1 + \gamma)$. Since $A(x) > 0$, one has $\gamma + c(x) = x/(1 + x) + \gamma > 0$ for $x \geqq \lambda^* > 0$. Let

$$f(x) = g_n(\lambda^*) \quad \text{for} \quad x < \lambda^*$$
$$= g_n(x) \quad \text{for} \quad \lambda^* \leqq x \leqq n.$$

Note that $g_n(\lambda^*)$ is a constant, exceeds $g_n(x)$ for all $x \,\varepsilon\, (0, n)$ and thus exceeds h, the excessive majorant to g_n. Let $\Gamma = [\lambda^*, n]$ and let $T(\Gamma)$ be the hitting time of Γ. Then for $x \,\varepsilon\, (0, \lambda^*]$ one has $h(x) \geqq E^x g_n(X_{T(\Gamma)}) = g_n(\lambda^*) = f(x)$. Thus $f(x) = h(x) = g_n(\lambda^*)$ for $x \,\varepsilon\, (0, \lambda^*]$, provided, of course that $\lambda^* < n$.

Again one can show that f is excessive. Let $G_1 = (0, \lambda^*)$ and $G_2 = (\lambda^* - \epsilon, n]$. Since f is constant in G_1, f is excessive in G_1. For $\epsilon < 0$ such that $x/(1+x) + \gamma > 0$ for $x \varepsilon G_2$ one has $g_n(x) = E^x \int_0^{T_n} [c(X_s) + \gamma]\, ds$ is excessive. To show this fix $x \varepsilon G_2$ and let U be an open neighborhood of x contained in G_2 with exit time $T(U)$. Then

$$E^x g_n(X_{T(U)}) = E^x E^{X_{T(U)}} \int_0^{T_n} [c(X_s) + \gamma]\, ds$$
$$\leqq E^x \int_0^{T_n} [c(X_s) + \gamma]\, ds = g_n(x).$$

Thus each g_n is excessive on $(\lambda^*, n]$ so that $f = g_n$ on $(\lambda^*, n]$ is excessive in this region. It remains only to consider a neighborhood about λ^*. Let $x \varepsilon G_2$ and $U = (r_1, r_2)$ be a neighborhood of x with $\lambda^* - \epsilon < r_1 < \lambda^*$. Let $T(U)$ be the exit time from U and T^* be the hitting time for $\{\lambda^*\}$. Then

$$E^x f(X_{T(U)}) \leqq E^x f(X_{T^* \wedge T(U)}) \leqq E^x g_n(X_{T^* \wedge T(U)}) \leqq f(x).$$

Thus f is excessive here also and hence f is excessive on $(0, n]$, and f is the excessive majorant to g_n and $T(\Gamma)$, the hitting time of $\Gamma = [\lambda^*, n]$, is optimal for the problem of maximizing over T the expression $\Theta_{T \wedge T_n}(\gamma, x)$. For initial points $x < \lambda^*$, the solution is independent of n and thus $T(\Gamma)$ is optimal for $\Theta_T(\gamma, x)$ at such initial points. According to Theorem 4, one should find a γ such that $\Theta_{T(\Gamma)}(\gamma, x) = 0$. In practice it's easier to list the optimal $T(\Gamma)$ or λ^* as γ is varied, and then also list the K such that $\Theta_{T(\Gamma)}(\gamma, x) = 0$. This is the approach that produced the charts in [22].

7. Remarks. While our theory applies to space-time processes, such as problems with a fixed finite time horizon and problems in which the reward function includes time as in the criterion $E^x[g(X_T)/T]$, it is often difficult to compute explicit solutions. We hope to consider such problems in the future.

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